Can Median-Maximizing Behavior Be Rational?

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Abstract

In this note, we consider a perennial problem in single-person choice theory, that is, characterizing choice under uncertainty. In particular, we consider a hypothesis put forward by Joseph Stiglitz (2005), suggesting that median-maximing behavior may be optimal under certain circumstances, and consider how it might best be rationalized within choice theory as it is currently conceived. As is well known, median-maximizing behavior is not generally optimal in the classical VNM framework. Our main result is that it is possible to rationalize the Stiglitz hypothesis in the Machina-Schmeidler (1992) framework of probabilistic sophistication.

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1 Introduction and Motivation

A perennial problem in single-person choice theory, variously known as utility theory or decision theory, concerns characterizing choice under uncertainty. Here, there is a gulf between results that have been established at the frontiers of economic theory, and the models that are generally deployed in applied theoretical work in fields such as, inter alia, industrial organization, economic development, and finance. Most applications still employ a version of the classical Von Neumann-Morgenstern (VNM) expected utility model, if it is assumed that probabilities (and, a fortiori, beliefs) are objectively determined, or, instead, the Savage subjective expected utility model, if beliefs (and the induced probabilities) are believed to be subjective. In this note, we consider a hypothesis about behavior under uncertainty, in a particular context, put forward by Joseph Stiglitz [6], and consider how it might best be rationalized within choice theory as it is currently conceived.

Consider, then, the following argument by Stiglitz, that appeared in the influential policy journal, Foreign Affairs. He comments on the fact that in recent times mean income has risen in the United States, while median income has actually fallen. He continues: "Consider the following thought experiment: If you could choose which country to live in but would be assigned an income randomly from within that country's income distribution, would you choose the

country with the highest GDP per capita? No. More relevant to that decision is median income ... As the income distribution becomes increasingly skewed, with an increasing share of the wealth and income in the hands of those at the top, the median falls further and further below the mean. That is why, even as per capita GDP has been increasing in the United States, U.S. median household income has actually been falling."

This rich quotation from Stiglitz contains, inter alia, an assertion about the evolution of the shape of the income distribution in the United States: whether this is accurate or not is an empirical question, that Dehejia and Marcel Voia [3] take up elsewhere. More relevant for our purpose, Stiglitz seems to be suggesting that, when faced with a choice over a set of income distributions that are rightward skewed, a (presumably) risk-averse individual would do better to pick the one with the highest median, not the highest mean. This would be, presumably, because the mean, in some sense, overestimates the "true" centre of the data, compared to the median. (The mean would, presumably, underestimate, in this sense, if the distribution were leftward skewed.) This would seem to accord with common sense. After all, a few very rich individuals raise US mean income, but do nothing to the median: so it is surely sensible to imagine that you are going to end up somewhere around the median, and discount the effect that Bill Gates and his ilk have upon the mean.

The first and most obvious question to ask is, how is one to understand this assertion? It can be read, at one level, as a claim about a "rule of thumb" behavior that an individual, either who is not fully rational, or who is operating under conditions of extreme uncertainty, will, or should follow: if "will", then it is an empirical claim; if "should", then a normative assertion. This would certainly be a legitimate reading, and could, if one takes the former interpretation, give rise to a research agenda within experimental, or behavioral, economics, to determine if individuals do, indeed, behave this way, under such conditions. Studying the behavior of putative migrants, choosing between two or more potential locations, would be the most natural application, although one could just as easily look at applications within, for instance, finance, such as portfolio choice, or, to take another example, occupation choice within labor economics. (This "rule of thumb" approach is pursued further in Dehejia [2].)

We will note, in passing, that there is also a more abstract, indeed philosophical, interpretation to the situation just described, if one takes the claim as a normative one. One can imagine, not a putative migrant, but a rational agent, behind a Rawlsian veil of ignorance, deciding on what sort of society he would like to live in. The only modification is that there would be now, not a finite, but a (hopefully countably) infinite number of distributions to consider, and the choice of a particular distribution represents, not the choice of a location where one would like to migrate, but the shape of a just society to which one wishes to belong. John Rawls' [5] celebrated investigation along these lines yielded the "maximin" rule: society should maximize the well-being of its least-well off member. One could imagine constructing a neo-Rawslian, call it Stiglitzian, political theory in which the chosen rule is to maximize the well-being of the median individual in society. We will not pursue these philosophical reflections

further in this paper.

Rather than pursuing either of the two avenues just noted, instead, we shall consider whether, and how, one might rationalize the Stiglitz intuition of median-maximizing behavior within contemporary choice theory. We see two potential approaches. The first is to restrict ourselves to the VNM approach. In this approach, it is known that a risk-averse individual, given a choice between two lotteries, will pick the one that exhibits second-order stochastic dominance over the other. (This is to leave aside the prior result, that any expected utility maximizer, risk averse or not, will prefer a lottery that first-order stochastically dominates another.) However, in general, there is no reason to expect that second-order stochastic dominance will be equivalent to a higher median, or vice-versa. The problem then reduces to finding restrictions on the utility function, or on the shape of distributions, to generate the result than a risk-averse expected utility maximizer will pick the highest median. No such general results are available, which takes us to our second approach, the one that we pursue here.

Our second approach is to move beyond the VNM model, and consider more general non-expected utility theory approaches to the choice-theoretic problem under uncertainty. In the next section, we shall formalize the problem, in a non-expected utility framework, and, using this, demonstrate that median-maximizing behavior can be rational. The final section offers some concluding observations.

2 The Main Result

We begin with some preliminaries. Let $\mathcal{X} \subset R^1_+$ be the set of outcomes. Define lottery Q as follows

$$Q = (x_1, q_1; x_2, q_2; \ldots; x_n, q_n)$$

where $x_i \in \mathcal{X}$, $q_i \geq 0$, $i = 1, 2, \ldots, n$, and

$$\sum_{i=1}^{n} q_i = 1$$

for $n = 1, 2, \ldots$.

Denote $D(\mathcal{X})$ the set of lotteries having finite support. For each lottery $Q \in D(\mathcal{X})$, define the median m_Q of Q by

$$\Pr(X_Q \leq m_Q) \geq \frac{1}{2} \text{ and } \Pr(X_Q \geq m_Q) \geq \frac{1}{2}. \tag{1}$$

It is noted that the solutions to above two inequalities may not be unique. If so, we assume the median m_Q of Q is the minimum of the solutions to the two inequalities (1).

Definition 1 *Given preferences $\succeq$ on $D(\mathcal{X})$, say an agent is an expected utility maximizer if there exists utility index $u : \mathcal{X} \longrightarrow R^1$ such that*

$$Q \succeq P \Longleftrightarrow \sum_{i=1}^{n} u(x_i) q_i \geq \sum_{i=1}^{n} u(x_i) p_i.$$

Question 2 *Suppose one agent has the following preferences*

$$Q \succeq P \Longleftrightarrow m_Q \geq m_P. \tag{2}$$

Is the agent an expected utility maximizer?

Unfortunately, as noted in our introduction, the answer is "no" in general.

The VNM model is linear in probability. Moreover, it specifies a particular functional form,

$$U(Q) = \sum_{i=1}^{n} u(x_i)\, q_i. \tag{3}$$

However, the particular functional form is not important for the underlying choice problem; the key issue is if the agent makes decision based on probabilities.

To overcome this difficulty with the VNM framework, Machina and Schmeidler [4] introduce the following probabilistic sophistication model.

Some further notions are required. A probability distribution $P = (x_1, p_1; ...; x_m,\ p_m)$ is said to *first-order stochastically dominate* $Q = (y_1,\ q_1; ...;\ y_n,\ q_n)$ over the outcome set $\mathcal{X}$ if

$$\sum_{\{i: x_i \leq x\}} p_i \leq \sum_{\{j: y_j \leq x\}} q_j \qquad \text{for all } x \in \mathcal{X}.$$

Use the term *strict dominance* if the above holds with strict inequality for some $x \in \mathcal{X}$.

Say that V is *monotonic* (with respect to stochastic dominance) if

$$V(P)(>)\ \geq\ V(Q)$$

whenever P (strictly) stochastically dominates Q, P and Q in $dom(V)$.

Given a real-valued function V defined on a mixture subspace $dom(V)$ of $D(\mathcal{X})$, say that V is *mixture continuous* if for any distributions P, Q and R in $dom(V)$, the sets

$$\{\lambda \in [0,\ 1] : V(\lambda P + (1-\lambda)Q) \geq V(R)\} \text{ and}$$
$$\{\lambda \in [0,\ 1] : V(\lambda P + (1-\lambda)Q) \leq V(R)\}$$

are closed.

Definition 3 *Given preferences $\succeq$ on $D(\mathcal{X})$, say an agent is probabilistically sophisticated if there exists a mixture continuous and monotonic function $V : D(\mathcal{X}) \longrightarrow R^1$ such that*

$$Q \succeq P \Longleftrightarrow V(Q) \geq V(P).$$

Roughly speaking, no stand is taken on the functional form of V, apart from monotonicity and mixture continuity, thus capturing primarily the decision-maker's reliance on probabilities for the evaluation of lotteries. VNM expected utility is merely one example, albeit an important one, in which V is an expected utility function on lotteries $D(\mathcal{X})$ and thus U has the familiar form (3).

Next, we ask

Question 4 *Suppose an agent has the following preferences defined in (2). Is the agent probabilistically sophisticated?*

Fortunately, the answer is "yes." The reasons are the following:
For any $Q = (x_1, q_1; x_2, q_2; \ldots; x_n, q_n) \in D(\mathcal{X})$, define

$$V(Q) = \arg\min_m \sum_{i=1}^{n} |x_i - m| \, q_i. \tag{4}$$

It is well known (see, for example, [1]) that $V(Q) = m_Q$. To prove that the preference relation $\succeq$ defined in (2) to be probabilistically sophisticated, we only need to prove V defined in (4) to be mixture continuous and monotonic.

Suppose that $P = (x_1,\, p_1; ...;\, x_m,\, p_m)$ first-order stochastically dominates $Q = (y_1,\, q_1; ...;\, y_n,\, q_n)$ over the outcome set $\mathcal{X}$. That is,

$$\sum_{\{i: x_i \le x\}} p_i \le \sum_{\{j: y_j \le x\}} q_j \qquad \text{for all } x \in \mathcal{X}.$$

Accordingly, $m_Q \le m_P$.

Next, we prove V is also mixture continuous. Without loss of generality, assume both Q and P have the same support. Thus,

$$V(\lambda Q + (1-\lambda) P) = \arg\min_m \sum_{i=1}^{n} |x_i - m| \left(\lambda q_i + (1-\lambda) p_i\right).$$

For given Q and P in $D(\mathcal{X})$ and $\lambda \in [0,\ 1]$, define

$$F(m) = \sum_{i=1}^{n} |x_i - m| \left(\lambda q_i + (1-\lambda) p_i\right).$$

Under the assumption made at the beginning, function F is continuous in m and $F(m)$ has a unique minimum $m(\lambda)$. By the maximum theorem (see, for example, [7]), the unique minimum $m(\lambda) = V(\lambda Q + (1-\lambda) P)$ is continuous in λ. Thus, V is mixture continuous. As a result, $\succeq$ defined in (2) is probabilistically sophisticated.

3 Concluding Remarks

In this note, we have considered a hypothesis of Joseph Stiglitz [6], arguing that, under certain, perhaps realistic, conditions, an agent may follow median-maximizing behavior. As is well known, such behavior is not generally optimal in the classical VNM framework. Our contribution is to demonstrate that it is possible to provide a firm choice-theoretic foundation for the Stiglitz hypothesis if we move beyond the restrictive VNM framework and consider the more general model of probabilistic sophistication of Machina-Schmeidler [4]. We prove that, with a particular functional form for the utility function in this framework, median-maximizing behavior emerges as the optimal choice rule.

We conclude with some open questions and remaining topics for research. First, while we have provided a formal rationale for the Stiglitz hypothesis, it might fairly be asked whether this matches the intuition of the type of problem that he described. An alternative approach would be to consider his hypothesis one about non-strictly-rational "rule of thumb" behavior, and to explore to what extent such behavior might, or might not match, what is known to be optimal in different theoretical frameworks. A tentative step in this direction is taken by Dehejia [2], who demonstrates that, at least within a particular application, that is, comparing income distributions in Canada and the United States over different years, an agent following a median-maximizing rule of thumb would, in fact, be picking the income distribution that exhibits second-order stochastic dominance, that is, it happens to coincide with the optimal choice of a risk-averse VNM expected utility maximizer.

A remaining question for future research is whether a rational agent, operating under conditions of extreme uncertainty, for instance, having knowledge only of summary statistics such as the mean, median, and mode, and with limited information about the shape of the distributions, might resort to a median-maximizing rule as an "approximation" to what would be optimal.

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