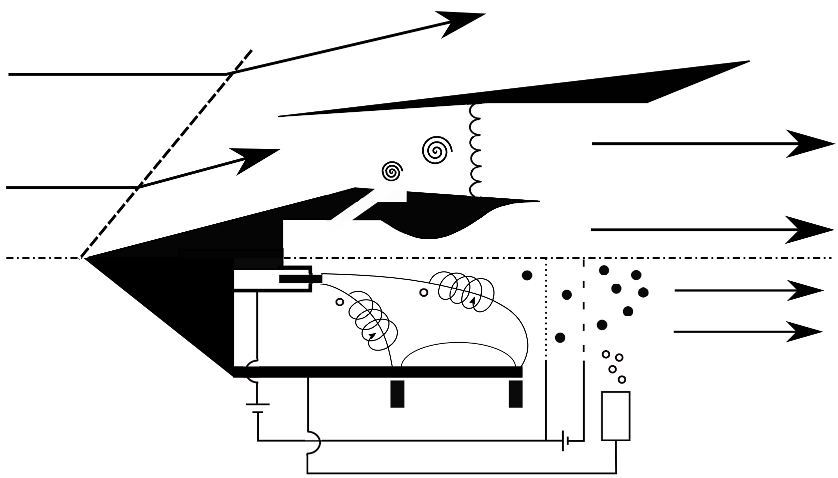
J. ETELE

FUNDAMENTALS OF TRANSATMOSPHERIC & SPACE PROPULSION



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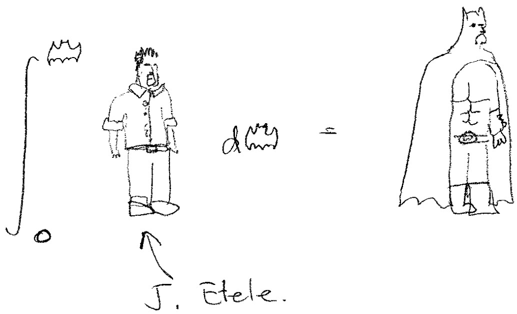

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To order books or to contact the author please email: thealdanoxgroup@gmail.com or call 1-613-327-3816.

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For everyone there exists a unique passion.
Lucky are those who find it;
Fortunate are those who pursue it;
Blessed are those that delight in it.

For my passion Judy (and my delights Dan, Orion, and Ally)

Nomenclature

A	area
a	acceleration, semi-major axis length, speed of sound
B_i	total number of atoms
B	magnetic induction/magnetic flux density
b	semi-minor axis length
c	speed of light [$299.792x10^6$ m/s]
c^*	characteristic velocity
c_D	coefficient of drag
c_k	massfraction
c_{L_α}	coefficient of lift
c_p	specific heat at constant pressure
c_T	thrust coefficient
c_v	specific heat at constant volume
D	drag
$\mathcal{E}$	expectation
E	electric field
E_{KE}	particle kinetic energy
E_{pot}	particle potential energy
e	specific mechanical energy, specific energy
F	force, focus
G	Gibbs free energy
G	Universal gravitational constant [$6.674x10^{-11}$ m^3/(kg s^2)], Gibbs free energy per unit volume, mass flux
g	specific Gibbs free energy
H	enthalpy
H	enthalpy per unit volume
h	specific angular momentum, specific enthalpy, height, Planck's constant [$6.626x10^{-34}$ Js]
I	impulse, moment of inertia, current
I_{sp}	specific impulse
i	inclination
J_2	non-spherical Earth zonal harmonic ($1.0826x10^{-3}$)
j	current density
K	burning surface area to nozzle throat area ratio
K_c	equilibrium constant based on concentrations
K_p	equilibrium constant based on partial pressures
KE	kinetic energy
k	equivalent spring constant
k_b	backwards reaction rate, Boltzmann constant [$1.380x10^{23}$ J/K]

k_f	forwards reaction rate
L	lift, rotation matrix
l_{gc}	great circle length
$\mathcal{M}$	molecular mass
M	Mach number
M^*	characteristic Mach number
m	mass
N_A	Avagadro's number [$6.022x10^{23}$ particles]
n	combustion index, number of stages, number density
$\mathcal{P}$	perveance
P	momentum, period, power per unit length
PE	potential energy
p	semi-latus rectum, pressure
p_u	probability function of u
Q	heat
Q	heat per unit volume
$Q_{e/a}$	collision cross section
q	heat per unit mass, elementary charge [$1.60218x10^{-19}$ C]
${}_1(\delta q)_2$	heat of reaction/energy of combustion
R	radius, gas constant
R_E	radius of Earth
R_L	Larmor radius
$\Re$	Universal gas constant [8314 J/(kmol K)]
r	distance in cylindrical and spherical coordinates
S	reference area, entropy per unit volume
s	specific entropy, distance
r	distance in cylindrical coordinates, surface recession rate
T	thrust, temperature
TOF	time of flight
t	time
u	x axis velocity
$\vec{u}$	unit vector
V	velocity
v	volume, y axis velocity
W	work
W	weight, work per unit volume
w	z axis velocity, work per unit mass
$[X_k]$	moles of species k per volume
x	Cartesian coordinate axis
y	Cartesian coordinate axis
z	Cartesian coordinate axis, height

Greek (and other) letters

α	angle of attack, cone half angle, Lagrange multiplier, air to rocket massflow ratio, fraction of total particles
α_i	ionized massflow fraction
β	shock angle, relaxation parameter, structural ratio, mixing parameter
β_m	fraction of particles in the m^{th} excited state
β_{rot}	fraction of particles with rotational state excited
β_{vib}	fraction of particles with vibrational state excited
γ	ratio of specific heats
δ	deflection angle
ϵ	eccentricity
ε	emissivity
ε_{diss}	dissociation energy
ε_{ion}	ionization energy
ε_m	energy of the m^{th} excited state
ε_o	permittivity of free space [$8.854x10^{-12}$ $\mathrm{C}^2/(\mathrm{N\ m}^2)$]
ζ	vorticity, ratio of fuel burned
ζ_{ab}	afterburner to core rocket massflow ratio
ζ_e	frozen flow parameter
η	Brayton efficiency, thrust vector angle
η_c	compression efficiency
η_{ik}	particles of atom i per particle of species k
θ	angle in cylindrical and spherical coordinates, angle with respect to vertical
Θ	latitude, air to rocket total specific enthalpy ratio, heat flux potential
ϑ	specific internal energy, eccentric anomaly
ϑ^f	specific internal energy of formation
$\tilde{\vartheta}_p^f$	heat of formation
κ	thermal conductivity
λ	divergence factor, payload ratio, plane change angle
λ_D	Debye length
μ	gravitational parameter, Mach wave angle, mass ratio
μ_e	electron mobility
ν	true anomaly
ν_k	stoichiometric mole number
Ξ	reaction rate
Π	rocket to entrained air total pressure ratio
Π_c	compression ratio
Π_m	mixed flow to entrained air total pressure ratio
Π_p	temperature sensitivity of pressure

Π_r	temperature sensitivity of the burning rate
ρ	density
ρ_q	charge density
ϱ	specific volume (=$1/\rho$)
σ	molar amount, rocket exhaust area to total inlet area ratio, conductivity
τ	shear stress, ratio of normal velocities across a shock
τ_c	characteristic time
ϕ	elevation angle, equivalence ratio, angle in spherical coordinates electric potential, gravitational potential
Φ	ratio of heat release compared to Chapman-Jouguet condition
χ	massfraction of particulate phase
χ_{++}	double ionization force correction factor
$\chi_{\dot{m}++}$	double ionization mass correction factor
$\chi_{B/T}$	beam to total potential ratio
χ_L	magnetized plasma length to total length ratio
χ_{Tr}	beam to total current ratio
Ψ	free fall angle, thrust augmentation ratio
ω	argument of perigee, rotation rate
ω_c	cyclotron frequency
$\omega_{e/a}$	collision frequency
Ω	Earth's rotation rate, resistance
Ω_{an}	right ascension of the ascending node
Ω_{Hall}	Hall parameter
Υ	vernal equinox

Contents

1

Introduction

Speed. Over the next little while as we consider the material covered in this book, we will find ourselves becoming devoted to the study of speed. We will become hooked on it, always wanting more, looking for different ways to get it. We will be sad when we don't have enough. We will feel triumphant when we discover a new way to get it. We will think about it day and night, almost to an obsession.....hold on a minute, this is beginning to sound a bit unhealthy....

Before we go and book ourselves into a self-help program, perhaps we should be a bit clearer. We're talking here about *velocity*. This book is all about how to generate velocity, specifically, how to generate the velocity required for various applications associated with launch and/or space propulsion. We need this limitation as without it, there are simply too many options to consider in a single textbook (turbines, internal combustion engines, propellers, wings, legs,...). Even limiting ourselves to space applications this still leaves a plethora of options to consider. For example, how do we want to get to space? Or perhaps, what do we want to do once we get there? Maybe we want to consider travelling great distances through space? Depending on how we choose to answer these questions different methods of propulsion (think generation of speed) should be considered.

However, a common thread among all the methods of propulsion we will consider in this book is their suitability for generating a desired velocity. And this is where the subtlety lies, the suitability. We can't JUST consider velocity. Velocity doesn't exist in a void (or does it?) but requires some sort of mass which then travels at this velocity. But a moving mass has momentum,

$$\vec{P} = m\vec{V} \tag{1.1}$$ $\left[\frac{\text{kg m}}{\text{s}}\right]$

which thanks to Sir Isaac Newton we know is something that is universally conserved *unless* the mass 'containing' this momentum is acted upon by a force in which case,

$$\vec{F} = \frac{d}{dt}(\vec{P}) \tag{1.2}$$ $\left[\frac{\text{kg m}}{\text{s}^2} = \text{N}\right]$

This expression, also known as *Newton's Second Law*, might be considered to be the 'Grand-Daddy' of all propulsion equations by some. The idea that force is required to generate a change in momentum (and thus velocity since mass cannot be created, or can it?) leads to another little subtlety. For very large masses, or very large changes in velocity, large forces are required. This is why, for getting to space, rockets have been extensively developed. They are capable of generating very large forces (on the order of millions of Newtons [N], or thousands of pounds for the Imperialists out there), which helps propel very heavy space ships to orbital velocities and beyond. Of course, there are some who might prefer alternative launch technologies (airbreathing rockets or scramjet aided engines to name but two), but why if rockets have proven to work so well? This leads us to another small subtlety, we can't JUST consider force.

For many propulsive devices force is generated by expelling mass which means that for a self contained vehicle there will be an upper limit as to how much can be expelled. Thus the ratio of force produced to the rate at which mass is ejected,

$$\frac{F}{\dot{m}_{pr}}$$ $\left[\frac{\text{N}}{\frac{\text{kg}}{\text{s}}} = \frac{\cancel{\text{kg}}\left(\frac{\text{m}}{\cancel{\text{s}^2}}\right)}{\frac{\cancel{\text{kg}}}{\cancel{\text{s}}}} = \frac{\text{m}}{\text{s}}\right]$

becomes a key consideration. Although this ratio adequately expresses the desired parameter, because of those pesky Imperialists again, the weight of things and the mass of things is often ambiguous ('pounds' anyone?). So to convert the proper *massflow* to a *weightflow* one must multiply by the acceleration of gravity (which can be problematic when in deep space but we'll leave that for now). In doing so one obtains the *Specific Impulse*, I_{sp},

$$I_{sp} = \frac{F}{\dot{m}_{pr} a_g} \tag{1.3}$$

$$\left[\frac{\cancel{m}}{\cancel{s}} \frac{1}{\left(\frac{\cancel{m}}{s^{\cancel{2}}}\right)} = \mathrm{s} \right]$$

With this we can now see why airbreathing engines might offer some advantages over traditional rockets. If one can obtain similar levels of thrust (F) using less weightflow of ejected mass/propellant ($\dot{m}_{pr} a_g$) then a higher I_{sp} is achieved. This might help get us to space more efficiently, but obviously once in space an airbreathing engine is simply not an option. And yet rockets clearly don't have the same dominance *in* space as they do *getting to* space. This leads us to yet another subtlety. We can't JUST consider I_{sp}.

Space missions can be notoriously long compared to a typical launch which is on the order of minutes. This idea of time can also play a key role in selecting the appropriate propulsion technology. Recalling Eqs.1.1 and 1.2 for a moment, written in our more familiar terrestrial form where mass is often treated as constant,

$$\vec{F} = \frac{d}{dt}(\vec{P}) = \frac{d}{dt}(m\vec{V}) = \frac{dm}{dt}\vec{V} + m\frac{d\vec{V}}{dt} = \dot{m}\vec{V} + m\vec{a}$$

and thus if $\dot{m} = 0$ then we obtain the familiar,

$$\vec{F} = m\vec{a} \tag{1.4}$$

Although quite familiar, Eq.1.4 reminds us that not only is force related to the velocity but also to the acceleration. In other words, how much time we have to generate the desired velocity is also an important factor that should be considered. In practical terms this means that since most, if not all, of our ideas concerning propulsion involve the expulsion of mass (propellant) over a long period of time, a large amount of mass would have to be expelled. If we are travelling in orbit, or through the galaxy, this means that all this mass has to first get to space. This can be quite expensive (try booking a flight on Virgin Galactic!). However, since once in space there is no atmospheric drag, this means that a continually applied force will result in a *continuous* acceleration, the integration of which yields the velocity.

From the point of view of the ejected mass the law of equal and opposite reaction (Newton's Third Law) requires that the magnitude of the force on our vehicle is the same as that on the propellant leaving a control volume surrounding our vehicle. Thus at a given instant in time (and dropping the vector notation which indicates that we accept that the acceleration will act in the same direction as the force),

$$\begin{aligned} m_{Vehicle}\, a_{Vehicle} = F &= \int_{Vol} a_{propellant}\, dm_{propellant} \\ &= \int_{Vol} \left\{ \frac{d\left(V_{pr}\right)}{dt} \right\} dm_{pr} \\ &= \int_{Vol} \frac{dV_{pr}}{dt} \left\{ \frac{dm_{pr}}{dt} dt \right\} \\ &= \dot{m}_{pr} \int_{Vol} dV_{pr} \\ m_{Vehicle} \left(\frac{dV}{dt} \right)_{Vehicle} = F &= \dot{m}_{pr} \Delta V_{pr} \end{aligned}$$

→ rephrasing to introduce the variable $\dot{m}_{pr} = \frac{dm_{pr}}{dt}$

→ if the propellant is ejected from the control volume at a constant rate then for any instant in time

→ assuming the propellant velocity is uniform along the exit area of the control volume

If we integrate the applied force over the time during which it acts we obtain the *Impulse*, I,

$$I = \int_t F dt \qquad (1.5)$$

[N s]

$$\begin{aligned} \int_t \left[m_{Vehicle} \left(\frac{dV}{dt} \right)_{Vehicle} \right] dt = \int_t F dt &= \int_t \left\{ \dot{m}_{pr} \Delta V_{pr} \right\} dt \\ &= \Delta V_{pr} \int_t \dot{m}_{pr} dt \end{aligned}$$

→ assuming the exit velocity of the propellant is constant over time

$$\overline{m}_{Vehicle}\, \Delta V_{Vehicle} = I = \Delta V_{pr} m_{pr} \qquad (1.6)$$

→ where $m_{pr} = \int_t \dot{m}_{pr} dt$ is simply the total amount of propellant mass ejected

From Eq.1.6 if the ejected propellant velocity is raised ($\Delta V_{pr} \uparrow$) then the ejected propellant mass (m_{pr}) can be reduced while still maintaining the same overall impulse. If the propellant mass is reduced, this in turn results in a lower overall vehicle mass (or average vehicle mass over the entire launch profile, $\overline{m}_{Vehicle}$) which for a given impulse will raise the obtainable vehicle velocity,

$$\Delta V_{Vehicle} = \frac{I}{\overline{m}_{Vehicle}} \tag{1.7}$$

From Eq.1.7 it becomes clearer why electrical propulsion technologies, be it electrostatic (ion and Hall thrusters) or electrothermal (resistojets and arcjets) are so attractive for in space applications. Since the ΔV of the vehicle is inversely proportional to the average vehicle mass being accelerated (which obviously includes the mass of any propellant to be ejected) then by decreasing the overall propellant mass it becomes possible to obtain large values of ΔV for a given impulse. Using the definition of the impulse (Eq.1.5) and replacing the force in terms of the specific impulse (Eq.1.3) yields,

$$I = \int_t F dt = \int_t \left\{ I_{sp} \dot{m}_{pr} a_g \right\} dt$$

$$I = \dot{m}_{pr} a_g \int_t I_{sp} dt \tag{1.8}$$

Since the specific impulse is inversely related to the mass of the propellant being accelerated, propulsion technologies which accelerate ions or electrons can result in specific impulses orders of magnitude larger than rockets (typical rockets have $I_{sp} \approx 300$ s, arcjets $I_{sp} \approx 700$ s, Hall thrusters $I_{sp} \approx 2000$ s, Ion engines $I_{sp} \approx 4000$ s!). It is also possible to theoretically accelerate ions to much higher velocities than by expanding gases through nozzles.

However, there are subtleties here as well. Since each particle being accelerated in an electric propulsion device is so light, the resulting impulse on the vehicle is correspondingly very small (Eq.1.8). Thus a very large number of particles and a lot of patience (i.e., time) is required to allow each individual 'push' to accumulate into a meaningful change in the overall vehicle velocity (the typical forces generated by these technologies is on the order of milli-Newtons). Throughout this book we will explore all these subtleties and more as we endeavour to derive, explain, and understand all things related to transatmospheric and space propulsion.

Textbook Philosophy

Before getting into the nitty-gritty, a small discussion of the layout and philosophy behind the writing of this book is offered here.

This book is organized so that it can be read like a novel in that it bounces linearly from one subject to another, where fundamentals are introduced as they are required. This is a departure from typical textbooks where the beginning chapters are devoted to fundamentals and the reader is forced to take for granted that this level of detail will be required later on but without the context as to why.

Instead, this book is structured in a such a way as to imagine someone who is interested in learning more about the idea of propulsion for getting to, and manoeuvring in, space. Therefore, it starts by looking into the requirements for getting to orbit and goes from there. For example, once the requirements are established for getting to orbit, how is this achieved? Well, that leads to an exploration of rockets, which in turn leads to a discussion of nozzle flows and combustion.

Once we understand the fundamental principles involved in making rockets work, we move on to how they are used which leads to subjects like multistaging, atmospheric properties, conical shocks. With all of that under our belt, we might feel like we're ready to consider more exotic concepts and so we move on to airbreathing engines like rocket based combined cycle engines, scramjets, and detonation wave engines. Having considered all of these things it would seem like we've exhausted getting to space and so we move on to consider space based engines, where we start with those that rely on many of the concepts we developed earlier like resistojets and arcjets.

In considering these electrothermal engines we must introduce the use of electromagnetic forces and thus this leads naturally to the consideration of engines based on the use of these physical principles such as ion and Hall thrusters. This leads us to realize the that relative magnitude of the forces these engines produce is quite a bit smaller than what we've been calculating for our previous propulsion concepts and so leads to a discussion of how these engines can be used (thus leading to an examination of orbital perturbations).

However, throughout this book the words "it can be shown...." do not appear anywhere. In other words, as each topic is introduced we consider the properties we are interested in finding values for, and develop the required equations for this purpose from the ground up.

Therefore, this book can also serve as a reference for students in more basic classes of fluid dynamics, orbital dynamics, thermodynamics, etc. As the fundamentals of these subjects are covered, they are done so after first introducing why we are spending our time examining these topics. Furthermore, as these fundamental subjects are explored, we keep a continuous eye on the end goal and in each case we circle back to the equations directly applicable to the engine concept of interest.

Therefore, as a reference text the index can be used to quickly find the part of the book relevant to something like Crocco's theorem, Chapman-Jouguet points, or the Child- Langmuir equation (sticking to the section on the letter 'C' for example), or the reader can simply bump into these equations as they are required for a given propulsion concept.

The typical reader of this book is likely a senior undergraduate or graduate student with a keen interest in propulsion and a solid foundation in first or second year engineering. Motivated students in the physical sciences may also enjoy the topics discussed as they will have the required background to follow the mathematics and might enjoy seeing how this math gets used on propulsion devices of practical interest.

Having said all of this, it's time to get down to business, so please continue reading and enjoy!

2

The Two Body Problem

Gravity can be our friend or our foe depending on what one is trying to do. If we want to get to space it is our enemy in that we must expend a great deal of energy trying to overcome it. However, as we sit in the observation room of our space cottage munching on some freeze dried takoyaki it is our friend in that it keeps us from flying off into deep space. But, for the purposes of space applications it is always a factor that must be considered and so this would seem to be a good place to start. Thanks to Sir Issac Newton we can start with the *Universal Law of Gravitation*,

$$\underline{\underline{\vec{F} = \frac{-Gm_1m_2}{r^2}\vec{u}_r}} \tag{2.1}$$

→ where the force field generated *by* m_1 *on* m_2 is the convention used here hence the "-" indicating the force on m_2 acts opposite the unit vector $\vec{u}_r$ (i.e., which points from m_1 to m_2, see Fig.2.1)

which is a convenient way to calculate the force of attraction felt between two masses (this force is always attractive, always acts in a line directly connecting the two masses, and has never been observed not to exist).

Having been reminded of Sir Isaac, we may also recall that he has three laws of motion the second of which we encountered in Eq.1.2 (or in the context of constant mass Eq.1.4) which applies to any force, including the one defined in Eq.2.1,

$$m_2\vec{a} = \vec{F} = \frac{-Gm_1m_2}{r^2}\vec{u}_r$$

→ where G is the Universal Gravitational constant

$$G = 6.674x10^{-11}\left[\frac{\mathrm{m}^3}{\mathrm{kg\ s}^2}\right]$$

To apply this relation one must choose our reference point. If we choose to consider ourselves as being located on m_2 then from our viewpoint it will appear as though the mass m_1 will be accelerated towards us by the force of gravity. Under these circumstances the vector $\vec{r}_1$ will change as the location of m_1 travels towards m_2 such that the resulting acceleration can be expressed as,

$$\vec{a}_1 = \frac{d\vec{V}_1}{dt} = \frac{d}{dt}\left\{\frac{d\vec{r}_1}{dt}\right\} = \frac{d^2\vec{r}_1}{dt^2} = \ddot{\vec{r}}_1$$

This can be used to replace the acceleration term in the universal law of gravitation to obtain,

$$\cancel{m_1}\{\ddot{\vec{r}}_1\} = \frac{+G\cancel{m_1}m_2}{r^2}\vec{u}_r$$

$$\ddot{\vec{r}}_1 = \frac{Gm_2}{r^2}\vec{u}_r \tag{2.2}$$

→ note that since we are now located on m_2 the gravitational force we observe is in the same direction as $\vec{u}_r$ and thus positive

Alternatively, if we imagine ourselves as being located on m_1 the same analysis will yield an expression for the acceleration of the vector $\vec{r}_2$,

$$\ddot{\vec{r}}_2 = \frac{-Gm_1}{r^2}\vec{u}_r \tag{2.3}$$

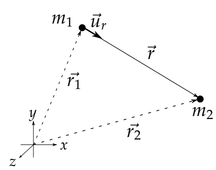

Figure 2.1: Line of action for the gravitational attraction between two masses (with positive defined as the direction from m_1 to m_2

→ $r = |\vec{r}|$ = distance between m_1 and m_2 [m]

→ $\vec{u}_r = \frac{\vec{r}}{r} = \frac{\vec{r}}{|\vec{r}|}$ = unit vector in the direction of $\vec{r}$

In Fig.2.1 both $\vec{r}_1$ and $\vec{r}_2$ are defined with respect to an arbitrary origin point. However, in most cases involving two masses we are more interested in the vector between these two masses ($\vec{r}$, which might for example be the distance between the Starship Enterprise and the planet Kronos) in which case we can note that,

$$\begin{aligned} \frac{d^2}{dt^2}(\vec{r}) &= \frac{d^2}{dt^2}(\vec{r}_2 - \vec{r}_1) \\ \ddot{\vec{r}} &= \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 \end{aligned}$$

Using Eqs.2.2 and 2.3 in the above expression while also recalling the definition of the unit vector $\vec{u}_r$ allows the acceleration along $\vec{r}$ to be written,

$$\begin{aligned} \ddot{\vec{r}} &= \left\{\frac{-Gm_1}{r^2}\vec{u}_r\right\} - \left\{\frac{Gm_2}{r^2}\vec{u}_r\right\} \\ &= \frac{-Gm_1}{r^2}\left\{\frac{\vec{r}}{r}\right\} - \frac{Gm_2}{r^2}\left\{\frac{\vec{r}}{r}\right\} \end{aligned}$$

$$\ddot{\vec{r}} = \frac{-G(m_1+m_2)}{r^3}\vec{r} \tag{2.4}$$

From this expression if one defines the *gravitational parameter*, μ,

$$\mu = G(m_1 + m_2) \tag{2.5}$$

$\left[\frac{m^3}{\cancel{kg}s^2}\cancel{kg} = \frac{m^3}{s^2}\right]$

then Eq.2.4 can be expressed as,

$$\underline{\underline{\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0}} \tag{2.6}$$

This is an interesting result. Equation 2.6 is an ordinary differential equation (hence solvable in many situations which is nice) that describes the relative position (and hence motion) of two bodies under the action of their mutual gravitational attraction. If there are no other forces acting on these bodies other than gravity (which is often a reasonable approximation in space) then this equation can be expected to yield reasonable results. Additionally, if one of the bodies is much more massive than the other (i.e. $m_1 >> m_2$) then the approximation that the gravitational parameter can be expressed as,

$$\mu \approx Gm_1 \approx Gm$$

$\rightarrow$ where m is simply the mass of the larger of the two bodies

means that Eq.2.6 allows one to determine the relative position of the two masses *independent* of the size of the smaller mass! Pausing here for a moment, it is interesting to note that Eqs.2.2 and 2.3 express the acceleration one observes due to the effects of gravity, while for our entire terrestrial careers we have continually made very good use of the expression,

$$F = ma_g$$

$\rightarrow$ $a_g = 9.81$ [m/s^2] where the direction of this force and acceleration is always assumed to act towards the centre of the Earth

This means that if m_1 is the Earth and m_2 represents us then we can write that the acceleration we undergo due to the Earth's mass is,

$$m_2 a_g = F = m_2|\ddot{\vec{r}}_2| = m_2\left\{\frac{-Gm_1}{r^2}\right\}$$

If we assume that the Earth is much more massive than ourselves then $Gm_1 = \mu_{Earth} = Gm_{Earth}$. Noting that the negative sign above simply indicates that the acceleration is towards m_1 which we have made the Earth, making the positive direction towards the Earth we can write,

$m_{Earth} = 5.972x10^{24}$ kg
$m_{me} = 73$ kg (160 lbs)
$\rightarrow \therefore \frac{m_{me}}{m_{Earth}} <<< 1$

$$
\begin{aligned}
m_2 a_g &= m_2 \tfrac{\mu_{Earth}}{r^2} \\
\therefore a_g = \frac{\mu_{Earth}}{r^2} & \qquad (2.7)
\end{aligned}
$$

Hmmm....this would seem to present a problem with treating the acceleration of gravity as a constant as this expression clearly shows that it depends on the distance between ourselves and the centre of the Earth! However, for most terrestrial applications (including even transport and spy aircraft) this value is for all intents and purposes constant. In fact, if we use a value of 9.81 m/s^2 and knowing the values for both G and m_{Earth} allows us to calculate the radius of the Earth as,

$$
r^2 = \frac{\mu_{Earth}}{a_g} = \frac{Gm_{Earth}}{a_g} = \frac{(6.674x10^{-11})(5.972x10^{24})}{9.81}
$$

$\rightarrow \mu_{Earth} = 3.986x10^{14}$ $[m^3/s^2]$

$$
\therefore R_{Earth} = 6374.09x10^3\text{m} = 6374\text{km}
$$

Clearly then, even when flying at an altitude of 30,000 ft ($\sim$ 9 km) this represents only a 0.14% change to the value of r and thus has no real effect on the resulting value of a_g. However, we've gotten distracted a bit and so let us get back to the two body problem.

To this stage we have considered two point masses separated by some distance represented by the vector $\vec{r}$. We have noted that in cases where one mass is significantly larger than the other (such as the case between planets and spacecraft) only the mass of the larger body is required to determine the value of the gravitational parameter μ (Eq.2.5). However, this assumption about the relative size of the masses yields another interesting consequence.

Consider every known object in the universe. As far as we know they all rotate and follow some sort of path (i.e., we have yet to encounter an object that is perfectly stationary). This is because, even for the simple case of only two objects existing in the universe, there is but a single initial condition that would result in both objects colliding under the influence of gravity and result in absolutely no motion (i.e., each body must have an initial velocity aligned with the vector $\vec{r}$ and the magnitude of the momentum must also be equal for the two bodies). In most other circumstances the two objects would travel towards each other under the influence of their mutual gravitational attraction but likely miss due to their differing initial momentum. As they pass each other the gravitational attraction would begin to pull the objects back together again allowing the process to repeat.

Furthermore, even if these two lone objects collide and stick together, similar to the linear momentum P in Eq.1.1 there is a quantity called angular momentum (which we spend a great deal of time thinking about in just a bit) which is also conserved unless acted upon. In broad strokes the angular momentum represents the spinning motion of a system and thus unless the collision is uniquely aligned as discussed above, the combined object created by the collision will contain the same angular momentum and thus spin.

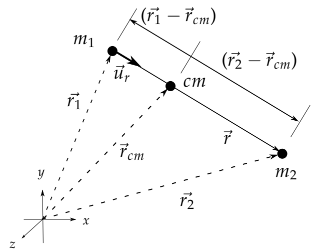

Figure 2.2: Centre of mass between two point masses

Having introduced this idea of rotational motion let's consider this a bit more. If we imagine two relatively equal masses attracting each other by gravity but avoiding a collision by virtue of their initial momentum, we likely see the resulting motion as the two masses rotating about some point between them. This centre of rotation is located at the centre of mass of the combined bodies where by definition this is the location about which mass is distributed everywhere equally,

$$\int \vec{r} dm = 0 \tag{2.8}$$

→ there is no uneven moment arm of mass if the origin for $\vec{r}$ is located at the centre of mass as in Fig.2.2

For our simple two point masses each dm is simply m_1 or m_2 while since they are point masses they are located at the unique points $\vec{r_1}$ and $\vec{r_2}$. Therefore, about the centre of mass we can write,

→ it does not matter which of the two masses is taken as the planet or the spacecraft. To illustrate this point we will assume that m_2 is the larger mass from this point on.

$$\begin{aligned}
\int \vec{r} dm = (\vec{r_1} - \vec{r}_{cm})\, m_1 + (\vec{r_2} - \vec{r}_{cm})\, m_2 &= 0 \\
-m_1\vec{r}_{cm} + m_1\vec{r_1} + m_2\vec{r_2} - m_2\vec{r}_{cm} &= 0 \\
-\vec{r}_{cm}\,(m_1 + m_2) + m_2\vec{r_2} + m_1\,\{\vec{r_2} - \vec{r}\} &= 0 \\
-\vec{r}_{cm}\,(m_1 + m_2) + \vec{r_2}\,(m_1 + m_2) - m_1\vec{r} &= 0
\end{aligned}$$

→ but $\vec{r} = \vec{r_2} - \vec{r_1}$ therefore $\vec{r_1} = \vec{r_2} - \vec{r}$

$$\therefore \vec{r_2} = \vec{r}_{cm} + \frac{m_1}{(m_1+m_2)}\vec{r}$$

This expresses the position of m_2 as a function of the location of the centre of mass for the two body system ($\vec{r}_{cm}$) and a fractional distance along the vector connecting the two masses.

In this form we can now begin to see the additional interesting consequence alluded to earlier. If $m_1 << m_2$ not only does that simplify the gravitational parameter μ, but from the above expression we can conclude that $\vec{r}_{cm} \approx \vec{r_2}$. In other words, the centre of mass (which is also the centre of rotation) is located at the *same* position as the larger of the two masses! In hindsight this might be a bit obvious, but it is nice to know the math backs us up!

At this point we have now established two of the main advantages allowed by considering a two body problem. In the absence of forces other than gravity (and more specifically, only the gravity between the two objects under consideration) when the mass of one body is much greater than the other the smaller mass can be neglected in the gravitational parameter,

$$\mu = G\left(m_1 + m_2\right) = Gm \tag{2.9}$$

while the centre of mass can be considered co-incident with that of the larger mass effectively making the larger mass *stationary*.

Therefore, we have now established that the expression governing the relative distance between our two masses (Eq.2.6) can be used to describe the motion of a moving smaller mass about a much larger stationary one (which would appear very useful for spacecraft travelling around planets). For example, since $m_{Earth} = 5.972x10^{24}$ kg while $m_{Sun} = 1.989x10^{30}$ kg (see Table 2.1),

→ In general, Eq.2.6 also applies to the motion of objects of relatively equal mass, but our focus here is on spacecraft about planets

$$\vec{r_2} = \vec{r}_{cm} + \frac{(5.972x10^{24})}{(5.972x10^{24}) + (1.989x10^{30})}\vec{r} = \vec{r}_{cm} + 0.000003\vec{r}$$

Given that the relative distance between the Sun and the Earth is $|\vec{r}| = 149.5x10^6$ km then the difference between the centre of the Sun and the centre of mass of the Earth-Sun system $(\vec{r_2} - \vec{r}_{cm})$ is $(3x10^{-6})(149.5x10^6) = 449$ km. Comparing this to the radius of the Sun which is $\sim$ 695,000 km we can see the validity of these approximations (indeed, the centre of mass of the two body system is well within the interior of the Sun itself).

Orbital Dynamics

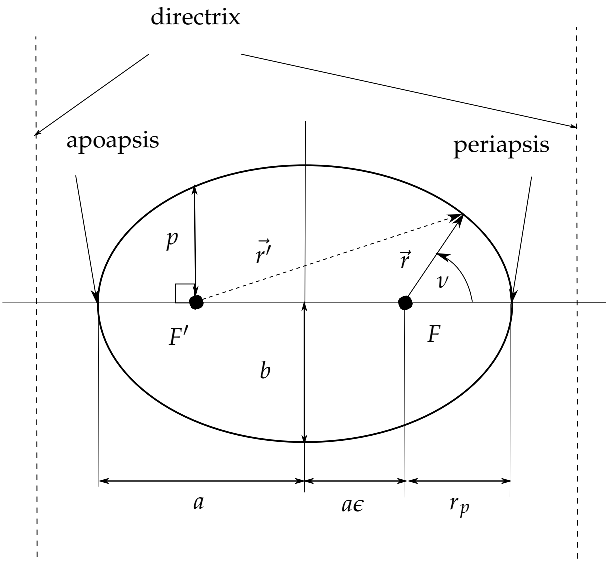

Figure 2.3: Geometry of an ellipse

F = primary focus (F' = secondary focus)

a = semi-major axis

b = semi-minor axis

ϵ = eccentricity

p = semilatus rectum

ν = angle as measured counterclockwise from periapsis

$\rightarrow$ the periapsis is the point closest to the primary focus (if Earth is located at F this is the *perigee* while if the Sun is located at F this is the *perihelion*)

$\rightarrow$ the apoapsis is the point farthest from the primary focus (if Earth is located at F this is the *apogee* while if the Sun is located at F this is the *aphelion*)

Having established the relevance of the two body problem with respect to small, light spacecraft travelling in the vicinity of much more massive planets, we now have a convenient framework for working out the motions of our spacecraft. If we can analyze the motion we can also calculate the speed and hence size our propulsion system accordingly.

However, before getting into those details let us more clearly establish what it is we know exactly. From our extensive knowledge of all things Star Wars and Star Trek we know that all objects in closed orbits travel paths described by circles, or more generally, ellipses. Lucky for us, we have learned quite a bit about these shapes in high school and may recall that for an ellipse the length (or distance) $|\vec{r} + \vec{r}'| = 2a$ is a constant (indeed, one can draw an ellipse using a string of length $2a$).

Given our love of formulas we may recall that for an ellipse any point can be located from the primary focus (or simply focus from now on) using,

$$r = \frac{p}{1 + \epsilon \cos \nu} \tag{2.10}$$

To get the minimum distance from the focus we need simply set $\nu = 0$ in Eq.2.10 to obtain,

$$\begin{aligned} r_p &= \frac{p}{1+\epsilon\cancel{\cos(0)}^{\,1}} \\ p &= r_p\,(1+\epsilon) \\ p &= \{a(1-\epsilon)\}\,(1+\epsilon) \end{aligned}$$

$\rightarrow$ but from Fig.2.3

$$\begin{aligned} r_p + a\epsilon &= a \\ r_p &= a(1-\epsilon) \end{aligned}$$

$$\therefore p = a\left(1-\epsilon^2\right) \tag{2.11}$$

Using Eq.2.11 to re-write Eq.2.10 yields an expression for the distance from the focus as a function of the semi-major axis length of the ellipse, its eccentricity, and the angle as measured from the periapsis,

$$\underline{\underline{|\vec{r}| = r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\nu}}} \tag{2.12}$$

Okay, so presumably if Star Wars/Trek can teach us anything then it would seem that Eq.2.12 could be used to determine the distance of our spacecraft from a given planet located at the focus provided the spacecraft is travelling in an elliptical path (or a circular path in which case $\epsilon = 0$ and thus $r = a = p =$ constant). Let's see if we are correct....

Angular Momentum

Recalling our two body analysis from earlier we had arrived at Eq.2.6 to describe the relative distance between two objects, where if one object is much more massive than the other, also described the motion of the smaller mass about the larger one,

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

In order to help solve this equation it is helpful to establish what quantities, if any, are constant. For our two body system one of these quantities is the *specific angular momentum*,

$$\underline{\underline{\vec{h} = \vec{r} \times \vec{V}}} \tag{2.13}$$

$\left[\frac{\mathrm{m}}{1}\frac{\mathrm{m}}{\mathrm{s}} = \frac{\mathrm{m}^2}{\mathrm{s}}\right]$

But we shouldn't just have to take this on faith....perhaps we can prove this to ourselves. If we take the moment of Eq.2.6 as written above then we can write,

$$\begin{aligned} \vec{r} \times \ddot{\vec{r}} &= \vec{r} \times \left\{ -\frac{\mu}{r^3} \vec{r} \right\} = \left(-\frac{\mu}{r^3} \right) (\vec{r} \times \vec{r}) \\ \vec{r} \times \ddot{\vec{r}} &= 0 \end{aligned}$$

→ for two general vectors $\vec{a}$ and $\vec{b}$ separated by an angle θ between them, by definition the magnitude of the cross product is

$$\vec{a} \times \vec{b} = |a||b| \sin\theta \tag{2.14}$$

and so for parallel vectors $\theta = 0$
$\therefore \vec{r} \times \vec{r} = |\vec{r}|^2 \cancelto{0}{\sin(\theta)} = 0$

To get the specific angular momentum into this expression we need to relate $\vec{r} \times \vec{V}$ to $\vec{r} \times \ddot{\vec{r}}$. To do this we note that $\vec{V} = \dot{\vec{r}}$ and so one can write,

→ for completeness (and use later on) the dot product is by definition

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta \tag{2.15}$$

$$\begin{aligned} \frac{d}{dt} \left(\vec{h} \right) &= \frac{d}{dt} \left(\vec{r} \times \dot{\vec{r}} \right) \\ &= \cancelto{0}{\dot{\vec{r}} \times \dot{\vec{r}}} + \vec{r} \times \ddot{\vec{r}} \end{aligned}$$

→ parallel vectors

$$\frac{d}{dt} \left(\vec{r} \times \dot{\vec{r}} \right) = \vec{r} \times \ddot{\vec{r}} = \dot{\vec{h}} \tag{2.16}$$

However, from the moment of the two body equation taken above, we note that $\vec{r} \times \ddot{\vec{r}} = 0$ and so this reduces to the result,

$$\frac{d}{dt} \left(\vec{h} \right) = 0 \tag{2.17}$$

which states that the time rate of change of the specific angular momentum is zero (or in other words, the specific angular momentum of our two body system is indeed a conserved quantity!). This is convenient in that if we can calculate this value at even just one point in an orbit, we will be able to use this value at all other points in the orbit as well.

Going back to Eq.2.6, multiplying through by the specific angular momentum (being careful to use the cross product as both the position and the angular momentum are *vectors*) one obtains,

$$\vec{h} \times \ddot{\vec{r}} + \frac{\mu}{r^3} \left(\vec{h} \times \vec{r} \right) = 0 \tag{2.18}$$

where solving Eq.2.18 or Eq.2.6 for the position $\vec{r}$ will yield the same result since $\vec{h}$ is constant. The first term can be re-expressed by noting that,

$$\begin{aligned} \tfrac{d}{dt}\left(\vec{h}\times\dot{\vec{r}}\right) &= \cancelto{0}{\tfrac{d}{dt}\left(\vec{h}\right)}\times\dot{\vec{r}}+\vec{h}\times\ddot{\vec{r}} \\ \vec{h}\times\ddot{\vec{r}} &= \tfrac{d}{dt}\left(\vec{h}\times\vec{V}\right) \end{aligned}$$

→ see Eq.2.17

and so we have,

$$\frac{d}{dt}\left(\vec{h}\times\vec{V}\right)+\frac{\mu}{r^3}\left(\vec{h}\times\vec{r}\right)=0 \tag{2.19}$$

Focusing on the second term we note that by definition $\vec{h}=\vec{r}\times\vec{V}=\vec{r}\times\dot{\vec{r}}$ (Eq.2.13) and so,

$$\vec{h}\times\vec{r}=\left(\vec{r}\times\dot{\vec{r}}\right)\times\vec{r}$$

Making use of some of our knowledge concerning cross products of vectors we may recall that for three general vectors $\vec{a},\vec{b},\vec{c}$ the following identity holds true,

$$\begin{aligned} \vec{a}\times\left(\vec{b}\times\vec{c}\right) &= \vec{b}\left(\vec{a}\cdot\vec{c}\right)-\vec{c}\left(\vec{a}\cdot\vec{b}\right) \\ \left\{-\left(\vec{b}\times\vec{c}\right)\right\}\times\vec{a} &= \vec{b}\left(\vec{a}\cdot\vec{c}\right)-\vec{c}\left(\vec{a}\cdot\vec{b}\right) \end{aligned}$$

$$\left(\vec{b}\times\vec{c}\right)\times\vec{a}=-\vec{b}\left(\vec{a}\cdot\vec{c}\right)+\vec{c}\left(\vec{a}\cdot\vec{b}\right) \tag{2.20}$$

→ vector triple product

→ recalling that $\vec{a}\times\vec{d}=-\vec{d}\times\vec{a}$ (since $\theta=180^o$ when one of the vectors is reversed, see Eq.2.14)

thus

$$\left(\vec{r}\times\dot{\vec{r}}\right)\times\vec{r} = -\vec{r}\left(\vec{r}\cdot\dot{\vec{r}}\right)+\dot{\vec{r}}\left(\vec{r}\cdot\vec{r}\right)$$

and so,

$$\vec{h}\times\vec{r}=\left(\vec{r}\times\dot{\vec{r}}\right)\times\vec{r}=-\vec{r}\left(\vec{r}\cdot\dot{\vec{r}}\right)+\dot{\vec{r}}\left(\vec{r}\cdot\vec{r}\right)$$

Before using this result in Eq.2.19 it is convenient to consider the elliptical path as traced out by the vector $\vec{r}$ from the focus expressed in polar coordinates (i.e., $\vec{r}=f(r,\theta)$) as shown in Fig.2.4. For a mass located at $\vec{r}$ with a velocity $\vec{V}$ we can rewrite the cross product of the specific angular momentum and the position obtained above as,

$$\vec{h}\times\vec{r}=-\vec{r}\left\{r\dot{r}\right\}+\dot{\vec{r}}\left\{|\vec{r}||\vec{r}|\cancelto{1}{\cos(\theta)}\right\}$$

→ using Eq.2.15 and Eq.2.24 from Fig.2.4

resulting in the relation,

$$\vec{h}\times\vec{r}=-r\dot{r}\vec{r}+r^2\dot{\vec{r}} \tag{2.21}$$

which allows Eq.2.19 to be re-expressed as,

$$\begin{aligned}
\tfrac{d}{dt}\left(\vec{h}\times\vec{V}\right)+\tfrac{\mu}{r^3}\left\{-r\dot{r}\vec{r}+r^2\dot{\vec{r}}\right\} &= 0\\
\tfrac{d}{dt}\left(\vec{h}\times\vec{V}\right)+\tfrac{\mu}{r}\dot{\vec{r}}-\tfrac{\mu}{r^2}\dot{r}\vec{r} &= 0\\
\tfrac{d}{dt}\left(\vec{h}\times\vec{V}\right)+\tfrac{d}{dt}\left(\tfrac{\mu}{r}\vec{r}\right) &= 0
\end{aligned}$$

$$\frac{d}{dt}\left[\left(\vec{h}\times\vec{V}\right)+\left(\frac{\mu}{r}\right)\vec{r}\right]=0 \tag{2.22}$$

→ from the Calculus one can write

$$\frac{d}{dt}\left(\frac{\mu}{r}\vec{r}\right)=$$

$$=\frac{\mu}{r}\overset{\dot{\vec{r}}}{\cancel{\frac{d}{dt}(\vec{r})}}+\vec{r}\left[-\frac{\mu}{r^2}\overset{\dot{r}}{\cancel{\frac{d}{dt}(r)}}\right]$$

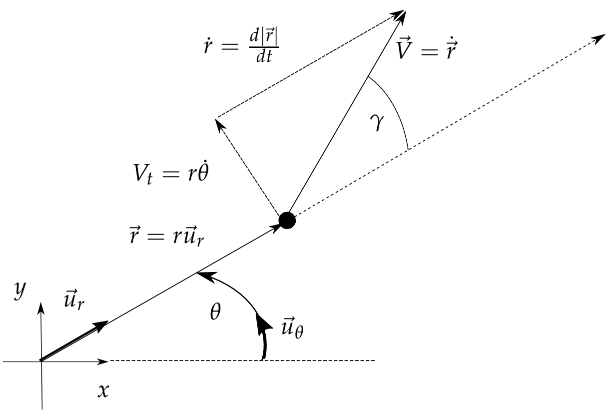

Figure 2.4: Point mass in polar coordinates with a velocity $\vec{V}$

→ recalling Eq.2.14

$$|\vec{r}\times\dot{\vec{r}}|=|\vec{r}\times\vec{V}|=|\vec{r}||\vec{V}|\sin\gamma=rV_t$$

$$\therefore |\vec{r}\times\vec{V}|=r^2\dot{\theta} \tag{2.23}$$

→ and similarly using the definition of the dot product (Eq.2.15)

$$\vec{r}\cdot\vec{V}=|\vec{r}||\vec{V}|\cos\gamma=r\frac{d|\vec{r}|}{dt}$$

$$\therefore \vec{r}\cdot\vec{V}=\vec{r}\cdot\dot{\vec{r}}=r\dot{r} \tag{2.24}$$

$$\rightarrow V=|\vec{V}|=\sqrt{\dot{r}^2+(r\dot{\theta})^2} \tag{2.25}$$

Okay, this is a lot of math so far so let us review what we have accomplished. Looking closely at Eq.2.22 we can see that the quantity in the square brackets does not change over time. Therefore, it can be replaced with a constant when integrated suggesting we have found yet another quantity that is constant over our orbit,

$$\underline{\underline{\vec{h}\times\vec{V}+\left(\frac{\mu}{r}\right)\vec{r}=-\vec{C}}} \tag{2.26}$$

where $\vec{C}$ is the vector sum of the two terms shown (the negative sign is not strictly necessary as the derivative of a positive or negative constant $\vec{C}$ is zero in both cases, but this sign will help us later on when we encounter the *eccentricity vector*).

If we take the dot product of the position of our spacecraft and this new constant vector $-\vec{C}$,

$$\begin{aligned}
\vec{r}\cdot\left(\vec{h}\times\vec{V}\right)+\left(\tfrac{\mu}{r}\right)\vec{r}\cdot\vec{r} &= -\vec{r}\cdot\vec{C} \\
\vec{r}\cdot\left(\vec{h}\times\vec{V}\right)+\left(\tfrac{\mu}{r}\right)\left\{r^2\right\} &= -rC\cos\beta \\
\left\{\vec{h}\cdot\left(\vec{V}\times\vec{r}\right)\right\}+\mu r &= -rC\cos\beta \\
\vec{h}\cdot\left\{-\left(\vec{r}\times\vec{V}\right)\right\}+\mu r &= -rC\cos\beta \\
-h^2+\mu r &= -rC\cos\beta
\end{aligned}$$

→ but $\vec{r}\cdot\vec{r} = |\vec{r}||\vec{r}|\cos(0) = r^2$

→ where β is the angle between the position vector r and the constant vector $\vec{C}$

→ recalling another of our vector identities (scalar triple product) which states $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{b}\cdot(\vec{c}\times\vec{a})$

→ as before we noted $\vec{a}\times\vec{d}=-\vec{d}\times\vec{a}$

→ using the definition of the specific angular momentum (Eq.2.13) and noting that $-\vec{h}\cdot\vec{h}=-h^2$

After all of this where have we ended up? Well, if we recall where we started (Eq.2.18) we had a second order differential equation for the position of our spacecraft ($\ddot{\vec{r}}$) obtained by considering it the smaller of two masses governed by the two body problem. Looking at the result we just obtained, we might be puzzled to not see any derivative terms......indeed, as we ponder this expression even further we might realize that for a given orbit many of these terms are constant.

We have already established that the specific angular momentum vector, $\vec{h}$, is conserved then so must be its magnitude h. Furthermore, we explicitly stated that the term $-\vec{C}$ is a constant since its derivative with respect to time is zero, and for a given planet the gravitational parameter, μ, is also fixed. Thus the only two terms that vary in our result are the distance of our spacecraft from the focus of the orbit, r, (which is co-incident with the centre of the larger mass or planet as we proved earlier) and the angle β. Therefore, it would seem prudent to consider this angle a little more carefully.

If we perform a little algebra we can re-arrange the above expression as,

$$\begin{aligned}
-h^2 &= -\left(\mu r + rC\cos\beta\right) \\
h^2 &= r\left(\mu + C\cos\beta\right)
\end{aligned}$$

which allows us to isolate for the distance r as,

$$\underline{\underline{r=\frac{h^2}{\mu+C\cos\beta}=\frac{\frac{h^2}{\mu}}{1+\left(\frac{C}{\mu}\cos\beta\right)}}} \tag{2.27}$$

Wait a minute....this looks a bit familiar.... Comparing this to Eq.2.12 we might notice that this expression is very similar in form to that for the distance of a point on an ellipse. Furthermore, we can see that when $\beta = 0$ the expression in Eq.2.27 is its smallest which means $r = r_p$ and hence $\nu = 0$. Therefore, the angles ν and β must be the same which means that our arbitrary constant of integration vector $-\vec{C}$ must point towards the periapsis of the elliptical orbit.

The angular position of our spacecraft as measured from this vector (or the position of the periapsis) is known as the *true anomaly*, ν. Furthermore, comparing the numerators between Eqs.2.12 and 2.27 we obtain the relation,

$$\frac{h^2}{\mu} = a(1 - \epsilon^2) \tag{2.28}$$

which allows us to relate the angular momentum of the orbit to the size and eccentricity of the orbit.

A final comparison also allows us to draw a relation between the magnitude of the arbitrary constant C and the eccentricity of the orbit as $C = \mu\epsilon$.

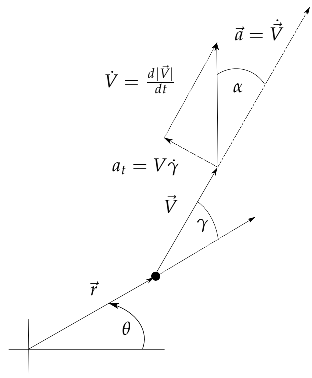

Figure 2.5: Acceleration in polar coordinates

$$\vec{V} \cdot \vec{a} = |\vec{V}| \cdot |\dot{\vec{V}}| \cos\alpha = V\dot{V}$$

$$\therefore \vec{V} \cdot \dot{\vec{V}} = V\dot{V} \tag{2.29}$$

At this point it might seem like a bit too much of a coincidence that there is some arbitrary value that remains constant for a given orbit and which is so useful. This constant of integration is actually a reflection of the fact that in addition to the specific angular momentum being a conserved quantity, so is the specific energy for a given orbit (which we shall demonstrate now).

Recalling that we proved $\vec{h}$ is constant by taking $\vec{r}\times$ of the governing two body equation (Eq.2.6), if we try taking $\vec{V}\cdot$ of the same governing equation perhaps we can obtain some more interesting results,

$$\begin{aligned}
\vec{V} \cdot \ddot{\vec{r}} + \tfrac{\mu}{r^3}\vec{V} \cdot \vec{r} &= 0 \\
\vec{V} \cdot \dot{\vec{V}} + \tfrac{\mu}{r^3}\{r\dot{r}\} &= 0 && \rightarrow \text{using Eq.2.24} \\
\{V\dot{V}\} + \tfrac{\mu}{r^2}\dot{r} &= 0 && \rightarrow \text{using Eq.2.29} \\
V\tfrac{dV}{dt} &= -\tfrac{\mu}{r^2}\tfrac{dr}{dt} \\
V dV &= -\mu r^{-2} dr && \rightarrow \text{for an incremental instant in time}
\end{aligned}$$

At this point some of you, I am sure, are relieved in that this is a simple enough expression that can be integrated with respect to time directly to yield,

$$\begin{aligned} \tfrac{1}{2}V^2 &= -\tfrac{\mu}{-1}r^{-1} + A \\ \tfrac{1}{2}V^2 - \tfrac{\mu}{r} &= A \end{aligned}$$

where A is a constant with respect to time. If we look at the terms on the left side of the above expression we see that the units of $(1/2)V^2$ are the same as those for specific energy (since $\mathrm{J} = \mathrm{N\ m} = \frac{\mathrm{kg\ m}}{\mathrm{s}^2}\mathrm{m} = \frac{\mathrm{kg\ m}^2}{\mathrm{s}^2}$ which when divided by kg to obtain a 'specific' value yields $\mathrm{m}^2/\mathrm{s}^2$) and thus this term is simply the *specific kinetic energy*.

Looking at the units of $-(\mu/r) = -\frac{Gm}{r}$ we again find the same units $\left(\frac{\mathrm{m}^{\cancel{3}\,2}}{\cancel{\mathrm{kg}}\ \mathrm{s}^2}\cancel{\mathrm{kg}}\frac{1}{\cancel{\mathrm{m}}} = \mathrm{m}^2/\mathrm{s}^2\right)$ and thus this too must be some sort of specific energy. When $r \to \infty$ this term goes to zero similar to the way the gravity force decreases with increasing distance between the masses. Conversely, as $r \to 0$ this value grows larger in the negative direction, again behaving in a manner similar to the magnitude of the gravitational force. Therefore, this term represents the *specific gravitational potential energy* if one assumes that we start in a negative 'hole' (i.e. $-\infty$) and go to zero as our distance from the focus of the orbit is increased. In this fashion an orbit can be thought of as being within a *gravity well*.

To summarize, the constant A is simply the sum of both the kinetic and potential energy of our spacecraft per unit mass and is thus properly referred to as the *specific mechanical energy*, e,

$$\underline{\underline{e = \frac{1}{2}V^2 - \frac{\mu}{r}}} \tag{2.30}$$

$[\mathrm{m}^2/\mathrm{s}^2]$

However, how does this help us with our constant C? Well, going back to when we first encountered the term $\vec{C}$ we had Eq.2.26. If we take the dot product of both sides of this expression we can write (this is similar to taking the square of each side of an equation, it does not alter the equality),

$$
\begin{aligned}
\left\{\vec{h}\times\vec{V}+\left(\tfrac{\mu}{r}\right)\vec{r}\right\}\cdot\left\{\vec{h}\times\vec{V}+\left(\tfrac{\mu}{r}\right)\vec{r}\right\} &= -\vec{C}\cdot-\vec{C} \\
(\vec{h}\times\vec{V})\cdot(\vec{h}\times\vec{V})+2\left(\tfrac{\mu}{r}\right)\underbrace{\left\{\vec{r}\cdot(\vec{h}\times\vec{V})\right\}}_{\text{see derivation of Eq.2.27}}+\left(\tfrac{\mu}{r}\right)^2\vec{r}\cdot\vec{r} &= |-\vec{C}||-\vec{C}|\cancelto{1}{\cos(0)} \\
\left\{|\vec{h}\times\vec{V}||\vec{h}\times\vec{V}|\right\}\cancelto{1}{\cos(0)}+2\left(\tfrac{\mu}{r}\right)\overbrace{\left\{-\vec{h}\cdot\vec{h}\right\}}+\tfrac{\mu^2}{r^2}|\vec{r}||\vec{r}|\cancelto{1}{\cos(0)} &= C^2 \\
\left\{|\vec{h}||\vec{V}|\cancelto{1}{\sin(90^o)}\right\}^2+2\left(\tfrac{\mu}{r}\right)\left\{|-\vec{h}||\vec{h}|\cancelto{-1}{\cos(180^o)}\right\}+\tfrac{\mu^2}{\cancel{r^2}}\cancel{r^2} &= C^2 \\
(hV)^2-\tfrac{2\mu h^2}{r}+\mu^2 &= C^2 \\
2h^2\cdot\underbrace{\left(\frac{1}{2}V^2-\frac{\mu}{r}\right)}_{\text{this exactly matches Eq.2.30!}}+\mu^2 &= C^2 \\
2h^2\left\{e\right\}+\mu^2 &= C^2
\end{aligned}
$$

$\rightarrow$ recalling that $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

$\rightarrow$ since by definition $\vec{h}=\vec{r}\times\vec{V}$ (Eq.2.13) then $\vec{h}$ is perpendicular to both $\vec{r}$ and $\vec{V}$

Dividing through by μ^2 allows one to obtain C/μ in Eq.2.27 and by analogy to Eq.2.12 the eccentricity (ϵ) as a sole function of values which remain constant for a given orbit,

$$\frac{C}{\mu}=\epsilon=\left(\frac{2eh^2}{\mu^2}+1\right)^{\frac{1}{2}} \tag{2.31}$$

which when substituted into Eq.2.27 yields,

$$r=\frac{\frac{h^2}{\mu}}{1+\left(\frac{2eh^2}{\mu^2}+1\right)^{\frac{1}{2}}\cos\nu} \tag{2.32}$$

where we have also explicitly inserted the true anomaly ν for the angle.

Velocity Requirements

This is a key result! Previously we managed to relate the position of our spacecraft anywhere in its orbit to properties of the orbit that are constant (Eq.2.27). In Eq.2.31 we have managed to relate the one seemingly arbitrary constant C to more physically meaningful variables like the specific mechanical energy

4
Mission Design

Now that we have a means of calculating all the required variables for determining the speed of the flow exiting from the rocket, we can return to Eq.3.8 to see what exactly our rocket can do. As most rockets are designed as non lifting bodies the main requirement for strength is to resist buckling (i.e., the major loads are aligned with the rocket axis) and thus the forces to consider are the thrust, drag, and gravity. Of these, although thrust vectoring can be used to guide a rocket, for stability purposes any non axial component of thrust generally evens itself out over a flight and thus the only force that continually acts out of line with the velocity vector is gravity. Under these circumstances recalling that θ can be used to represent the rocket axis angle with respect to vertical (i.e., the gravity vector) Eq.3.8 can be written,

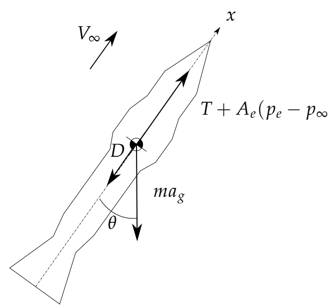

Figure 4.1: Major forces acting on a rocket during flight

$$\frac{\partial}{\partial t}(mV) = \{\dot{m}u_e\} + A_e(p_e - p_\infty) - D - ma_g \cos\theta$$

→ using Eq.3.9 to replace the thrust

$$\frac{\partial}{\partial t}(mV) = \dot{m}_{pr}u_{eq} - D - ma_g \cos\theta \tag{4.1}$$

→ recalling Eq.3.63 $\dot{m}u_{eq} = \dot{m}u_e + A_e(p_e - p_\infty)$

where $\dot{m}_{pr} = \dot{m}$ is used to specifically identify the massflow as that of the *pr*opellant leaving the vehicle.

Before getting into what can be learned from this expression it will be convenient if we establish a few terms relevant to rocket launch vehicles. As with any space access system weight is of critical importance. In particular, how much weight is actual payload as compared to how much is fuel and structure are key performance factors. Therefore, let us define several useful weight or mass ratios as follows. The first is the *mass ratio*, μ,

$$\mu = \frac{m_I}{m_F} = \frac{\text{Initial mass}}{\text{Final mass}} \tag{4.2}$$

→ a large value for μ reflects a large amount of propellant since over a given stage propellant consumption represents the only mass lost during flight

where this can be defined over an entire launch, or over a given stage of a multistage vehicle.

Another useful ratio is the *payload ratio*, λ,

$$\lambda = \frac{m_{pay}}{m - m_{pay}} = \frac{\text{payload mass}}{\text{total mass without payload}} \tag{4.3}$$

where the total mass m can be defined as,

→ m_{pr} = mass of propellant

→ m_s = mass of structure

→ m_{pay} = mass of payload

$$m = m_{pr} + m_s + m_{pay} \tag{4.4}$$

which allows the payload ratio to be written as,

$$\lambda = \frac{m_{pay}}{m_s + m_{pr}} \tag{4.5}$$

→ the payload can be the mass of whatever the customer wants to put into orbit, or the total mass of a higher stage on a lower one (i.e., the higher stage is the 'payload' of the lower stage)

The skill of the structural designer can be represented by the *structural ratio*, β, defined as,

$$\beta = \frac{m_s}{m - m_{pay}} = \frac{\text{structural mass}}{\text{total mass without payload}} \tag{4.6}$$

→ small values of β indicate that there is little mass devoted to the vehicle structure and are thus generally valued (large boosters can have ratios of $\sim$ 0.1 while this ratio generally gets larger for upper stages with typical values between 0.6 and 0.8)

where again from Eq.4.4 one can write,

$$\beta = \frac{m_s}{m_s + m_{pr}} \tag{4.7}$$

These terms can be related to the mass ratio if one assumes that the total mass is equal to the initial mass while the final mass at burnout is simply the mass of both the payload and the structure such that,

$$\begin{aligned}
\mu &= \frac{m}{m_s + m_{pay}} \\
&= \frac{\left\{m_{pr} + m_s + m_{pay}\right\}}{m_s + m_{pay}} \\
&= \frac{\cancelto{1}{\frac{m_{pr}+m_s}{m_{pr}+m_s}} + \cancelto{\lambda}{\frac{m_{pay}}{m_{pr}+m_s}}}{\cancelto{\beta}{\frac{m_s}{m_{pr}+m_s}} + \cancelto{\lambda}{\frac{m_{pay}}{m_{pr}+m_s}}}
\end{aligned}$$

→ using Eq.4.4

→ recalling Eqs.4.7 and 4.5

$$\underline{\underline{\mu = \frac{\lambda + 1}{\lambda + \beta}}} \tag{4.8}$$

With these ratios defined let us return to Eq.4.1 and let m be the instantaneous mass of the vehicle at any point within the flight. This allows it to be removed from the time derivative leaving,

$\rightarrow$ since the propellant is solely responsible for any change to the instantaneous mass (leaving for the moment dropping stages),

$$\dot{m}_{pr} = -\frac{dm}{dt} \tag{4.9}$$

$$\begin{aligned} m\tfrac{dV}{dt} &= \dot{m}_{pr} u_{eq} - D - m a_g \cos\theta \\ &= \left\{ -\tfrac{dm}{dt} \right\} u_{eq} - D - m a_g \cos\theta \end{aligned}$$

which allows one to express the instantaneous change in velocity of the vehicle as,

$$\underline{\underline{dV = -u_{eq}\frac{dm}{m} - \frac{D}{m}dt - a_g \cos\theta dt}} \tag{4.10}$$

Integrating this result between any two points in the flight yields,

$$V_2 - V_1 = -u_{eq}\left[\ln(m_2) - \ln(m_1)\right] - \int_1^2 \tfrac{D}{m} dt - \int_1^2 a_g \cos\theta dt$$

which for a single integration between liftoff and burnout becomes,

$\rightarrow$ if one assumes that initially we are at the launch pad then $V_1 = 0$ while at burnout $V_2 = V_{bo}$

$\rightarrow$ where

$\int \frac{D}{m} dt = \Delta V_{drag}$ $[\frac{\mathrm{N\,s}}{\mathrm{kg}} = \frac{\mathrm{kg\,m}}{\mathrm{s}^2}\frac{\mathrm{s}}{\mathrm{kg}} = \frac{\mathrm{m}}{\mathrm{s}}]$

$\int a_g \cos\theta dt = \Delta V_{gravity}$ $[\frac{\mathrm{m}}{\mathrm{s}^2}\frac{\mathrm{rad \cdot s}}{1} = \frac{\mathrm{m}}{\mathrm{s}}]$

$$\cancelto{V_{bo}}{V_2} - \cancelto{0}{V_1} = u_{eq}\ln\left(\frac{\cancelto{m_I}{m_1}}{\cancelto{m_F}{m_2}}\right) - \int_I^F \tfrac{D}{m} dt - \int_I^F a_g \cos\theta dt$$

$$\underline{\underline{V_{bo} = \Delta V = u_{eq}\ln\mu - \Delta V_{drag} - \Delta V_{gravity}}} \tag{4.11}$$

If one neglects the effects of drag and gravity the result becomes,

$\rightarrow$ where from Eq.3.65 $u_{eq} = I_{sp} a_g$

$\rightarrow$ using Eq.4.8 to re-express the mass ratio

$$\underline{\underline{V_{bo} = \Delta V = \ln\mu^{u_{eq}} = \ln\mu^{I_{sp}a_g} = \ln\left(\frac{\lambda+1}{\lambda+\beta}\right)^{I_{sp}a_g}}} \tag{4.12}$$

This is sometimes referred to as the Tsiolkovsky rocket equation. Although neglecting the effects of gravity is a severe restriction, Eq.4.12 provides an interesting upper limit on the achievable burnout or change in velocity that can be imparted to the vehicle. With the mass ratio expressed as a function of both the payload and structural ratios, in the limit of no payload as $\lambda \to 0$ the mass ratio is reduced to a sole function of the structural ratio. This allows the maximum change in velocity to be written as,

$$\Delta V_{max} = \ln\left(\frac{1}{\beta}\right)^{u_{eq}} \qquad \text{or} \qquad \frac{\Delta V_{max}}{u_{eq}} = \ln\left(\frac{1}{\beta}\right) \quad (4.13)$$

$\to \Delta V_{drag} = \Delta V_{gravity} = \lambda = 0$

which shows that the maximum obtainable ΔV, which if from launch is equal to the maximum burnout velocity $V_{bo,max}$, is directly related to the structural mass ratio. Alternatively, one can find the maximum allowable structural mass ratio to obtain a given burnout or change in velocity from Eq.4.13 as,

$$e^{\frac{\Delta V}{u_{eq}}} = \frac{1}{\beta_{max}}$$

$$\beta_{max} = e^{-\frac{\Delta V}{u_{eq}}} = e^{-\frac{\Delta V}{I_{sp} a_g}} \quad (4.14)$$

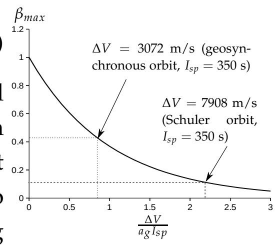

Figure 4.2: Maximum structural ratio as a function of non-dimensional orbital velocity

Eq.4.14 is plotted over over a range of non dimensional orbital velocities in Fig.4.2. As can be seen, as the required non dimensional orbital velocity increases this places a significant restriction on the maximum allowable structural mass ratio (while this result excludes the effects of both gravity and drag which act to further increase the velocity requirement).

Recalling our earlier discussions of orbital velocities we showed that to simply orbit along the surface of the Earth requires a velocity of 7908 m/s. If one assumes a rocket with a specific impulse of 350 s this yields (as shown by the dashed line in Fig.4.2),

$$\frac{\Delta V}{I_{sp} a_g} = \frac{(7908)}{(350)(9.81)} = 2.3$$

$$\beta_{max} = e^{-2.3} = 0.10$$

Thus there is a significant challenge in keeping the structural mass to a maximum of 10% of the total vehicle mass neglecting payload. For higher orbits, although the required velocity is less (as shown by the dotted line in Fig.4.2 for a circular geosynchronous orbit) which in turn requires less structural efficiency, the issue then becomes one of total overall weight. If the structure is on the order of 40% of the vehicle weight, with the required mass of fuel to reach these higher orbits (recalling that these are higher energy orbits despite the decreased velocity) the overall vehicle weight becomes an issue.

Furthermore, all the results represented by Eq.4.14 neglect the velocity required to overcome both drag and gravity during the ascent. Additionally, they assume there is no actual payload (we're just going to space for the thrill of it)! Clearly a single stage to orbit rocket is a challenging task and hence the abundance of multi-stage launch vehicles. On the other hand, this does not restrict the use of single stage rockets for uses other than orbital insertion (ballistic trajectories for example).

However, before considering either multi-stage rockets or sub-orbital trajectories, we really should consider the effects of both drag and gravity on the rocket.

Starting with drag, this force can be expressed as a function of the flight conditions as represented by the atmospheric density (ρ) and the flight speed, the size of the vehicle (using a reference area S), and the shape as represented by the *drag co-efficient* (also referred to as the *co-efficient of drag*, c_D),

$$\Delta V_{drag} = \int \frac{D}{m} dt = \int \frac{\left\{ c_D \frac{1}{2} \rho V^2 S \right\}}{m} dt$$

→ by definition

$$c_D = \frac{D}{\frac{1}{2}\rho V^2 S} \quad (4.15)$$

To perform this integration all the terms on the right must be examined for any time dependence. For rockets the reference area S is usually taken as the cross sectional area of the cylindrical body and is thus constant with time (it should also be noted that the reference area can actually be anything so long as it is defined consistent with the definition of the co-efficient of drag, i.e., when non-dimensionalizing the drag to obtain c_D, the same reference area must be used to re-dimensionalize c_D into D).

→ often the co-efficient of drag is broken down into components representing different types of drag such as skin friction, wave, interference, and base

→ typical rockets can have overall drag coefficients between 0.1 and 0.7 depending on the altitude, flight Mach number, and alignment with the flow direction

→ a bullet has a typical drag co-efficient (based on the cross sectional area) of approximately 0.3

Although the local atmospheric density is a function of time as the weather changes, even in an idealized atmosphere without changing weather one cannot consider density constant with time. This is due to the fact that the purpose of any launch vehicle is to increase altitude while the atmospheric density is a function of the weight of air above a given location and thus will continuously change with time.

Consider the infinitesimal element of atmospheric fluid shown in Fig.4.3. If the forces acting on this small element are in balance in the vertical direction then the pressure acting on the lower surface times the area dA must be equal to the pressure force acting on the upper surface *plus* the weight of the fluid contained in the column of height dh above the lower surface,

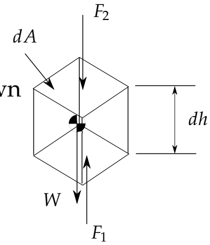

Figure 4.3: Static element of atmospheric fluid

$$\begin{aligned} F_1 &= F_2 + W \\ p_1 \cancel{dA} &= p_2 \cancel{dA} + \{\rho a_g (dh \cancel{dA})\} \\ p_1 - p_2 = -dp &= \rho a_g dh \end{aligned}$$

→ $W = (\rho a_g)$(Volume) $= \rho a_g (dhdA)$

With this force balance one arrives at the differential expression,

$$\frac{dp}{dh} = -\rho a_g = -\left\{\frac{p}{RT}\right\} a_g$$

→ recalling the perfect gas equation of state (Eq.3.26)

$$\frac{dp}{p} = -a_g \frac{dh}{RT}\left\{\frac{dT}{dT}\right\} = -\frac{a_g}{R}\left(\frac{dh}{dT}\right)\frac{dT}{T}$$

$$\frac{dp}{p} = \frac{-a_g}{R\left(\frac{dT}{dh}\right)}\frac{dT}{T} \tag{4.16}$$

→ $\left(\frac{dT}{dh}\right)$ is the lapse rate [K/m]

Integrating Eq.4.16 between two points within the atmosphere will yield an expression for the change in pressure between these two points provided one knows the manner in which temperature varies with altitude (known as the lapse rate). As it turns out, the manner in which the temperature changes with altitude has been studied extensively and this relationship can be determined for a number of different scenarios (dry/moist air, the inclusion of radiation and convection effects, an average of observed conditions....). However, in the standard atmosphere defined by the *International Civil Aviation Organization (ICAO)* which contains no moisture the lapse rates for the various sections of the atmosphere are shown in Fig.4.4.

Therefore, between any of the atmospheric layers for which the lapse rate is constant the integration becomes,

$$\int_1^2 \frac{dp}{p} = \frac{-a_g}{R\left(\frac{dT}{dh}\right)} \int_1^2 \frac{dT}{T}$$

$$\ln\left(\frac{p_2}{p_1}\right) = \frac{-a_g}{R\left(\frac{dT}{dh}\right)}\ln\left(\frac{T_2}{T_1}\right)$$

Letting point 1 be Sea Level and point 2 be any point within the troposphere this becomes,

$$\ln\left(\frac{p_{h<11\text{ km}}}{p_{S/L}}\right) = \ln\left(\frac{T_{h<11\text{ km}}}{T_{S/L}}\right)^{\frac{-a_g}{R\left(\frac{dT}{dh}\right)}}$$

$$\left(\frac{p_{h<11\text{ km}}}{p_{S/L}}\right) = \left[\frac{\left\{T_{S/L}+\left(\frac{dT}{dh}\right)(h-h_{S/L})\right\}}{T_{S/L}}\right]^{\frac{-a_g}{R\left(\frac{dT}{dh}\right)}}$$

$\rightarrow$ but $\underline{\underline{T_{h<11\text{ km}} = T_{S/L} + (dT/dh)(h-h_{S/L})}}$ (4.17)

$$\underline{\underline{p_{h<11\text{ km}} = p_{S/L}\left[1+\frac{\left(\frac{dT}{dh}\right)(h-h_{S/L})}{T_{S/L}}\right]^{\frac{-a_g}{R\left(\frac{dT}{dh}\right)}}}} \qquad (4.18)$$

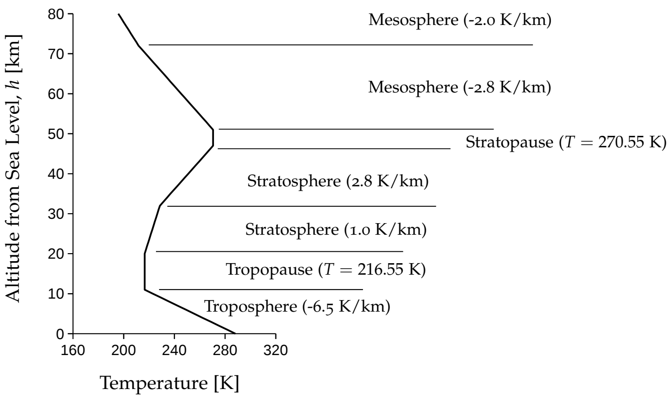

Figure 4.4: ICAO standard atmosphere (Sea Level Temperature 288.16 K)

Table 4.1: Lapse rates

Altitude [km]	dT/dh [K/km]
S/L to 11	-6.5
11 to 20	0
20 to 32	1.0
32 to 47	2.8
47 to 51	0
51 to 72	-2.8
72 to 86	-2.0

With both the pressure (Eq.4.18) and the temperature (Eq.4.17) as functions of altitude the perfect gas equation of state (Eq.3.26) can be used to obtain the density at any point within the troposphere,

$$\frac{p}{p_{S/L}} = \frac{\rho \cancel{R} T}{\rho_{S/L}\cancel{R} T_{S/L}}$$

$$\left\{\left[1+\frac{\left(\frac{dT}{dh}\right)(h-h_{S/L})}{T_{S/L}}\right]^{\frac{-a_g}{R\left(\frac{dT}{dh}\right)}}\right\} = \frac{\rho}{\rho_{S/L}}\left\{\left[1+\frac{\left(\frac{dT}{dh}\right)(h-h_{S/L})}{T_{S/L}}\right]\right\}$$

$\rightarrow$ where Eqs.4.18 and 4.17 have been used to replace the pressure and temperature ratios respectively

$$\rho_{h<11\text{ km}} = \rho_{S/L}\left[1+\frac{\left(\frac{dT}{dh}\right)(h-h_{S/L})}{T_{S/L}}\right]^{-\left(\frac{a_g}{R\left(\frac{dT}{dh}\right)}+1\right)} \tag{4.19}$$

For portions of the atmosphere where the temperature remains constant the dp/dh relation obtained from our static atmospheric fluid element can be integrated directly,

$$\begin{aligned}
\frac{dp}{dh} &= -a_g\frac{p}{RT} \\
\int\frac{dp}{p} &= -\frac{a_g}{RT}\int dh \\
\ln\left(\frac{p}{p_{11\text{km}}}\right) &= -\frac{a_g}{RT}(h-h_{11\text{km}})
\end{aligned}$$

→ inserting the limits of the tropopause

$$p_{11\text{km}<h<20\text{km}} = p_{11\text{km}}e^{-\frac{a_g(h-h_{11\text{km}})}{RT}} \tag{4.20}$$

Equation 4.20 applies within the tropopause which along with the known static temperature can be used to find the density at any location within this region of the atmosphere. By changing the limits for each particular atmospheric region equations similar to those in Eqs.4.18 and 4.20 can be obtained for any altitude range.

Getting back to our drag analysis, since density decreases with increasing altitude there will come a point where eventually ΔV_{drag} becomes negligible despite the increasing velocity. For example, at approximately 30 km ($\approx$ 100,000 ft) the atmospheric density decreases to around 1% of the Sea Level value while at 75 km (the edge of the sensible atmosphere) it has decreased to just 0.004% of $\rho_{S/L}$.

At lower altitudes where one may still want to evaluate ΔV_{drag} one must say something about the manner in which the vehicle mass changes with time. Having already noted that the change in the instantaneous vehicle mass is due to the rate of propellant massflow (Eq.4.9) one can write,

$$m = m_I - \dot{m}_{pr}t = m_I\left(1-\frac{\dot{m}_{pr}}{m_I}t\right) \tag{4.21}$$

allowing the velocity required to overcome the effects of drag to be written,

$$\Delta V_{drag} = \frac{S}{2m_I} \int \frac{\rho V^2 c_D}{1 - \left(\frac{\dot{m}_{pr}}{m_I}\right) t} dt \tag{4.22}$$

The only term in this expression yet to be considered is the drag co-efficient. For most low speed flight vehicles this term is broken down into two components, parasite and induced drag. This is because the parasite drag component is generally fairly constant while the induced drag can be directly related to the lift co-efficient of the vehicle (which in turn is directly related to the angle, α, the vehicle makes with the flow velocity),

$$c_D = \underbrace{c_{D_o}}_{\text{parasite}} + \underbrace{A \frac{dc_L}{d\alpha} \alpha}_{\text{induced}} \tag{4.23}$$

→ the variable A is a constant related to the geometry of the lifting surfaces

As we have already noted that a rocket generally remains aligned with the flow velocity this means that $\alpha \approx 0$ and thus only the parasite drag term need be considered. However, the approximation that this term is constant applies only to flight speeds below Mach one. As sonic conditions are reached shock waves begin to form around the vehicle giving rise to the appearance of *wave drag*. This dramatically increases the drag and is a function of the loss of momentum through the shocks. The result of wave drag is to make the parasite drag term vary with Mach number as shown in Fig.4.5.

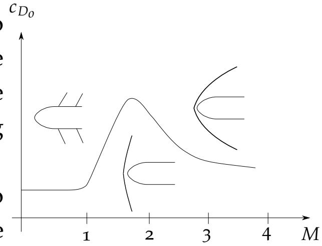

Figure 4.5: Variation of parasite drag co-efficient with Mach number

At speeds slightly below Mach 1 small shocks can start to form due to local areas of flow acceleration. As the vehicle reaches sonic velocity a bow shock forms whose angle depends on the design of the leading edge of the vehicle. The more perpendicular this bow shock is to the flow direction, the larger the loss of momentum through the shock and thus the larger the drag. As the vehicle speed increases beyond Mach one, the bow shock begins to bend back towards the vehicle and thus becomes more aligned with the flow. This in turn decreases the momentum loss in the direction of the flow and thus decreases the drag. However, the drag always remains above the value without shocks present.

(i.e., $\overline{V}^2 = \overline{V_r}^2 + \overline{V_\theta}^2$).

To dimensionalize these values one can use the freestream conditions and our previously derived oblique shock equations to get any of the conditions directly behind the shock. Once this has been done, given that we have assumed both homentropic and homenthalpic flow behind the shock, the stagnation conditions remain constant and thus V_{max} can be found using Eq.4.45. Therefore, not only can we solve for the shock angle, but we now have the complete flowfield between this shock and the cone surface!

Given that we started all this conical shock business because of wave drag, it might be helpful to remind ourselves how this helps us. Well, if we know the entire pressure field between the cone and the shock, we can examine the ray along the cone surface and find $p(\theta = \delta_c)$. In turn, this can be multiplied by the area to obtain the force acting on the cone due to pressure. Resolving this force in the flight direction yields drag and when non-dimensionalized gets us c_D *including the effects of the shock*, i.e., wave drag.

Burnout Conditions

Having established a means of estimating c_{D_o} for conical nose cones as a function the vehicle Mach number, we have clearly shown that the drag co-efficient is a function of time and thus cannot be treated as constant. Clearly, with $\rho(t)$, $V(t)$, and $c_D(t)$, analytic solutions for ΔV_{drag} depend on how these variables are modelled, where often look up tables are employed in a numerical integration approach. All of that being said, ΔV_{drag} is on the order of between 300 and 1000 m/s and applies only to the lower stages of a typical launch (although if the launch profile is changed for something like an airbreathing engine this can change).

Moving next to the gravity effects, recalling from Eq.4.11 we had written,

$$\Delta V_{gravity} = \int a_g \cos\theta dt$$

but from our consideration of orbits we established that the acceleration due to gravity is a function of our distance away from the centre of the Earth and so we can write,

$$\frac{a_g}{a_{g_{S/L}}} = \frac{\frac{\cancel{\mu_E}}{(R_E+h)^2}}{\frac{\cancel{\mu_E}}{R_E^2}} = \frac{R_E^2}{(R_E+h)^2}$$

$$a_g = a_{g_{S/L}}\left[\frac{R_E^2}{(R_E+h)^2}\right] \tag{4.58}$$

→ where the acceleration due to gravity has been replaced using Eq.2.7

and so the gravity term can be expressed as,

→ assuming sea level is equal to R_E thus $h = 0$

→ recall

$$a_{g_{S/L}} = 9.81\text{m/s}^2$$

$$\Delta V_{gravity} = \int \left\{ a_{g_{S/L}}\left[\frac{R_E^2}{(R_E+h)^2}\right]\right\}\cos\theta dt$$

$$\underline{\underline{\Delta V_{gravity} = a_{g_{S/L}} R_E^2 \int \frac{\cos\theta}{(R_E+h)^2}dt}} \tag{4.59}$$

Since the radius of the Earth is so large ($R_E = 6375$ km) this means that even at altitudes of 200 km $R_E^2/(R_E+h)^2 = 0.94$ and thus often this ratio is approximated as unity leaving,

→ $\overline{\cos\theta} = \int_0^{t_{bo}} \cos\theta dt$

→ t_{bo} is the time at burnout or simply the burn time t_b

$$\underline{\underline{\Delta V_{gravity} \approx a_{g_{S/L}}\overline{\cos\theta}\ t_b}} \tag{4.60}$$

This loss is generally on the order of 1000 to 3000 m/s and depends directly on the burn time. This in turn depends directly on the massflow of propellant where at any time from launch the instantaneous mass is expressed by Eq.4.21,

$$m = m_I - \dot{m}_{pr}t$$

therefore,

→ where at $t = t_{bo} = t_b$, $m = m_{bo}$

→ where Eq.1.3 has been used to replace the propellant massflow

→ noting that $a_g m_I$ is simply the initial weight of the rocket and that F is the total force acting on the rocket (due to both momentum and pressure, see Eq.3.7)

$$\begin{aligned} t_b &= \frac{m_{bo}-m_I}{-\dot{m}_{pr}} \\ &= \frac{m_I - m_{bo}}{\left\{\frac{F}{I_{sp}a_g}\right\}} \\ &= \frac{\frac{1}{m_I}(m_I - m_{bo})}{\frac{1}{m_I}\left[\frac{F}{I_{sp}a_g}\right]} = \frac{\left(1-\frac{m_{bo}}{m_I}\right)I_{sp}}{\frac{F}{W}} \end{aligned}$$

At burnout the rocket has reached its final mass and thus $m_{bo} = m_F$ allowing the burn time to be expressed as,

→ where the definition of the mass ratio (Eq.4.2) has been used

$$\underline{\underline{t_b = \frac{I_{sp}}{\left(\frac{F}{W}\right)}\left(1-\frac{1}{\mu}\right)}} \tag{4.61}$$

In Eq.4.61 often the force F is simply taken as the thrust thus the denominator contains the *thrust to weight* ratio. However, it should be kept in mind that there can be a difference between the thrust T (Eq.3.9) and the total propulsive force acting on the rocket (Eq.3.7).

As most rockets are launched in the vertical direction this implies a force (or thrust) to weight ratio greater than unity. The larger this value, the smaller the burn time for a given I_{sp} engine. This is beneficial for reducing the gravity loss term, however, large thrust to weight ratios imply large initial accelerations. This in turn increases the velocity at lower altitudes and thus the drag loss term increases. On balance, a reduction in gravity losses usually outweighs an increase in drag losses and thus shorter burn times generally prevail.

Adding the effects of gravity (Eq.4.60) to our expression for the velocity at burnout in Eq.4.12 while still neglecting drag one can write,

$$\underline{\underline{\Delta V = V_{bo} = u_{eq} \ln \mu - a_{g_{S/L}} \overline{\cos\theta}\ t_b}} \tag{4.62}$$

At this point we have everything we need to calculate the burnout velocity. However, recalling our orbit analyses we require both the burnout velocity, and radius, to determine the resulting orbit (along with the burnout elevation angle, but we will return to that later). The burnout radius is equal to the burnout height plus the radius of the Earth, while for vertical or near vertical ascents the rate of change of height is equal to the velocity....

$$V - \cancelto{0}{V_I} = u_{eq} \ln \mu - a_{g_{S/L}} \cancelto{1}{\overline{\cos\theta}}\ t$$

→ assume a vertical ascent where $\theta = 0$ at all times

$$\left\{\frac{dh}{dt}\right\} = u_{eq} \ln \left\{\frac{m_I}{m}\right\} - a_{g_{S/L}} t$$

→ using Eq.4.2 to replace μ for a particular instant in time t

$$dh = \left[u_{eq} \ln m_I - u_{eq} \ln m - a_{g_{S/L}} t\right] dt$$

$$\int_0^{h_{bo}} dh = \int_0^{t_{bo}} \left[u_{eq} \ln m_I - u_{eq} \ln m - a_{g_{S/L}} t\right] dt$$

$$h_{bo} = -u_{eq} \int_0^{t_b} \ln m dt + u_{eq} \ln m_I \int_0^{t_b} dt - a_{g_{S/L}} \int_0^{t_b} t dt \tag{4.63}$$

→ noting that at burnout $t_{bo} = t_b$ while assuming the engine produces a constant I_{sp} during the burn time (this is the same as saying it yields a constant equivalent velocity u_{eq}, see Eq.3.65)

In order to evaluate all the integrals on the right hand side of Eq.4.63 one must say something about the manner in which the instantaneous mass varies with time. Recalling that it is the ejection of propellant mass which is solely responsible for any vehicle mass changes, using the definition in Eq.4.9 one can write,

→ let $\ln m = u$ (thus $du = dm/m$) and $dv = dm$ (thus $v = m$)

→ from the Calculus we might recall that integration by parts yields the result

$$\int udv = uv - \int vdu \quad (4.64)$$

$$\int_0^{t_b} \ln m dt = \int_0^{t_b} \ln m \left\{ \frac{-1}{\dot{m}_{pr}} dm \right\} = \frac{-1}{\dot{m}_{pr}} \int_0^{t_b} \ln m dm$$

$$\int_{m_I}^{m_F} \ln m dm = \ln(m) m|_{m_I}^{m_F} - \int_{m_I}^{m_F} \cancel{m} \left(\frac{dm}{\cancel{m}} \right)$$

where at $t = 0$ the mass is equal to the initial mass m_I and at burnout the mass equals m_F. Completing the integration as a function of dm yields,

$$\int_{m_I}^{m_F} \ln m dm = [m \ln m - m]\,|_{m_I}^{m_F} = [m(\ln m - 1)]\,|_{m_I}^{m_F}$$

$$\begin{aligned} \int_{m_I}^{m_F} \ln m dm &= m_F(\ln m_F - 1) - m_I(\ln m_I - 1) \\ &= (m_I - m_F) + m_F \ln m_F - m_I \ln m_I \end{aligned}$$

Using this result in our original time integral yields,

→ from Eq.4.21 one can note that over the entire burn time $m_F = m_I - \dot{m}_{pr} t_b$ which can be rearranged to replace the propellant massflow

$$\begin{aligned} \int_0^{t_b} \ln m dt &= \frac{1}{-\dot{m}_{pr}} \int_0^{t_b} \ln m dm \\ &= \frac{1}{-\dot{m}_{pr}} \{(m_I - m_F) + m_F \ln m_F - m_I \ln m_I\} \\ &= \left\{ \frac{t_b}{(m_F - m_I)} \right\} [(m_I - m_F) + m_F \ln m_F - m_I \ln m_I] \\ &= \frac{t_b \cancel{(m_I - m_F)}}{-\cancel{(m_I - m_F)}} - \frac{t_b m_F \ln m_F}{(m_I - m_F)} + \frac{t_b m_I \ln m_I}{(m_I - m_F)} \\ &= -t_b - \frac{t_b \cancel{m_F} \ln m_F}{\cancel{m_F}\left(\frac{m_I}{m_F} - 1\right)} + \frac{t_b m_I \ln m_I}{m_F\left(\frac{m_I}{m_F} - 1\right)} \\ \int_0^{t_b} \ln m dt &= -t_b - \frac{t_b \ln m_F}{(\mu - 1)} + \frac{t_b \mu \ln m_I}{(\mu - 1)} \end{aligned}$$

→ recalling the definition of the mass ratio (Eq.4.2)

With this result we can now return to Eq.4.63 to complete the integration process (where the other two integrals are reasonably straightforward after having solved this one!),

$$
\begin{aligned}
h_{bo} &= -u_{eq}\left\{-t_b - \tfrac{t_b \ln m_F}{(\mu-1)} + \tfrac{t_b \mu \ln m_I}{(\mu-1)}\right\} + u_{eq}\ln m_I\left\{t_b\right\} - a_{g_{S/L}}\left\{\tfrac{1}{2}t_b^2\right\} \\
&= u_{eq}t_b + \tfrac{u_{eq}t_b \ln m_F}{(\mu-1)} - \tfrac{u_{eq}t_b\mu \ln m_I}{(\mu-1)} + \left\{\tfrac{(\mu-1)}{(\mu-1)}\right\} u_{eq}\ln m_I t_b - \tfrac{1}{2}a_{g_{S/L}}t_b^2 \\
&= u_{eq}t_b\left\{1 + \tfrac{\ln m_F - \cancel{\mu \ln m_I} + \cancel{\mu \ln m_I} - \ln m_I}{(\mu-1)}\right\} - \tfrac{1}{2}a_{g_{S/L}}t_b^2
\end{aligned}
$$

→ where

$m_F = m_s + m_{pay}$

and

$-(\ln m_I - \ln m_F)$
$= -\ln(\frac{m_I}{m_F})$
$= -\ln \mu$

$$
\underline{\underline{h_{bo} = u_{eq}t_b\left[1 - \frac{\ln \mu}{(\mu - 1)}\right] - \frac{1}{2}a_{g_{S/L}}t_b^2}} \tag{4.65}
$$

→ assuming a constant gravitational acceleration (due to the use of Eq.4.60, otherwise Eq.4.59 must be used)

This expression gives us the burnout height (from the surface of the Earth) of a vertical or near vertical ascent neglecting the effects of atmospheric drag on the vehicle. Therefore, Eq.4.65 can be used to determine r_{bo} in all our previous orbit expressions by adding the result to the radius of the Earth.

This expression can also be used to estimate the maximum height obtainable by the vehicle in something like a sounding rocket application. Even though after burnout no more energy is being added to the vehicle, energy can be re-distributed between kinetic and potential forms. If one trades all the kinetic energy at burnout for potential energy one can write,

$$
\begin{aligned}
\cancelto{0}{KE_{h_{max}}} + PE_{h_{max}} &= KE_{bo} + PE_{bo} \\
KE_{bo} &= PE_{h_{max}} - PE_{bo} \\
\tfrac{1}{2}\cancel{m_F}V_{bo}^2 &= \cancel{m_F}a_g\left(h_{max} - h_{bo}\right)
\end{aligned}
$$

$$
\underline{\underline{h_{max} = h_{bo} + \frac{V_{bo}^2}{2a_g}}} \tag{4.66}
$$

We might notice that in Eq.4.66 we have not specified the Sea Level value of the acceleration due to gravity even though we have assumed that it is constant to obtain the above expression. Indeed, if one is trying to obtain as much height as possible then heights on the order of $h \approx R_E$ can be obtained (and thus from Eq.4.58 a_g can vary considerably). However, during this process since no energy is being added the trajectory will follow that of a given orbit thus our orbital expression should apply!

If we consider a very, very, very, elliptical orbit where it is so elliptical that it looks like a straight line then several approximations can be made. First, since we are trying to reach a maximum height our trajectory should be vertical and thus at burnout $\phi_{bo} = 90^o$ allowing the eccentricity to be calculated using Eq.2.52 as,

$$\epsilon^2 = \left(\frac{r_{bo}V_{bo}^2}{\mu} - 1\right)^2 \cancelto{0}{\cos^2(90^o)} - \cancelto{0}{\cos^2(90^o)} + 1$$

$$\therefore \epsilon = 1$$

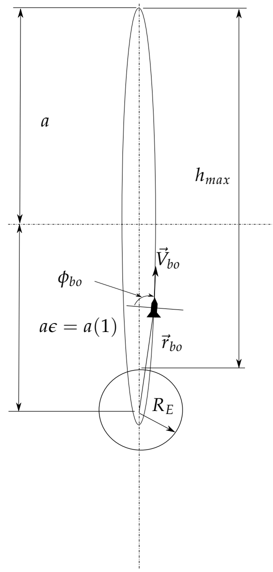

Figure 4.13: Vertical trajectory as a highly elliptical orbit

If the eccentricity is equal to unity then the focus of the ellipse, which is the location from which $\vec{r}_{bo}$ is measured (i.e., the centre of mass of the body being orbited), co-incides with the periapsis and thus from the geometry in Fig.4.13 one can write,

$$2a = R_E + h_{max} \tag{4.67}$$

Recalling Eq.2.30 which at burnout conditions can be written,

$$\begin{aligned} e &= \frac{1}{2}V_{bo}^2 - \frac{\mu}{r_{bo}} \\ \left\{\frac{\cancel{-\mu}}{\cancel{2}a}\right\} &= \frac{\cancel{-\mu}}{\cancel{2}}\left(-\frac{V_{bo}^2}{\mu} + \frac{2}{r_{bo}}\right) \\ \frac{1}{a} &= \frac{1}{r_{bo}}\left(2 - \frac{r_{bo}V_{bo}^2}{\mu}\right) \end{aligned}$$

→ where Eq.2.33 has been used to relate the semi-major axis length a to the specific mechanical energy e

allows the semi-major axis length to be expressed as,

$$a = \frac{r_{bo}}{2 - \frac{r_{bo}V_{bo}^2}{\mu}} \tag{4.68}$$

This is an alternative expression to that obtained in Eq.2.53 that depends on knowing the burnout velocity instead of the angular distance from periapsis at burnout. Using Eq.4.68 in Eq.4.67 allows the maximum height from the surface to be written,

$$h_{max} = \cancel{2}\left\{\frac{r_{bo}}{\cancel{2}\left(1-\frac{r_{bo}V_{bo}^2}{2\mu}\right)}\right\} - R_E$$

$$= \frac{r_{bo}}{\left(1-\frac{r_{bo}V_{bo}^2}{2\mu}\right)} - R_E$$

$$h_{max} = \frac{\{R_E + h_{bo}\}}{1 - \frac{V_{bo}^2}{2\mu}\{R_E + h_{bo}\}} - R_E \qquad (4.69)$$

$\rightarrow$ noting that

$$r_{bo} = R_E + h_{bo} \quad (4.70)$$

To evaluate Eq.4.69 the burnout height h_{bo} can be found using Eq.4.65 while the burnout velocity can be calculated using Eq.4.62 (where for a vertical ascent $\theta = 0$ and thus $\overline{\cos\theta} = 1$). Therefore, if one knows the mass ratio of the rocket and the burn time then Eq.4.69 can be used to determine the maximum obtainable height while accounting for the variation in the strength of gravity (unlike Eq.4.66 where it is assumed constant).

Ballistic Trajectories

Noting the usefulness of our orbital equations in that they do not assume a constant gravitational acceleration, if the eccentricity condition is such that $\epsilon < 1$ but a portion of the orbital path passes through the body being orbited, perhaps this approach can be used to find the distance travelled using a ballistic trajectory ....

For example, if we assume that after burnout from some launch point on the Earth a rocket has a velocity, position, and elevation angle such that it cannot completely orbit the Earth then it will re-enter the Earth's atmosphere at some point and land once again. Under these circumstances, from burnout to re-entry the rocket will travel an elliptical 'free fall' path covering an angle Ψ as shown in Fig.4.14.

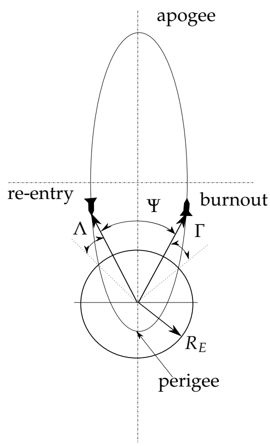

Figure 4.14: Ballistic trajectory

In addition to the free fall path the rocket will travel some angular distance during both the exit from, and re-entry to, the atmosphere depicted in Fig.4.14 as Γ and Λ respectively. Under these circumstances the total distance over the Earth's

8
Epilogue

And so finally we have come to the end! As we look back over the spectrum of topics covered, I hope that you now feel confident in front of your friends and family offering critiques of movies like Star Wars with a significant degree of expertise. Indeed, when you see Darth Vader in his Twin Ion Engine fighter (or simply TIE fighter for those who haven't read this book) you can wonder aloud if his ship is using Xenon as a propellant.....

Having gotten to the end I hope you have enjoyed the unique style with which this book was written and appreciate the manner in which the material was presented. Every effort was made to keep our focus on the production of thrust for all manner of engines, so that as propulsion enthusiasts we would always be able to offer a reasonable answer to the question "will this engine design produce the force required?" However, this did not mean that we shied away from digging deeper into more fundamental concepts like thermal equilibrium, internal energy, unit vectors, or even coordinate systems and transformations to name but a few.

However, unlike many books, these more fundamental adventures were presented as they were needed. Hopefully this makes the reader more inclined to examine them closely given that their use is imminent. Furthermore, these adventures are presented in a significant amount of detail when deriving our expressions of interest. This is done for two reasons. One, should there be any typos or omissions (despite my best efforts), the reader should be capable of spotting and correcting these with less difficulty than in books where equations are simply presented in the form required.

Second, this material can act as a reference for the assumptions, simplifications, and approximations used to arrive at many useful formulae. This way, should the reader want to eliminate a given approximation in order to apply one of these formulae to a new scenario (say, for example, for your revolutionary new launch engine....), they will be able to quickly see where the given approximation is applied. In turn, they can then see what the result of this approximation is on the formula in question and modify the derivation as appropriate. The result is then the proper version of the desired expression applicable to the new scenario!

On a final note, thank you for investing your time, effort, and money, in this book. Much of this material has been presented in classes I have taught over the past two decades in various Aerospace Engineering courses. As such, I feel compelled to add that should you have any questions regarding anything in this book, please feel free to contact me through the publisher's email. Indeed, for those of you thinking 'I'd really like a chance to test out some of these equations', it is possible I could be convinced to send out an exam or two for die hard fans.....

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Index