Approximation of Asymmetric Multivariate Return Distributions

Ba Chu*

Carleton University

October 29, 2011

Abstract

We develop a new method to approximate the asymmetric multivariate probability density function (pdf) of financial asset returns by using series expansions; a rate of convergence for the mean absolute error of this approximation is also provided. We then propose the method of maximum likelihood and the generalized method of moments to estimate the parameters of the approximated pdf. A Monte-Carlo experiment corroborates the feasibility of our approach.

JEL *classification:* C60; C13; G11

Key words: Edgeworth expansion; Series approximation; Asymmetric dependence; Tail risk; Mixture of the Gamma distributions; Laguerre polynomials

1 INTRODUCTION

Modeling joint distributions of asset returns has been considered as a core task in determining optimal strategies and managing risk for investment under uncertainty. In this sprit, many risk

*Department of Economics, Carleton University, B-857 Loeb Building, 1125 Colonel By Drive, Ottawa, ON K1S 5B6, Canada. Phone: +1 (613) 520 2600 (ext. 1546). Fax: +1 (613) 520 3906. E-mail: ba_chu@carleton.ca. Web Page: http://http-server.carleton.ca/~bchu/.

measures and portfolio performance measures are naturally embedded into the moments of the portfolio return distribution. For instance, Malevergne and Sornette (2006) propose a large choice of relevant risk measures congruent with high-order moments, which a portfolio manager is free to choose from, depending on his/her own risk aversion to small versus large risks. In addition, a recent issue of asymmetric dependence, possibly explained by the tendency for stock market declines to be much more rapid than rises, is particularly important for risk control and policy management. This is because dependence structure across risky assets is a key feature of the portfolio choice problem and optimal asset allocation depends on forecasts of asset dependence (see, e.g., Cont (2001) and Okimoto (2008)).

The methods of constructing joint distributions for asset returns characterized by asymmetric dependence include, among many others, the copula approach and series approximations. Copula is conceptually a device to effectively capture a variety of nonlinear relationships which are usually ignored by the multivariate Gaussian distribution. (See, e.g., Patton (2004), Chu (2011b), and references therein for further details on this subject.) The Edgeworth expansion and its variant – so-called the saddlepoint method – has been widely used in a variety of practical investment problems, such as the calculation of shortfall probability and expected shortfall in loss distributions (see, e.g., Sargan (1976), Barndorff-Nielsen and Cox (1979) and Martin (2006), among many others). In the sequel, the purpose of the present paper will be to develop a new approach based on series expansion, which has a natural congruence with the Edgeworth expansion, to approximate the asymmetric joint probability density function (pdf) of asset returns.

In this paper, we shall primarily be concerned with the issue of approximating the multivariate pdf of asset returns by using the series expansion technique. It is to be stressed at this point that we shall not strive for conditions under which a series approximation of a multivariate pdf is nonnegative on its domain because of the complicated nature of this problem; instead we shall assume that the asset returns are uniformly bounded so that a second-order series approximation is legitimate in the sense that, on a compact subset of the positive real surface $\mathbb{R}^{+2}$, the first low-order orthogonal polynomials have dominant influence in a series expansion. Given the validity of this series approximation on some compact subset in $\mathbb{R}^{+2}$, we shall propose methods to estimate the parameters (*or* the correlation coefficients) of this approximated pdf.

The remainder of this paper is organized as follows. Section 2 is mainly concerned with a series approximation for a joint pdf marginalized by either mixtures of the Gamma distributions (see, e.g., Knight et al. (1995)) or the Gaussian distributions; and the formula representing the rate of convergence of the approximation is also provided in this section as Theorem 1. Section 3 proposes the maximum likelihood estimation (MLE) method and the generalized method of moments (GMM) to estimate the parameters of the approximated pdf. These methods are shown to be potentially useful from the practical point of view. Finally, a summary of a simulation study comparing the small-sample properties of the ML and GMM estimators is contained in Section 4. Overall, it was found that the GMM performs relatively well vis-à-vis the ML estimator. Section 5 concludes this paper.

2 MULTIVARIATE DISTRIBUTIONS OF ASSET RETURNS

2.1 Approximate A Bivariate Gamma Probability Density Function with the Laguerre Polynomials

One of many stylized facts about equity returns is gain/loss asymmetry in the return distribution (i.e., large drawdowns in stock prices are not equal to large upward movements.) More often, an asymmetric return distribution can be parametrically modeled with a mixture of the Gamma distributions – in what follows, we shall denote by $\mathrm{G}(\theta, \kappa)$ a Gamma distribution with the scale parameter θ and the location parameter κ. Let (X, Y) represent a vector of stock returns. In view of Knight et al. (1995), the probability density functions (pdfs) of the mixtures of the Gamma distributions of (X, Y) are then given by

$$f_X(x) = p \frac{1}{\theta_{11}^{\kappa_{11}} \Gamma(\kappa_{11})} x^{\kappa_{11}-1} \exp\left\{-\frac{x}{\theta_{11}}\right\} \mathbb{I}(x \geq 0)$$

$$+ \quad (1-p)\frac{1}{\theta_{12}^{\kappa_{12}}\Gamma(\kappa_{12})}(-x)^{\kappa_{12}-1}\exp\left\{\frac{x}{\theta_{12}}\right\}\mathbb{I}(x<0) \tag{2.1}$$

and

$$\begin{aligned} f_Y(y) &= q\frac{1}{\theta_{21}^{\kappa_{21}}\Gamma(\kappa_{21})}y^{\kappa_{21}-1}\exp\left\{-\frac{y}{\theta_{21}}\right\}\mathbb{I}(y\geq 0) \\ &+ (1-q)\frac{1}{\theta_{22}^{\kappa_{22}}\Gamma(\kappa_{22})}(-y)^{\kappa_{22}-1}\exp\left\{\frac{y}{\theta_{22}}\right\}\mathbb{I}(y<0), \end{aligned} \tag{2.2}$$

where $\mathbb{I}(x \geq 0)$ is an indicator function with its value equal to 1 if $x \geq 0$ and to 0 otherwise; $\mathbb{I}(x < 0) = 1 - \mathbb{I}(x \geq 0)$; $\Gamma(x)$ denotes the Gamma function; p and q are the probability weights; θ_{11}, θ_{12}, θ_{21}, and θ_{22} signify the scale parameters of the Gamma distributions; and κ_{11}, κ_{12}, κ_{21}, and κ_{22} represent the shape parameters of the Gamma distributions. Note that these scale and shape parameters are, by definition, positive and finite. The closed-form maximum likelihood estimates (MLEs) of these parameters are obtained by Chu et al. (2010).

In the sequel, we propose to construct a bivariate pdf marginalized by mixtures of the Gamma distributions as follows:

$$\begin{aligned} f_{XY}(x,y) &= P\{x\geq 0, y\geq 0\}f_X(x^+)f_Y(y^+)\left(1+g(x^+,y^+,\boldsymbol{\rho}_{++})\right) \\ &+ P\{x< 0, y< 0\}f_X(x^-)f_Y(y^-)\left(1+g(x^-,y^-,\boldsymbol{\rho}_{--})\right) \\ &+ P\{x\geq 0, y< 0\}f_X(x^+)f_Y(y^-)\left(1+g(x^+,y^-,\boldsymbol{\rho}_{+-})\right) \\ &+ P\{x< 0, y\geq 0\}f_X(x^-)f_Y(y^+)\left(1+g(x^-,y^+,\boldsymbol{\rho}_{-+})\right), \end{aligned} \tag{2.3}$$

where $X^+ = X\mathbb{I}(X \geq 0)$ and $X^- = X\mathbb{I}(X < 0)$; and $g(X, Y, \boldsymbol{\rho})$ is some analytic function with $\boldsymbol{\rho}$ representing linear and nonlinear correlation coefficients. The intuition for this functional specification is as follows: If X and Y are independent, then the joint pdf of X and Y is merely the product of the two mixtures of the Gamma pdfs. Thus, in the presence of dependency, it is natural to specify the joint pdf as a separable function with two components (the product of the two mixtures of the Gamma pdfs and a coupling function, $g(X, Y, \boldsymbol{\rho})$, used to capture asymmetric dependence).

For illustration purposes, we shall provide an approximation to $g(X^+, Y^+, \boldsymbol{\rho}_{++})$, because the other coupling functions can be approximated in the same manner. (For future references, it is helpful to note that, in what follows, the superscript + will be suppressed unless confusion is likely.) First, we need some preliminary results on sequences of orthonormal polynomials:

Definition 1. *(Parthasarathy, 1977, p. 219) Let $(\mathcal{X}, \mathcal{B}, \mu)$ represent a probability space with a measure, μ. Then $g_n(x)$ is a sequence of orthonormal polynomials on a Hilbert space, say $L_2(\mu)$, if and only if $\int_{\mathcal{X}} g_n(x)g_m(x)d\mu(x) = \delta_{n,m}$, where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ otherwise.*

Lemma 1. *Let $\{(\mathcal{X}_i, \mathcal{B}_i, \mu_i)\}_{i=1}^k$ denote a sequence of probability spaces; and $\mu = \mu_1 \times \cdots \times \mu_k$ represents a product measure on the product space $(\mathcal{X}_1 \times \cdots \times \mathcal{X}_k, \mathcal{B}_1 \times \cdots \times \mathcal{B}_k)$. Let $\{f_{i,n_i}\}_{i=1}^k$ be an orthonormal basis for $L_2(\mu_i)$, then the tensor products $g_{n_1,\dots,n_k}(x_1, \dots, x_k) = f_{1,n_1}(x_1) \times \cdots \times f_{k,n_k}(x_k)$ become an orthonormal basis, which contains orthonormal multivariate polynomials that linearly span $L_2(\mu)$.*

Proof. An application of the Fubini theorem yields

$$\delta = \int_{\mathcal{X}_1\times\cdots\times\mathcal{X}_k} g_{n_1,\dots,n_k}(x_1, \dots, x_k)g_{m_1,\dots,m_k}(x_1, \dots, x_k)d\mu(x_1, \dots, x_k) = \prod_{i=1}^{k} \int_{\mathcal{X}_i} f_{i,n_i}(x_i)f_{i,m_i}(x_i)d\mu_i(x_i).$$

It then follows that $\delta = 1$, if $n_i = m_i \ \forall \ i = 1, \dots, k$, and $\delta = 0$ otherwise. □

Lemma 2. *Suppose that the probability measure μ (as given in Definition 1) is the Gamma distribution, then the orthonormal basis for $L_2(\mu)$ is the following sequence of* normalized Laguerre polynomials*:*

$$p_n^{(\kappa)} = \left\{ \frac{1}{n!} \frac{\Gamma(\kappa)}{\Gamma(k+n)} \right\}^{\frac{1}{2}} L_n^{(\kappa)}(x),$$

where

$$L_n^{(\kappa)}(x) = (-1)^n x^{1-\kappa} \exp(x) \frac{\partial^n}{\partial x^n} \left(x^{n+\kappa-1} \exp(-x) \right), \tag{2.4}$$

such that

$$\int_0^\infty \frac{1}{\Gamma(\kappa)} x^{\kappa-1} \exp(-x) p_n^{(\kappa)}(x) p_m^{(\kappa)}(x) dx = \delta_{m,n}.$$

Proof. The proof immediately follows from Rade and Westergren (1999, p. 263). □

Eq. (2.4) yields the first three Laguerre polynomials: $L_0^{(\kappa)}(x) = 1$, $L_1^{(\kappa)}(x) = x - \kappa$, $L_2^{(\kappa)}(x) = x^2 - 2(\kappa+1)x + \kappa(1+\kappa)$, and $L_3^{(\kappa)}(x) = x^3 - 3(\kappa+2)x^2 + 3(\kappa+1)(\kappa+2)x - \kappa(\kappa+1)(\kappa+2)$.

Now we can proceed to approximate the joint Gamma pdf $f(x,y)$. Lemma 1 yields a sequence of the bivariate orthogonal Laguerre polynomials,

$$L_{n,m}^{(\kappa_{11},\kappa_{21})}(x,y) = (-1)^{m+n} x^{1-\kappa_{11}} y^{1-\kappa_{21}} \exp(x+y) \frac{\partial^{n+m}}{\partial x^n \partial y^m} \left(x^{n+\kappa_{11}-1} y^{m+\kappa_{21}-1} \exp(-(x+y)) \right).$$

By virtue of Parseval's theorem (see, e.g., (Parthasarathy, 1977, Proposition 6.3.4)), upon normalizing the Gamma random variables by their scale parameters, we obtain the following approximation:

$$f_{NM}(x,y) \doteq \frac{1}{\Gamma(\kappa_{11})\Gamma(\kappa_{12})} x^{\kappa_{11}-1} y^{\kappa_{21}-1} \exp\{-(x+y)\} \left\{ 1 + \sum_{i=1}^{N} \sum_{j=1}^{M} A_{ij} L_{i,j}^{(\kappa_{11},\kappa_{21})}(x,y) \right\} \tag{2.5}$$

for some integers $N > 1$ and $M > 1$, where

$$\begin{aligned} A_{ij} &= \frac{1}{i!j!} \frac{\Gamma(\kappa_{11})\Gamma(\kappa_{21})}{\Gamma(\kappa_{11}+i)\Gamma(\kappa_{21}+j)} \int_{[0,\infty]^2} L_{ij}^{(\kappa_{11},\kappa_{21})}(x,y) f(x,y) dx dy \\ &= \frac{1}{i!j!} \frac{\Gamma(\kappa_{11})\Gamma(\kappa_{21})}{\Gamma(\kappa_{11}+i)\Gamma(\kappa_{21}+j)} \left(g, L_{ij}^{(\kappa_{11},\kappa_{21})} \right)^*, \end{aligned} \tag{2.6}$$

where $\left(g, L_{ij}^{(\kappa_{11},\kappa_{21})} \right)^*$ represents the inner product between $g(x,y,\boldsymbol{\rho}_{++})$ and $L_{ij}^{(\kappa_{11},\kappa_{21})}(x,y)$, taken under the measure $d\mu = \frac{1}{\Gamma(\kappa_{11})\Gamma(\kappa_{12})} x^{\kappa_{11}-1} y^{\kappa_{21}-1} \exp\{-(x+y)\}$, on the Hilbert space $L_2(\mu)$. Note at this point that Eq. (2.6) is derived by using the following orthonormality property: $\int_{[0,\infty]^2} L_{ij}^{(\kappa_{11},\kappa_{21})}(x,y) f(x,y) dx dy = A_{ij} i! j! \frac{\Gamma(\kappa_{11}+i)\Gamma(\kappa_{21}+j)}{\Gamma(\kappa_{11})\Gamma(\kappa_{21})}$.

Kibble (1941) propose an alternative decomposition of a bivariate Gamma distribution as given by $f(x,y) = \frac{1}{\Gamma(\kappa_{11})\Gamma(\kappa_{12})} x^{\kappa_{11}-1} y^{\kappa_{21}-1} \exp\{-(x+y)\} \left\{ 1 + \sum_{i=1}^{\infty} \rho^i L_{i,i}^{(\kappa_{11},\kappa_{21})}(x,y) \right\}$, where ρ represents the simple linear correlation between two Gamma random variables.[1] However, it can be immediately seen that this formulation does not allow for nonlinear correlations, which are the main characteristics of asymmetric dependence in asset returns, albeit its parametric parsimony,

[1] I am indebted to a referee for pointing out this simplified formulation.

while our formulation can be implicitly represented in terms of nonlinear correlations.

Theorem 1 (below) provides the asymptotic mean absolute error (MAE) of the series approximation [defined in Eq. (2.5)] in terms of N and M.

Theorem 1. *Let $\Lambda^2_{r\omega}\left(M_0, \ldots, M_{r+1}; \mathbb{R}^{+2}\right)$, where $\mathbb{R}^{+2}$ is the positive half-plane, represent a class of smooth bivariate functions, say $\ell(x,y)$ for every $(x,y) \in \mathbb{R}^{+2}$, with the continuous partial derivatives of orders, $j = 0, \ldots, r$, that satisfy the following conditions: For each partial derivative of order j, $\|D^j \ell\|_2 \leq M_j$ with $j = 0, \ldots, r$, where the L_2- norm is taken with respect to the joint pdf, $f(x,y)$; and in addition, for each partial derivative of order r, $\omega_x\left(D^r_x \ell, h\right) \leq M_{r+1}\omega_x(\ell, h)$ and $\omega_y\left(D^r_y \ell, h\right) \leq M_{r+1}\omega_y(\ell, h)$, where $D^r_x \ell$ indicates the $r-th$ order partial derivative of ℓ with respect to x; and $\omega_x(\ell, h) \doteq \max_{\substack{x,y \\ |t|\leq h}} |\ell(x+t, y) - \ell(x,y)|$ is the* modulus of continuity *of ℓ with respect to x.*

Suppose that the bounded function $g(x,y) = g(x,y,\boldsymbol{\rho}_{++})$ such that $\sup_{x,y} g(x,y) \in (-1, \infty)$ (cf. Eq. (2.3)) belongs to the class $\Lambda^2_{r\omega}\left(M_0, \ldots, M_{r+1}; \mathbb{R}^{+2}\right)$; and the random variables (X,Y) satisfy the joint-moment condition $\max_{\substack{1\leq i\leq N \\ 1\leq j\leq M}} E\left[|X|^{pi}|Y|^{pj}\right] < \infty$ for some $p > 1$, where the expectation is taken with respect to the joint pdf, $f(x,y)$, then one has that

- *If $N = M$ and $r \geq 1$, then, for some generic constants, $m > \frac{3p}{p-1}$ and $a \in \left(0, 1 - \frac{3p}{m(p-1)}\right)$,*

$$\|f_{NN}(X,Y) - f(X,Y)\|_1 = O\left(\max\left(N^{3 - m\frac{(p-1)(1-a)}{p}}, \frac{\omega_x\left(\frac{N^{-a}}{2}\right)}{N^{r-1}} + \frac{\omega_y\left(\frac{N^{-a}}{2}\right)}{N^{r-1}}\right)\right),$$

  where the L_1- norm is taken with respect to the joint pdf, $f(x,y)$; $\omega_x(.)$ and $\omega_y(.)$ signify the moduli of continuity of the function g with respect to x and y respectively.

- *If $N^\alpha < M < N^\beta$, with $0 < \alpha < \beta$ being given, and $r > \max\left(\frac{\beta+1}{2}, \frac{\alpha+1}{2\alpha}\right)$, then, for some generic constants, $m > \frac{3}{2}\max\left(\frac{p(\beta+1)}{p-1}, \frac{p(\alpha+1)}{\alpha(p-1)}\right)$, $a \in \left(0, \min\left(1, 1 - \frac{3}{2}\frac{p(\beta+1)}{m(p-1)}\right)\right)$ and $b \in \left(0, \min\left(1, 1 - \frac{3}{2}\frac{p(\alpha+1)}{\alpha m(p-1)}\right)\right)$,*

$$\|f_{NM}(X,Y) - f(X,Y)\|_1 \quad = \quad O\Big(\max\Big(N^{\frac{3(\beta+1)}{2} - \frac{m(p-1)(1-a)}{p}}, M^{\frac{3(\alpha+1)}{2\alpha} - \frac{m(p-1)(1-b)}{p}},$$

$$N^{\frac{\beta+1}{2}-r}\omega_x\left(\frac{N^{-a}}{2}\right)+M^{\frac{\alpha+1}{\alpha}-r}\omega_y\left(\frac{M^{-b}}{2}\right)\Bigg)\Bigg).$$

Proof. See Appendix. □

Remark 2.1. *We shall now discuss on the behavior of the asymptotic series, $f_{NM}(x,y)$, defined in Eq.* (2.5)*. Theorem 1 asserts that the MAE of the series approximation converges to zero as the orders of approximation (N and M) become large; and the rate of convergence essentially depends on the ratios (M/N^{α}, M/N^{β}) and the degree of smoothness of the function $g(x,y)$ [as measured by the moduli of continuity, $\omega_x(.)$ and $\omega_y(.)$] as well as the order of partial differentiation, r.*

In particular, given some finite, but sufficiently large, orders of approximation, this series is always convergent on a certain compact subset (for instance, a parallelepiped) of $\mathbb{R}^{+2}$; and, close in spirit to the Edgeworth expansion, the first few Laguerre polynomials will contribute most of the influence in this expansion – that is, higher-order terms will have smaller impact.[2] *Moreover, since X and Y are both standardized by their scale parameters, it is possible in a number of cases to approximate the joint pdf with a few low-order Laguerre polynomials if the scale parameters are sufficiently large. However, we are agnostic ex ante as to which orders of the Laguerre polynomial may optimally approximate the joint pdf. In general, we believe that the choice of finite, but not too large, N and M would be sufficient to obtain a good approximation as the scale parameters are large enough.*

Theorem 2. *Let (X^+, Y^+) be a vector of positive Gamma random variables. The second-order series approximation of the joint pdf, $f(x,y)$, is given by*

$$\begin{aligned} f_{22}(x,y) &= \frac{1}{\Gamma(\kappa_{11})\Gamma(\kappa_{21})}x^{\kappa_{11}-1}y^{\kappa_{21}-1}\exp\{-(x+y)\}\Big\{1+a_1L_1^{(\kappa_{11})}(x)L_1^{(\kappa_{21})}(y) \\ &+ a_2L_1^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)+a_3L_2^{(\kappa_{11})}(x)L_1^{(\kappa_{21})}(y)+a_4L_2^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)\Big\}, \qquad (2.7)\end{aligned}$$

[2] Reminiscent that the Edgeworth expansion of the pdf of $S_n^* = \sum_{i=1}^n X_i/\sqrt{n}$ utilizes the Hermite polynomials. The truncation at the sixth-order Hermite polynomial yields an error term, $O_p(n^{-3/2})$, because the fourth-order cummulant of S_n^* is $O(n^{-3/2})$. (See, e.g., Small (2010, p. 261).)

where $x = x^+/\theta_{11}$, $y = y^+/\theta_{21}$,

$$
\begin{aligned}
a_1 &= \frac{1}{\sqrt{\kappa_{11}\kappa_{21}}}\rho_{xy}, \\
a_2 &= \frac{1}{2\kappa_{11}\kappa_{21}(\kappa_{21}+1)}\left\{\sqrt{m_2^{(x)}m_4^{(y)}}\rho_{xy^2} - 2\sqrt{m_2^{(x)}m_2^{(y)}}\rho_{xy}\right\}, \\
a_3 &= \frac{1}{2\kappa_{11}\kappa_{21}(\kappa_{11}+1)}\left\{\sqrt{m_4^{(x)}m_2^{(y)}}\rho_{x^2y} + 2\sqrt{m_2^{(x)}m_2^{(y)}}\rho_{xy}\right\}, \\
a_4 &= \frac{1}{4\kappa_{11}\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)}\left\{\sqrt{m_4^{(x)}m_4^{(y)}}\rho_{x^2y^2} - 2\sqrt{m_4^{(x)}m_2^{(y)}}\rho_{x^2y} - 2\sqrt{m_2^{(x)}m_4^{(y)}}\rho_{xy^2}\right. \\
&+ \left. 4\sqrt{m_2^{(x)}m_2^{(y)}}\rho_{xy} - \kappa_{11}\kappa_{21}\right\},
\end{aligned}
$$

with $\rho_{xy} = Corr(X,Y)$, $\rho_{x^2y^2} = Corr(X^2,Y^2)$, $\rho_{xy^2} = Corr(X,Y^2)$, $\rho_{x^2y} = Corr(X^2,Y)$; *and* $m_n^{(x)}$ *and* $m_n^{(y)}$ *respectively represent the* n*-th central moments of* X *and* Y.

Proof. The proof immediately follows from Eqs. (2.5) and (2.6). □

Remark 2.2. *Like the Edgeworth expansion, the approximator* $f_{NM}(x,y)$ *can also take negative values. Thus the validity of the proposed series approximation for approximating the joint pdf* $f(x,y)$ *with small orders of approximation,* N *and* M*, can be legitimately questioned. In fact the function* $f_{NM}(x,y)$ *will be nonnegative if the supremum of the series* $\left|\sum_{i=N}^{\infty}\sum_{j=M}^{\infty} A_{ij}L_{ij}^{(\kappa_{11},\kappa_{21})}(x,y)\right|$ *converges to zero as* $(N,M) \longrightarrow \infty$*, and so we shall insist that*

$$\lim_{(N,M)\to\infty}\ \sup_{(x,y)\in\mathbb{R}^2}\left|\sum_{i=N}^{\infty}\sum_{j=M}^{\infty} A_{ij}L_{ij}^{(\kappa_{11},\kappa_{21})}(x,y)\right| = 0.$$

This certainly holds under the following two conditions: $\sup_{i\geq N, j\geq M}\sup_{(x,y)\in\mathbb{R}^2}\left|L_{ij}^{(\kappa_{11},\kappa_{21})}(x,y)\right| < \infty$ *and* $\sum_{i=N}^{\infty}\sum_{j=M}^{\infty}|A_{ij}| < \infty$*. In this spirit, a feasible way to deal with the non-positiveness of* $f_{22}(x,y)$ *in an estimation procedure is possibly to restrict* $\{a_i\}_{i=1}^4$ *to be infinitesimally small relative to the sample size.*

Remark 2.3. *In view of Lemma 1 the above approximation can be generalized to the* n*-dimensional case without any significant difficulty. We shall present this generalization in Section 2.2.*

In addition, the correlation coefficients ρ_{xy}*,* $\rho_{x^2y^2}$*,* ρ_{xy^2}*, and* ρ_{x^2y} *are fully identified from*

a_1, a_2, a_3, and a_4 which can then be efficiently estimated by MLE or the generalized method of moments (GMM).

The pdfs of (X^-, Y^-), (X^+, Y^-), and (X^-, Y^+) can be approximated by using the same method, which gives rise to rather cumbersome expressions, so we shall allow ourself to dispense with the need of presenting them here. However, all the details are given in Chu (2011a).

An example:

It is necessary to be stressed at this point that the accuracy of the above approximation, defined in Eq. (2.7), can be legitimately questioned. We shall now examine the validity of this approximation with a simple example. In what follows, we use an exact bivariate Gamma pdf with the marginal Gamma pdf (see, e.g., Johnson and Kotz (1972)). The method introduced in Section 2.1 is then applied to construct the second-order series approximation of this bivariate pdf. As shown, this approximation is parameterized by $(\rho_{xy}, \rho_{xy^2}, \rho_{x^2y}, \rho_{x^2y^2})$. We then use Monte Carlo data simulated from the exact pdf to estimate these parameters by the MLE and GMM methods. Hence, the accuracy of the approximated pdf can be evaluated by computing the differences between the theoretical correlation coefficients and their estimates.

Johnson and Kotz (1972) show that a bivariate vector of dependent, positive Gamma random variables, (Y_1^+, Y_2^+), can be constructed by $Y_1^+ = X_0^+ + X_1^+$ and $Y_2^+ = X_0^+ + X_2^+$, where X_0^+, X_1^+, and X_2^+ have the pdfs $\mathrm{G}(\kappa_0)$, $\mathrm{G}(\kappa_1)$, and $\mathrm{G}(\kappa_2)$ respectively. It then immediately follows that $Y_1^+ \sim \mathrm{G}(\kappa_0 + \kappa_1)$ and $Y_2^+ \sim \mathrm{G}(\kappa_0 + \kappa_2)$. Since the joint characteristic function of (Y_1^+, Y_2^+) is

$$E\left[\exp(t_1 Y_1^+ + t_2 Y_2^+)\right] = (1 - (t_1 + t_2))^{-\kappa_0}(1 - t_1)^{-\kappa_1}(1 - t_2)^{-\kappa_2},$$

the Fourier inversion yields the following joint pdf

$$\begin{aligned} f(y_1, y_2) &= (\Gamma(\kappa_0)\Gamma(\kappa_1)\Gamma(\kappa_2))^{-1}\exp(-y_1 - y_2) \\ &\quad \int_0^{\min(y_1, y_2)} x^{\kappa_0 - 1}(y_1 - x)^{\kappa_1 - 1}(y_2 - x)^{\kappa_2 - 1}\exp(x)dx. \end{aligned} \tag{2.8}$$

The correlation coefficients are given by

$$
\begin{aligned}
\rho_{xy}^{++} &= \frac{1}{\sqrt{\theta_1\theta_2}}\Big\{E[Y_1^+Y_2^+] - \theta_1\theta_2\Big\},\\
\rho_{x^2y}^{++} &= \frac{1}{\sqrt{\theta_2(3\theta_1^2+6\theta_1)}}\Big\{E[Y_1^{+2}Y_2^+] - \theta_2\theta_1(\theta_1+1) - 2\theta_1 E[Y_1^+Y_2^+] + 2\theta_1^2\theta_2^+\Big\},\\
\rho_{xy^2}^{++} &= \frac{1}{\sqrt{\theta_1(3\theta_2^2+6\theta_2)}}\Big\{E[Y_1^+Y_2^{+2}] - \theta_1\theta_2(\theta_2+1) - 2\theta_2 E[Y_1^+Y_2^+] + 2\theta_1\theta_2^2\Big\},\\
\rho_{x^2y^2}^{++} &= \frac{1}{\sqrt{(3\theta_1^2+6\theta_1)(3\theta_2^2+6\theta_2)}}\Big\{E[Y_1^{+2}Y_2^{+2}] - 2\theta_2 E[Y_2^+Y_1^{+2}] - 2\theta_1 E[Y_1^+Y_2^{+2}]\\
&+ \theta_1^2\theta_2(\theta_2+1) + \theta_2^2\theta_1(\theta_1+1) - 3\theta_1^2\theta_2^2\Big\},
\end{aligned}
\tag{2.9}
$$

where $\theta_1 = \kappa_0 + \kappa_1$; $\theta_2 = \kappa_0 + \kappa_2$; $E[Y_1^+Y_2^+] = (1+\kappa_0)\kappa_0 + \kappa_0\kappa_1 + \kappa_0\kappa_2 + \kappa_1\kappa_2$; $E[Y_1^{+2}Y_2^+] = (2+\kappa_0)(1+\kappa_0)\kappa_0 + 2(1+\kappa_0)\kappa_0\kappa_1 + \kappa_0(1+\kappa_1)\kappa_1 + (1+\kappa_0)\kappa_0\kappa_2 + 2\kappa_0\kappa_1\kappa_2 + (1+\kappa_1)\kappa_1\kappa_2$; $E[Y_1^+Y_2^{+2}] = \kappa_0\big((2+\kappa_0)(1+\kappa_0) + 2(1+\kappa_0)\kappa_2 + (1+\kappa_2)\kappa_2\big) + \kappa_1\big((1+\kappa_0)\kappa_0 + 2\kappa_0\kappa_2 + (1+\kappa_2)\kappa_2\big)$; and $E[Y_1^{+2}Y_2^{+2}] = (1+\kappa_0)\kappa_0\big((3+\kappa_0)(2+\kappa_0) + 2(2+\kappa_0)\kappa_2 + (1+\kappa_2)\kappa_2\big) + 2\kappa_0\kappa_1\big((2+\kappa_0)(1+\kappa_0) + 2(1+\kappa_0)\kappa_2 + (1+\kappa_2)\kappa_2\big) + (1+\kappa_1)\kappa_1\big((1+\kappa_0)\kappa_0 + 2\kappa_0\kappa_2 + (1+\kappa_2)\kappa_2\big)$.

The bivariate Gamma random variables (Y_1^-, Y_2^-), (Y_1^-, Y_2^+), and (Y_1^+, Y_2^-) can be immediately constructed by using the same method, so we shall not present them here. However, the details are given in Chu (2011a).

Substituting Eq. (2.9) into Eq. (2.7) yields the second-order approximation of the exact pdf defined in Eq. (2.8). In Section 4, we shall provide some Monte Carlo simulations to examine the accuracy of the proposed approximations.

2.2 Approximation of the Multivariate Distribution

Suppose that we have a n-dimensional vector of asset returns, say $\boldsymbol{X} = (X_1, .., X_n)$. Then there are 2^n quadrants of multivariate probability densities. In order to make the analytics tractable, we shall consider only the total loss of individual assets, $\boldsymbol{X}^- = (X_1^-, \cdots, X_n^-)$. We then approximate the joint pdf of $\boldsymbol{X}^-$ by using the Laguerre series expansion. This approximation can be potentially used for approximating downside-risk or tail-risk measures.

Theorem 3. *Suppose that the losses, $\boldsymbol{X}^-$, from investment assets have Gamma distributions, the n-dimensional loss pdf can be approximated as*

$$
\begin{aligned}
f(x_1,\ldots,x_n) &\approx \frac{1}{\prod_1^n \Gamma(\kappa_i)}(-1)^{\prod_1^n(\kappa_i-1)}\prod_1^n (x_i)^{\kappa_i-1}\exp\left\{\sum_1^n x_i\right\}\Bigg\{A+B(x_1,\ldots,x_n)\rho_1 \\
&+ C\left(x_1,\ldots,x_n\right)\rho_2 + \sum_{\substack{\{(J,K)\subset I \text{ and } J\bigcap K=\varnothing\}\\ \bigcup\{J=\varnothing \text{ and } K\subset I\}\\ \bigcup\{J\subset I \text{ and } K=\varnothing\}}} D_{K,J}\left(x_1,\ldots,x_n\right)\rho_{JK}\Bigg\}, \quad (2.10)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= 1+(-1)^n\prod_1^n \kappa_i, \\
B\left(x_1,\cdots,x_n\right) &= (-1)^n\frac{1}{\sqrt{\prod_1^n \kappa_i}}L_1^{(\kappa_1,\ldots,\kappa_n)}(x_1,\ldots,x_n) + \frac{1}{2^n\prod_1^n\sqrt{\kappa_i}(\kappa_i+1)}L_2^{(\kappa_1,\ldots,\kappa_i)}\left(x_1,\cdots,x_n\right), \\
C\left(x_1,\ldots,x_n\right) &= \frac{\prod_1^n\sqrt{3\kappa_i^2+6\kappa_i}}{4^n\prod_1^n\kappa_i(\kappa_i+1)}L_2^{(\kappa_1,\ldots,\kappa_n)}(x_1,\ldots,x_n), \\
D_{K,J}\left(x_1,\ldots,x_n^+\right) &= (-1)^{|I\setminus J\bigcup K|}2^{|K|}\prod_{i\in J}\sqrt{3\kappa_i^2+6\kappa_i}\prod_{i\in K}\sqrt{\kappa_i}\prod_{i\in I\setminus J\bigcup K}\kappa_i L_2^{(\kappa_1,\ldots,\kappa_n)}\left(x_1,\ldots,x_n\right), \\
\rho_1 &= E\left[\prod_1^n(X_i+\kappa_i)\right], \\
\rho_2 &= E\left[\prod_1^n(X_i+\kappa_i)^2\right], \\
\rho_{JK} &= E\left[\prod_{i\in J}(X_i+\kappa_i)\prod_{i\in K}(X_i+\kappa_i)\right],
\end{aligned}
$$

where $L_i^{(\kappa_1,\ldots,\kappa_n)}(\bullet)$ denotes a Laguerre polynomial such that

$$
L_m^{(\kappa_1,\ldots,\kappa_n)}\left(x_1,\ldots,x_n\right) = \prod_1^n (x_i)^{1-\kappa_i}\exp\left(\sum_1^n x_i\right)\prod_{i=1}^n\frac{\partial^m}{\partial x_i^m}\left\{(x_i)^{m+\kappa_i-1}\exp\left(-\sum_1^n x_i\right)\right\}.
$$

For instance, $L_1(x_1,\ldots,x_n) = (-1)^n\prod_1^n(\kappa_i - x_i)$ and $L_2 = \prod_1^n\left(x_i^2+2(\kappa_i+1)x_i+\kappa_i(\kappa_i+1)\right)$.

The notation $\sum_{\substack{\{(J,K)\subset I \text{ and } J\bigcap K=\varnothing\}\\ \bigcup\{J=\varnothing \text{ and } K\subset I\}\\ \bigcup\{J\subset I \text{ and } K=\varnothing\}}}$ *signifies a summation taken over all the possible, different subsets,* K *and* J*, of the index set* $I=\{1,\ldots,n\}$ *and* $|K|$ *denotes the cardinality of the subset* K*.*

Proof. Available upon request. □

For example, if there are three risky assets, say $I=\{1,2,3\}$, then $\{J,K\}=\{1,2\},\{2,3\},\{1,3\}$, $\{3,2\},\{3,1\},\{2,1\},\{(1,3),2\},\{(1,2),3\}$, and $\{(2,3),1\}$. Substituting these (J,K) into Eq. (2.10) yields an approximation for the trivariate loss pdf.

3 ESTIMATION OF CORRELATION COEFFICIENTS

3.1 Maximum Likelihood (ML) Estimation

In this section, we shall provide the ML estimates of $\left(\rho_{x^+y^+},\rho_{x^{+2}y^{+2}},\rho_{x^{+2}y^+},\rho_{x^+y^{+2}}\right)$. (Note that the estimates of the correlation coefficients in other probability quadrants can be similarly derived. Eq. (2.7) can be rewritten as

$$f_{X^+Y^+}(x,y)\approx \mathrm{G}(x,\kappa_{11})\mathrm{G}(y,\kappa_{21})\Big\{B_0(x,y)+B_1(x,y)\rho_{x^2y^2}+B_2(x,y)\rho_{xy}+B_3(x,y)\rho_{xy^2}+B_4(x,y)\rho_{x^2y}\Big\},$$

where $B_0(x,y)=1+\kappa_{11}\kappa_{21}L_2^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)$; $B_1(x,y)=\frac{\sqrt{m_4^{(x)}m_4^{(y)}}}{4\kappa_{11}\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)}L_2^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)$; $B_2(x,y)=\frac{1}{\sqrt{\kappa_{11}\kappa_{21}}}\Big(L_1^{(\kappa_{11})}(x)L_1^{(\kappa_{21})}(y)-\frac{1}{\kappa_{21}+1}L_1^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)-\frac{1}{\kappa_{11}+1}L_2^{(\kappa_{11})}(x)L_1^{(\kappa_{21})}(y)$ $+\frac{1}{(\kappa_{11}+1)(\kappa_{21}+1)}L_2^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)\Big)$; $B_3(x,y)=\frac{\sqrt{m_2^{(x)}m_4^{(y)}}}{2\kappa_{11}\kappa_{21}(\kappa_{21}+1)}\Big(L_1^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)-\frac{1}{\kappa_{11}+1}L_2^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)\Big)$; and $B_4(x,y)=\frac{\sqrt{m_4^{(x)}m_2^{(y)}}}{\kappa_{11}\kappa_{21}(\kappa_{11}+1)}\Big(L_2^{(\kappa_{11})}(x)L_1^{(\kappa_{21})}(y)-\frac{1}{\kappa_{21}}L_2^{(\kappa_{11})}(x)L_2^{(\kappa_{21})}(y)\Big)$.

Theorem 4. *The ML estimates of* $\left(\rho_{x^+y^+},\rho_{x^{+2}y^{+2}},\rho_{x^{+2}y^+},\rho_{x^+y^{+2}}\right)$ *solve the following system:*

$$\sum_{i=1}^{n}B_k(x_i,y_i)\Bigg\{\sum_{\substack{(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{G},\boldsymbol{H})\subset\{1,2,\cdots,i-1,i+1,\cdots,n\}\\ \boldsymbol{I}\bigcap\boldsymbol{J}\bigcap\boldsymbol{K}\bigcap\boldsymbol{G}\bigcap\boldsymbol{H}=\varnothing\\ |\boldsymbol{I}|+|\boldsymbol{J}|+|\boldsymbol{K}|+|\boldsymbol{G}|+|\boldsymbol{H}|=n-1}}\prod_{i\in\boldsymbol{I}}B_0^i(x,y)\prod_{\boldsymbol{J}\subset\{1,2,\cdots,i-1,i+1,\cdots,n\backslash\boldsymbol{I}\}}B_1^j(x,y)$$

$$(\rho_{x^2y^2})^{|\boldsymbol{J}|} \prod_{\boldsymbol{K} \subset \{1,2,\cdots,i-1,i+1,\cdots,n \backslash \boldsymbol{J}\}} B_2^j(x,y)\,(\rho_{xy})^{|\boldsymbol{K}|} \prod_{\boldsymbol{H} \subset \{1,2,\cdots,i-1,i+1,\cdots,n \backslash \boldsymbol{K}\}} B_3^j(x,y)$$

$$\left.(\rho_{xy^2})^{|\boldsymbol{H}|} \prod_{\boldsymbol{G} \subset \{1,2,\cdots,i-1,i+1,\cdots,n \backslash \boldsymbol{H}\}} B_4^j(x,y)\,(\rho_{x^2y})^{|\boldsymbol{G}|} \right\} = 0, \qquad (3.1)$$

where $k = 1, 2, 3, 4$ and $A \| B$ denotes the complement of B in A.

Proof. See Chu (2011a) for details. □

In reality, it is very complicated to solve Eq. (3.1), even numerically. Hence, we have to rely on a recursive algorithm such that, given a starting point, the system will converge to a fixed point which is a unique solution to Eq. (3.1); see Chu (2011a) for details.

Remark 3.1. *The conditions required for the existence of the ML estimates are rather strong so that the MLE may not be achieved in practice. In the example presented below, the MLE does not work because the pdf defined in Eq. (2.7) is underidentified.*

First, let's rewrite Eq. (2.7) as

$$f_{X^+Y^+}(x,y) \approx \mathrm{G}(x,\kappa_{11})\mathrm{G}(y,\kappa_{21}) \sum_{i,j=0}^{2} c_{ij} x^i y^j,$$

where $c_{00} = 1 + a_1\kappa_{11}\kappa_{21} - a_2\kappa_{11}\kappa_{21}(\kappa_{21}+1) - a_3\kappa_{11}\kappa_{21}(\kappa_{11}+1) + a_4\kappa_{11}\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)$; $c_{10} = \kappa_{21}(\kappa_{21}+1)a_2 + 2\kappa_{21}(\kappa_{11}+1)a_3 - \kappa_{21}a_1 - 2\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)a_4$; $c_{01} = \kappa_{11}(\kappa_{11}+1)a_3 + 2\kappa_{11}(\kappa_{21}+1)a_2 - \kappa_{11}a_1 - 2\kappa_{11}(\kappa_{11}+1)(\kappa_{21}+1)a_4$; $c_{20} = \kappa_{21}(\kappa_{21}+1)a_4 - \kappa_{21}a_3$; $c_{02} = \kappa_{11}(\kappa_{11}+1)a_4 - \kappa_{11}a_2$; and $c_{11} = a_1 - 2(\kappa_{21}+1)a_2 - 2(\kappa_{11}+1)a_3 + 4(\kappa_{11}+1)(\kappa_{21}+1)a_4$. Setting $\theta = \sqrt{n}$, where n denotes the sample size of the data, a Taylor approximation to the log-likelihood function yields

$$\begin{aligned}\mathcal{L}(\Theta) &= \sum_{i=1}^{n} \log Pdf(x_i^+, y_i^+|\Theta) \propto \sum_{k=1}^{n} \log \sum_{i,j=0}^{2} c_{ij} x_k^i y_k^j \propto c_{10}\bar{x} + c_{01}\bar{y} + \frac{1}{2}\Big(c_{20} - \frac{c_{10}^2}{c_{00}}\Big)\bar{x^2} \\ &+ \frac{1}{2}\Big(c_{02} - \frac{c_{01}^2}{c_{00}}\Big)\bar{y^2} + \Big(c_{11} - \frac{c_{10}c_{01}}{c_{00}}\Big)\bar{xy} + \mathcal{O}(n^{-3/2}), \qquad (3.2)\end{aligned}$$

where $\bar{X}$ denotes the sample average of X; and Θ represents the set of parameters to be estimated. In order to facilitate our analysis, we assume that the leading term of the log-likelihood function

(c_{00}) satisfies $c_{00} - 1 = \mathcal{O}(1)$ such that it has small impact on the value of the log-likelihood function, i.e we set $c_{00} = 1 + \epsilon$. Taking first-order derivatives of the approximated log-likelihood function in Eq. (3.2) yields

$$
\begin{aligned}
\frac{\mathcal{L}(\Theta)}{\rho_{ij}} &\propto \left(\overline{C}_0^{(ij)} - \frac{1}{4(\kappa_{11}+1)(\kappa_{21}+1)}\overline{C}_4^{(ij)}\right) + \left\{\frac{1}{\sqrt{\kappa_{11}\kappa_{21}}}\overline{C}_1^{(ij)} + \frac{1}{\sqrt{\kappa_{11}\kappa_{21}}(\kappa_{21}+1)}\overline{C}_2^{(ij)}\right. \\
&+ \left.\frac{1}{\sqrt{\kappa_{11}\kappa_{21}}(\kappa_{11}+1)}\overline{C}_3^{(ij)} - \frac{1}{\sqrt{\kappa_{11}\kappa_{21}}(\kappa_{11}+1)(\kappa_{21}+1)}\overline{C}_4^{(ij)}\right\}\rho_{xy} \\
&- \left\{\overline{C}_2^{(ij)}\frac{\sqrt{\kappa_{11}(3\kappa_{21}^2 - 6\kappa_{21})}}{2\kappa_{11}\kappa_{21}(\kappa_{21}+1)} - \overline{C}_4^{(ij)}\frac{\sqrt{\kappa_{11}(3\kappa_{21}^2 - 6\kappa_{21})}}{2\kappa_{11}\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)}\right\}\rho_{xy^2} \\
&- \left\{\overline{C}_3^{(ij)}\frac{\sqrt{\kappa_{21}(3\kappa_{11}^2 - 6\kappa_{11})}}{2\kappa_{11}\kappa_{21}(\kappa_{11}+1)} - \frac{\sqrt{\kappa_{21}(3\kappa_{11}^2 - 6\kappa_{11})}}{2\kappa_{11}\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)}\overline{C}_4^{(ij)}\right\}\rho_{x^2y} \\
&- \frac{\sqrt{(3\kappa_{11}^2 - 6\kappa_{11})(3\kappa_{21}^2 - 6\kappa_{21})}}{4\kappa_{11}\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)}\overline{C}_4^{(ij)}\rho_{x^2y^2} \\
&= \widehat{C}_0^{(ij)} - \widehat{C}_1^{(ij)}\rho_{xy} - \widehat{C}_2^{(ij)}\rho_{xy^2} - \widehat{C}_3^{(ij)}\rho_{x^2y} - \widehat{C}_4^{(ij)}\rho_{x^2y^2} = 0,
\end{aligned}
\tag{3.3}
$$

where $i, j = \{1, 2\}$; $\widehat{C}_\bullet^{(ij)}$ has an obvious meaning; and $\overline{C}_\bullet^{(ij)}$ are defined as $\overline{C}_0^{(ij)} = C_0^{(ij)}$, $\overline{C}_1^{(ij)} = \kappa_{21}C_1^{(ij)} + \kappa_{11}C_2^{(ij)}$, $\overline{C}_2^{(ij)} = (\kappa_{21}+1)(\kappa_{21}C_1^{(ij)} + 2\kappa_{11}C_2^{(ij)})$, $\overline{C}_3^{(ij)} = (\kappa_{11}+1)(2\kappa_{21}C_1^{(ij)} + \kappa_{11}C_2^{(ij)})$, and $\overline{C}_4^{(ij)} = 2(\kappa_{11}+1)(\kappa_{21}+1)(\kappa_{21}C_1^{(ij)} + \kappa_{11}C_2^{(ij)})$, with $C_0^{(ij)} = c_{10}^{(ij)}\bar{x} + \frac{1}{2}(c_{01}^{(ij)}\bar{y} + c_{20}^{(ij)}\bar{x^2}) + c_{02}^{(ij)}\bar{y^2} + c_{11}^{(ij)}\bar{xy}$, $C_1^{(ij)} = \frac{1}{\epsilon}(c_{10}^{(ij)}\bar{x^2} + c_{01}^{(ij)}\bar{xy})$, and $C_2^{(ij)} = \frac{1}{\epsilon}(c_{01}^{(ij)}\bar{y^2} + c_{10}^{(ij)}\bar{xy})$, where the mathematical expressions of $c_{10}^{(ij)}$, $c_{01}^{(ij)}$, $c_{20}^{(ij)}$, $c_{02}^{(ij)}$, and $c_{11}^{(ij)}$, for $i, j = \{1, 2\}$, are rather cumbersome, thus we shall resist the temptation to tax the readers' patience with a burden of notations. However, all the details are given in Chu (2011a).

By virtue of Eq. (3.3), one can show that $\{\rho_{xy}, \rho_{xy^2}, \rho_{x^2y}, \rho_{x^2y^2}\}$ solves the following system:

$$
\begin{bmatrix}
\widehat{C}_1^{(11)} & \widehat{C}_2^{(11)} & \widehat{C}_3^{(11)} & \widehat{C}_4^{(11)} \\
\widehat{C}_1^{(12)} & \widehat{C}_2^{(12)} & \widehat{C}_3^{(12)} & \widehat{C}_4^{(12)} \\
\widehat{C}_1^{(21)} & \widehat{C}_2^{(21)} & \widehat{C}_3^{(21)} & \widehat{C}_4^{(21)} \\
\widehat{C}_1^{(22)} & \widehat{C}_2^{(22)} & \widehat{C}_3^{(22)} & \widehat{C}_4^{(22)}
\end{bmatrix}
\begin{bmatrix}
\rho_{xy} \\ \rho_{xy^2} \\ \rho_{x^2y} \\ \rho_{x^2y^2}
\end{bmatrix}
=
\begin{bmatrix}
\widehat{C}_0^{(11)} \\ \widehat{C}_0^{(12)} \\ \widehat{C}_0^{(21)} \\ \widehat{C}_0^{(22)}
\end{bmatrix}
\tag{3.4}
$$

Since, as shown, $\overline{C}_4^{(ij)} = 2(\kappa_{11}+1)(\kappa_{21}+1)\overline{C}_1^{(ij)}$ and $\frac{\overline{C}_2^{(ij)}}{\kappa_{21}+1} + \frac{\overline{C}_3^{(ij)}}{\kappa_{11}+1} = 3\overline{C}_1^{(ij)}$, in view of Eq.

(3.3), the columns $\widehat{C}_1^{(ij)}$ and $\widehat{C}_4^{(ij)}$ of Eq. (3.4) satisfies $\widehat{C}_1^{(ij)} = \text{const.}\widehat{C}_4^{(ij)}$. It then follows that the determinant in Eq. (3.4) is zero. That is, there exists a redundant equation; and thus the approximated log-likelihood function, defined in Eq. (3.2), contains a unidentified parameter. This can be summarized in the following remark:

Remark 3.2. *Suppose that the logarithm of the second-order series approximation defined in Eq. (2.7) has a sufficiently small leading term and the second-order Taylor expansion of this logarithm is valid, then the ML estimates are not feasible.*

3.2 GMM Estimation

In this section, we present a GMM estimator for $\boldsymbol{\rho} = (\rho_{xy}, \rho_{x^2y}, \rho_{xy^2}, \rho_{x^2y^2})$. Note that, with a little generalization, this estimator can be applied to estimate the parameters of a higher-order series approximation of the joint pdf.

The intuition for this estimator is that the approximation given by Eq. (2.7) is fully characterized by complete orthogonal polynomials in the sense that each joint moment evaluated under this approximated pdf can be represented as a linear combination of the expectations of these orthogonal polynomials. This assertion is summarized in Lemma 2 in Chu and Satchell (2003). We shall restate it in the following lemma:

Lemma 3. *The pdf defined in Eq. (2.7) can be characterized via the following four moment conditions:*

$$\begin{aligned}
\frac{1}{\kappa_{11}\kappa_{21}} E_{f_{X^+Y^+}}[L_1^{(\kappa_{11})}(X^+)L_1^{(\kappa_{21})}(Y^+)] &= \boldsymbol{a}_1, \\
\frac{1}{2\kappa_{11}\kappa_{21}(\kappa_{21}+1)} E_{f_{X^+Y^+}}[L_1^{(\kappa_{11})}(X^+)L_2^{(\kappa_{21})}(Y^+)] &= \boldsymbol{a}_2, \\
\frac{1}{2\kappa_{11}\kappa_{21}(\kappa_{11}+1)} E_{f_{X^+Y^+}}[L_2^{(\kappa_{11})}(X^+)L_1^{(\kappa_{21})}(Y^+)] &= \boldsymbol{a}_3, \\
\frac{1}{4\kappa_{11}\kappa_{21}(\kappa_{11}+1)(\kappa_{21}+1)} E_{f_{X^+Y^+}}[L_2(X^+)L_2(Y^+)] &= \boldsymbol{a}_4. \qquad (3.5)
\end{aligned}$$

Eq. (3.5) implies that the expectation, taken under $f_{X^+Y^+}(x,y)$, of any function defined on $\mathbb{R}^{+2}$, say $g(x^+, y^+)$, in the Hilbert space $L_2(\mu)$ can be uniquely expressed as a linear combination *of* $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, *and* $\boldsymbol{a}_4$. *This is because* $g(x^+, y^+)$ *always adopts the following representation:* $g(x^+, y^+) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\omega_{ij}L_i^{(\kappa_{11})}(x^+)L_j^{(\kappa_{21})}(y^+)$, *where* $\omega_{ij} = E_{f_{X^+Y^+}}\left[L_i^{(\kappa_{11})}(X^+)L_j^{(\kappa_{21})}(Y^+)\right]$.

Proof. See Lemma 2 in Chu and Satchell (2003). □

Applying the Gram-Schmidth orthogonalization technique, one can derive four orthogonal conditions for the standard GMM procedure. First let's write Eq. (3.5) as follows:

$$
\begin{aligned}
E_{f_{X^+Y^+}}[\boldsymbol{h}_1(X, Y, \rho_{xy})] &= 0, \\
E_{f_{X^+Y^+}}[\boldsymbol{h}_2(X, Y, \rho_{xy}, \rho_{xy^2})] &= 0, \\
E_{f_{X^+Y^+}}[\boldsymbol{h}_3(X, Y, \rho_{xy}, \rho_{x^2y})] &= 0, \\
E_{f_{X^+Y^+}}[\boldsymbol{h}_4\left(X, Y, \rho_{xy}, \rho_{x^2y}, \rho_{x^2y^2}, \rho_{xy^2}\right)] &= 0,
\end{aligned}
$$

where the meanings of notations, $\boldsymbol{h}_i(x, y, \rho)$, are clear from the context. It then follows that

$$
\begin{aligned}
\boldsymbol{h}_1^o(x, y, \rho_{xy}) &= \boldsymbol{h}_1(x, y, \rho_{xy}), \\
\boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2}) &= \boldsymbol{h}_2(x, y, \rho_{xy}, \rho_{xy^2}) \\
&- \frac{<\boldsymbol{h}_2(x, y, \rho_{xy}, \rho_{xy^2}), \boldsymbol{h}_1(x, y, \rho_{xy})>}{<\boldsymbol{h}_1(x, y, \rho_{xy}), \boldsymbol{h}_1(x, y, \rho_{xy})>}\boldsymbol{h}_1^o(x, y, \rho_{xy}), \\
\boldsymbol{h}_3^o(x, y, \rho_{xy}, \rho_{x^2y}) &= \boldsymbol{h}_3(x, y, \rho_{xy}, \rho_{x^2y}) \\
&- \frac{<\boldsymbol{h}_3(x, y, \rho_{xy}, \rho_{x^2y}), \boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2})>}{<\boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2}), \boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2})>}\boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2}) \\
&- \frac{<\boldsymbol{h}_3(x, y, \rho_{xy}, \rho_{x^2y}), \boldsymbol{h}_1^o(x, y, \rho_{xy})>}{<\boldsymbol{h}_1^o(x, y, \rho_{xy}), \boldsymbol{h}_1^o(x, y, \rho_{xy})>}\boldsymbol{h}_1^o(x, y, \rho_{xy}), \\
\boldsymbol{h}_4^o(x, y, \rho_{xy}, \rho_{x^2y}, \rho_{x^2y^2}, \rho_{xy^2}) &= \boldsymbol{h}_4(x, y, \rho_{xy}, \rho_{x^2y}, \rho_{x^2y^2}, \rho_{xy^2}) \\
&- \frac{<\boldsymbol{h}_4(x, y, \rho_{xy}, \rho_{x^2y}, \rho_{x^2y^2}, \rho_{xy^2}), \boldsymbol{h}_3^o(x, y, \rho_{xy}, \rho_{x^2y})>}{<\boldsymbol{h}_3^o(x, y, \rho_{xy}, \rho_{x^2y}), \boldsymbol{h}_3^o(x, y, \rho_{xy}, \rho_{x^2y})>}\boldsymbol{h}_3^o(x, y, \rho_{xy}, \rho_{x^2y}) \\
&- \frac{<\boldsymbol{h}_4(x, y, \rho_{xy}, \rho_{x^2y}, \rho_{x^2y^2}, \rho_{xy^2}), \boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2})>}{<\boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2}), \boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2})>}\boldsymbol{h}_2^o(x, y, \rho_{xy}, \rho_{xy^2}) \\
&- \frac{<\boldsymbol{h}_4(x, y, \rho_{xy}, \rho_{x^2y}, \rho_{x^2y^2}, \rho_{xy^2}), \boldsymbol{h}_1^o(x, y, \rho_{xy})>}{<\boldsymbol{h}_1^o(x, y, \rho_{xy}), \boldsymbol{h}_1^o(x, y, \rho_{xy})>}\boldsymbol{h}_1^o(x, y, \rho_{xy}),
\end{aligned}
$$

where $< X, Y >$ denotes the inner product of X and Y in the Hilbert space. Hence, we obtain four orthogonal conditions:

$$\begin{aligned} E_{f_{X+Y+}}[\boldsymbol{h}_1^o(X,Y,\rho_{xy})] &= 0, \\ E_{f_{X+Y+}}[\boldsymbol{h}_2^o(X,Y,\rho_{xy},\rho_{xy^2})] &= 0, \\ E_{f_{X+Y+}}[\boldsymbol{h}_3^o(X,Y,\rho_{xy},\rho_{x^2y})] &= 0, \\ E_{f_{X+Y+}}[\boldsymbol{h}_4^o(X,Y,\rho_{xy},\rho_{x^2y},\rho_{x^2y^2},\rho_{xy^2})] &= 0. \end{aligned}$$

Let $\boldsymbol{H}^o(X,Y,\boldsymbol{\rho}) = \{\boldsymbol{h}_1^o(X,Y,\rho_{xy}), \boldsymbol{h}_2^o(X,Y,\rho_{xy},\rho_{xy^2}), \boldsymbol{h}_3^o(X,Y,\rho_{xy},\rho_{x^2y}), \boldsymbol{h}_4^o(X,Y,\rho_{xy},\rho_{x^2y},\rho_{x^2y^2},\rho_{xy^2})\}'$. The GMM estimates of $\boldsymbol{\rho} = \{\rho_{xy},\rho_{x^2y},\rho_{xy^2},\rho_{x^2y^2}\}'$ are given by

$$\widehat{\boldsymbol{\rho}} = \operatorname{argmin}_{\boldsymbol{\rho}\in[-1,1]^4} \underset{1\times 4}{\widehat{\boldsymbol{H}^o}(\boldsymbol{\rho})'} \underset{4\times 4}{\widehat{\boldsymbol{W}}^{-1}(\boldsymbol{\rho})} \underset{4\times 1}{\widehat{\boldsymbol{H}^o}(\boldsymbol{\rho})}, \tag{3.6}$$

where

$$\widehat{\boldsymbol{\rho}} \overset{d}{\approx} N\left(0, \underbrace{[T \underset{4\times 4}{\boldsymbol{D}(\widehat{\boldsymbol{\rho}})'} \underset{4\times 4}{\widehat{\boldsymbol{W}}^{-1}(\widehat{\boldsymbol{\rho}})} \underset{4\times 4}{\boldsymbol{D}(\widehat{\boldsymbol{\rho}})^T}]^{-1}}_{\text{Asymptotic Variance}}\right) \tag{3.7}$$

together with

$$\begin{aligned} \widehat{\boldsymbol{H}^o}(\boldsymbol{\rho}) &= \frac{1}{T}\sum_{i=1}^{T} \underset{4\times 1}{\boldsymbol{H}^o(X_i,Y_i,\boldsymbol{\rho})}, \\ \underset{4\times 4}{\widehat{\boldsymbol{W}}(\boldsymbol{\rho})} &= \frac{1}{T}\sum_{i=1}^{T} \underset{1\times 4}{\boldsymbol{H}^o(X_i,Y_i,\boldsymbol{\rho})} \underset{4\times 1}{\boldsymbol{H}^{o'}(X_i,Y_i,\boldsymbol{\rho})}, \\ \boldsymbol{D}(\boldsymbol{\rho}) &= \frac{\partial \widehat{\boldsymbol{H}}^o(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}. \end{aligned}$$

It is helpful to note at this point that many numerical optimization techniques such as the Nelder-Mead simplex algorithm can be used to find $\widehat{\boldsymbol{\rho}}$.

4 SIMULATION STUDY

In this section, we shall numerically examine the accuracy of the estimators proposed in Section 2. Since Monte Carlo simulation only requires random data simulated from a pdf, one can effectively compare the performance of different theoretical models.

4.1 GMM

First, we assign some numbers to the parameters of the density (2.7), then use the von Neumann acceptance/rejection technique to draw several batches of random data from this density. Next the GMM is applied to generate estimates from these batches of data. To measure the accuracy of the proposed GMM estimator, we compute the differences between the assigned parameters and the empirical means of the GMM estimates. The steps are as follows:

STEP I: Assign the following values to the density (2.7):

$$\kappa_{11} = 1.2,\ \kappa_{12} = 1.5,\ \kappa_{21} = 1.4,\ \kappa_{22} = 1.8;$$

$$p = q = 0.5\ (\textit{equal shifting positive-negative probabilities});$$

$$\{\rho_{11}^{++}, \rho_{12}^{++}, \rho_{21}^{++}, \rho_{22}^{++}\} = \{0.39, 0.09, -0.25, 0.52\};$$

$$\{\rho_{11}^{--}, \rho_{12}^{--}, \rho_{21}^{--}, \rho_{22}^{--}\} = \{0.69, 0.08, 0.62, 0.51\};$$

$$\{\rho_{11}^{+-}, \rho_{12}^{+-}, \rho_{21}^{+-}, \rho_{22}^{+-}\} = \{0.01, 0.11, 0.23, 0.29\};$$

$$\{\rho_{11}^{-+}, \rho_{12}^{-+}, \rho_{21}^{-+}, \rho_{22}^{-+}\} = \{-0.18, -0.29, 0.98, 0.42\}.$$

STEP II: Use the von Neumann algorithm to generate pseudo-random numbers in $(-1, 1)$ for the pdf defined in Eq. (2.7). Interested readers are referred to Chu (2011a) for the details of this algorithm.[3] (See, e.g., Niederreiter (1992).)

STEP III: Use the algorithm described in **STEP II** to draw 21 batches of 200 random data points and input them into the GMM in Section 3.2. This yields 21 vectors of the GMM estimates in Table 1. The absolute biases are reported in Table 3. These small biases confirm

[3] The SAS/IML code for generating these random numbers is available upon request.

that the GMM estimator for the density (2.7) is legitimately efficient. Moreover, this density can be identified by its parameters.

4.2 MLE

The GMM fits the sample moments of orthogonal restrictions to the true moments. Thus, it may still produce good estimates, even when the model is slightly misspecified. Meanwhile, the MLE fits the likelihood function of data to the true likelihood function; that is to say the data generation step is crucial in the MLE. We propose the following strategy:

1. As described in the example in Section 2.1, we draw three Gamma random variables to construct a bivariate Gamma variable as follows: $X = X_1 + X_2$ and $Y = X_1 + X_3$.

2. We then use the method in Section 2.1 to approximate the density of (X, Y) up to second order. Since the mgf of (X, Y) has a closed form, we can easily derive closed-form expressions for the nonlinear correlations.

3. We apply the MLE described in Section 3.1 to estimate these correlations.

True values of the parameters and their estimates are reported in Table 2. As usual the accuracy of statistical estimates is measured through biases, as reported in Table 3. Notice that the second quadratic correlations' estimates perform very poorly, possibly because this model is under-identified when the scale parameters of the Gamma densities become large, or because there is a redundant parameter (cf. Remark 3.2). For this reason the GMM performs better than the MLE.

5 CONCLUSION

This paper is related to the recent, rapidly expanding, literature on modeling joint probability distribution for equity returns in risk management and optimal asset allocation. We construct a series approximation, based on a sequence of Laguerre polynomials, of a joint pdf. The MLE and

GMM methods to estimate the correlation coefficients of the approximated pdf from empirical data are then proposed. We also provide a simulation study to compare the performance of these two methods of estimation. Overall, it was found that the GMM method outperforms the MLE method.

References

Barndorff-Nielsen, O. and Cox, D. R. (1979). Edgeworth and saddle-point approximations with statistical applications, *Journal of the Royal Statistical Society: Series B* **41**(3): 279–312.

Chu, B. (2011a). Approximation of asymmetric multivariate return distributions. Working paper.

Chu, B. (2011b). Recovering copulas from limited information and an application to asset allocation, *Journal of Banking & Finance* **35**: 1824 – 1842.

Chu, B., Knight, J. and Satchell, S. (2010). Optimal investment and asymmetric risk: a large deviations approach, *Optimization* **59**(1): 3 – 27.

Chu, B. and Satchell, S. (2003). Optimal portfolio choice under linex utility function: A large deviation approach. Working paper.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues, *Quantitative Finance* **1**: 223–236.

DeVore, R. A. and Lorentz, G. G. (1993). *Constructive Approximation*, Spinger-Verlag, Berlin, Heidelberg, New York.

Johnson, N. L. and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions*, John Wiley & Sons.

Kibble, W. F. (1941). A two-variate gamma type distribution, *Sankhyā* **5**: 137–150.

Knight, J., Satchell, S. and Tran, K. (1995). Statistical modeling of asymmetric risk in asset returns, *Applied Mathematical Finance* **2**(3): 155–172.

Lorentz, G. G. (1966). *Approximation of Functions*, Holt,Rinehart and Winston, New York, Chicago, London.

Malevergne, Y. and Sornette, D. (2006). Multi-moments method for portfolio management: generalized capital asset pricing model in homogeneous and heterogeneous markets, *in* B. Maillet and E. Jurczenko (eds), *Multi-moment Asset Allocation and Pricing Models*, John Wiley & Sons, pp. 165–193.

Martin, R. (2006). The saddlepoint method and portfolio optionalities, *Risk* **19**(12): 93–95.

Natanson, I. P. (1964). *Constructive Function Theory*, Vol. I, Frederick Ungar Publishing Co., New York.

Niederreiter, H. (1992). *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF regional conference series in applied mathematics, SIAM, Philadelphia.

Okimoto, T. (2008). New evidence of asymmetric dependence structures in international equity markets, *Journal of Financial and Quantitative Analysis* **43**(3): 787–816.

Parthasarathy, K. R. (1977). *Introduction to Probability and Measure*, Macmillan Publishers India.

Patton, A. J. (2004). On the out-of-sample importance of skewness and asymmetric dependence for asset allocation, *Journal of Financial Econometrics* **2**(1): 130–168.

Pötscher, B. M. and Prucha, I. R. (1997). *Dynamic Nonlinear Econometric Models*, Springer-Verlag, Berlin Heidelberg.

Rade, L. and Westergren, B. (1999). *Mathematics Handbook for Science and Engineering*, 4 edn, Springer-Verlag.

Sargan, J. D. (1976). Econometric estimators and the Edgeworth approximation, *Econometrica* **44**(3): 421–448.

Small, C. G. (2010). *Expansions and Asymptotics for Statistics*, CRC Press.

Appendix A: MAIN PROOFS

Proof of Theorem 1: Define the linear operator $U_{NM}g = \sum_{i=1}^{N}\sum_{j=1}^{M} A_{ij} L_{i,j}^{(\kappa_{11},\kappa_{21})}(x,y)$ of the *bounded* function g, where A_{ij} is given in Eq. (2.6), as a map from the functional space $\Lambda^2_{r\omega}\left(M_0,\ldots,M_{r+1};\mathbb{R}^{+2}\right)$, which is a subspace of the infinite-dimensional Hilbert space $L_2(\mu)$, into a Hilbert space, $L_2^{(NM)}(\mu)$, of dimension NM. The space $L_2^{(NM)}(\mu)$ uses the Laguerre polynomials as its basis. In view of Eq. (2.5), it is obvious that $\|f_{NM}(X,Y) - f(X,Y)\|_1 \propto \|U_{NM}g - g\|_1$. [Note at this point that, in what follows, all the quasi-norms are taken with respect to the pdf $f(x,y)$, unless otherwise indicated.]

An application of Lebesgue's lemma (see, e.g., DeVore and Lorentz (1993)) yields $|U_{NM}g - g| \leq \left(1 + \|U_{NM}\|_{L_2^{(NM)}(\mu)}\right) \inf_{P_{NM}\in L_2^{(NM)}(\mu)} |P_{NM} - g|$ *pointwise*, where $\|U_{NM}\|_{L_2^{(NM)}(\mu)}$ is the norm of U_{NM}, and P_{NM} represents an algebraic polynomial in $L_2^{(NM)}(\mu)$. It then follows that

$$\|U_{NM}g - g\|_1 \leq \left(1 + \|U_{NM}\|_{L_2^{(NM)}(\mu)}\right) \left\| \inf_{P_{NM}\in L_2^{(NM)}(\mu)} |P_{NM} - g| \right\|_1 . \tag{A-1}$$

Because

$$\begin{aligned}
\|U_{NM}g\|_{L_2(\mu)} &= \left(\int_{\mathbb{R}^{+2}} \left(\sum_{i=1}^{N}\sum_{j=1}^{M} \left(g, L_{i,j}^{(\kappa_{11},\kappa_{21})}\right)^{*} L_{i,j}^{(\kappa_{11},\kappa_{21})}(x,y) \right)^2 \right)^{1/2} \\
&\propto \left(\sum_{i=1}^{N}\sum_{j=1}^{M} \left(g, L_{i,j}^{(\kappa_{11},\kappa_{21})}\right)^{*2} \right)^{1/2} \\
&\leq \|g\|_{L_2(\mu)} \left(\sum_{i=1}^{N}\sum_{j=1}^{M} \left\| L_{i,j}^{(\kappa_{11},\kappa_{21})} \right\|^2_{L_2(\mu)} \right)^{1/2},
\end{aligned}$$

where the last inequality follows from Hölder's inequality, together with $\|g\|_{L_2(\mu)} < \infty$ and $\left\| L_{i,j}^{(\kappa_{11},\kappa_{21})} \right\|_{L_2(\mu)} < \infty$, we have

$$\|U_{NM}\|_{L_2^{(NM)}(\mu)} = O\left((NM)^{1/2}\right). \tag{A-2}$$

By using a truncation argument and Minkowski's inequality, one can derive

$$\begin{aligned}\|P_{NM}(X,Y)-g(X,Y)\|_1 &\le \left\|[P_{NM}(X,Y)-g(X,Y)]\mathbb{I}(X\le\tau_N\bigcap Y\le\tau_M)\right\|_1\\ &+ \left\|[P_{NM}(X,Y)-g(X,Y)]\mathbb{I}(X>\tau_N\bigcup Y>\tau_M)\right\|_1 = I+II,\end{aligned}$$

where τ_N and τ_M are the truncation parameters depending on N and M respectively. An application of Hölder's inequality, Tchebyshev's inequality, and an elementary inequality $|a_1+\cdots+a_m|^r \le m^{r-1}\left(|a_1|^r+\cdots+|a_m|^r\right)$ (see, e.g., Pötscher and Prucha (1997, p. 273)) yields, for some $p>1$ and $m>1$,

$$\begin{aligned}II &\le (NM)\max_{\substack{1\le i\le N\\1\le j\le M}}\left(E[|X|^{pi}|Y|^{pj}]\right)^{1/p}\left(\frac{E^{\frac{p-1}{p}}[|X|^m]}{\tau_N^{m\frac{p-1}{p}}}+\frac{E^{\frac{p-1}{p}}[|Y|^m]}{\tau_M^{m\frac{p-1}{p}}}\right)\\ &\le 2\times\text{Const.}\times\max_{\substack{1\le i\le N\\1\le j\le M}}\left(E[|X|^{pi}|Y|^{pj}]\right)^{1/p}\max\left(\frac{NM}{\tau_N^{m\frac{p-1}{p}}},\frac{NM}{\tau_M^{m\frac{p-1}{p}}}\right). \qquad \text{(A-3)}\end{aligned}$$

- If $N=M$, then, by setting $\tau_N=\tau_M=N^{1-a}$ with $0<a<1-\frac{3p}{m(p-1)}$ for some $m>\frac{3p}{p-1}$, we have that, in view of Eq. (A-3),

$$II \le \text{Const.}\times N^{2-m\frac{(p-1)(1-a)}{p}}. \qquad \text{(A-4)}$$

  Moreover, in view of Lemma 4, for some $r\ge 1$, imposing Hilbert norms on the variable quantities yields

$$I \le M_r^*\frac{1}{N^r}\left(\omega_x\left(\frac{N^{-a}}{2}\right)+\omega_y\left(\frac{N^{-a}}{2}\right)\right). \qquad \text{(A-5)}$$

  Hence, in view of Eqs. (A-1), (A-2), (A-4) and (A-5), one can immediately obtain

$$\|f_{NN}(X,Y)-f(X,Y)\|_1 = O\left(\max\left(N^{3-m\frac{(p-1)(1-a)}{p}},\frac{\omega_x\left(\frac{N^{-a}}{2}\right)}{N^{r-1}}+\frac{\omega_y\left(\frac{N^{-a}}{2}\right)}{N^{r-1}}\right)\right).$$

- If $N^\alpha<M<N^\beta$ for some $0<\alpha<\beta$, then, by setting $\tau_N=N^{1-a}$ and $\tau_M=N^{1-b}$

with $0 < a < \min\left(1, 1 - \frac{3}{2}\frac{p(\beta+1)}{m(p-1)}\right)$ and $0 < b < \min\left(1, 1 - \frac{3}{2}\frac{p(\alpha+1)}{\alpha m(p-1)}\right)$ for some $m > \frac{3}{2}\max\left(\frac{p(\beta+1)}{p-1}, \frac{p(\alpha+1)}{\alpha(p-1)}\right)$, we have that, in view of Eq. (A-3),

$$II \le \text{Const.} \times \max\left(N^{\beta+1-m\frac{(p-1)(1-a)}{p}}, M^{\frac{\alpha+1}{\alpha}-m\frac{(p-1)(1-b)}{p}}\right). \tag{A-6}$$

Moreover, an application of Lemma 4 together with a Hilbert norm argument yield, for some $r > \max\left(\frac{\beta+1}{2}, \frac{\alpha+1}{2\alpha}\right)$,

$$I \le M_r^*\left(\frac{1}{N^r}\omega_x\left(\frac{N^{-a}}{2}\right) + \frac{1}{M^r}\omega_y\left(\frac{M^{-a}}{2}\right)\right). \tag{A-7}$$

Hence, reminiscent of Eqs. (A-1), (A-2), (A-6) and (A-7), by noting that $(NM)^{1/2} < N^{\frac{\beta+1}{2}}$ and $(NM)^{1/2} < M^{\frac{\alpha+1}{2\alpha}}$, we obtain

$$\begin{aligned}\|f_{NM}(X,Y) - f(X,Y)\|_1 \quad = \quad & O\left(\max\left(N^{\frac{3(\beta+1)}{2}-\frac{m(p-1)(1-a)}{p}}, M^{\frac{3(\alpha+1)}{2\alpha}-\frac{m(p-1)(1-b)}{p}},\right.\right. \\ & \left.\left. N^{\frac{\beta+1}{2}-r}\omega_x\left(\frac{N^{-a}}{2}\right) + M^{\frac{\alpha+1}{\alpha}-r}\omega_y\left(\frac{M^{-b}}{2}\right)\right)\right). \ \blacksquare\end{aligned}$$

Appendix B: AUXILIARY RESULTS

Lemma 4. *Let $\Lambda^2_{r\omega}(M_0, \ldots, M_{r+1}; \mathfrak{B})$, where $\mathfrak{B} = [a_1, b_1] \times [a_2, b_2]$ with $a_1 < b_1$ and $a_2 < b_2$ is a parallelepiped, represent a space of smooth bivariate functions, $f(x,y)$, with the continuous partial derivatives of orders, $j = 0, \ldots, r$, that satisfy the following conditions: For each partial derivative of order j, $|D^j f| \le M_j$ for every $j = 0, \ldots, r$, and in addition, for each partial derivative of order r, $\omega_x(D^r_x f, h) \le M_{r+1}\omega_x(f,h)$ and $\omega_y\left(D^r_y f, h\right) \le M_{r+1}\omega_y(f,h)$, where $D^r_x f$ represents the r-th order partial derivative of f with respect to x; $\omega_x(f,h) \doteq \max_{\substack{x,y \\ |t|\le h}} |f(x+t,y) - f(x,y)|$ is the* modulus of continuity *of f with respect to x. Then there exists a constant, M^*_r, with the property that, for each $f \in \Lambda^2_{r\omega}(M_0, \ldots, M_{r+1}; \mathfrak{B})$, there are algebraic polynomials, $P_{NM}(x,y)$, of degrees*

N in x and M in y, for which

$$\max_{(x,y)\in\mathfrak{B}} \inf_{P_{NM}} |P_{NM}(x,y) - f(x,y)| \leq M_r^* \left(\frac{1}{N^r}\omega_x \left(f, \frac{|b_1 - a_1|}{2N} \right) + \frac{1}{M^r}\omega_y \left(f, \frac{|b_2 - a_2|}{2M} \right) \right),$$

where M_r^* depends only on r.

Proof. For each $(x, y) \in \mathfrak{B}$, there exists a pair of $(u, v) \in [-1, 1]^2$, such that $x = \frac{(b_1-a_1)u+(a_1+b_1)}{2}$ and $y = \frac{(b_2-a_2)v+(a_2+b_2)}{2}$. It can then be shown that $f(x, y) = f\left(\frac{(b_1-a_1)u+(a_1+b_1)}{2}, \frac{(b_2-a_2)v+(a_2+b_2)}{2}\right) = \phi(u, v) = \phi(\cos\theta_1, \cos\theta_2) = \psi(\theta_1, \theta_2)$, where $\psi(., .)$ is a periodic function with the period 2π, defined for all real values of (θ_1, θ_2). Since $f \in \Lambda^2_{r\omega}(M_0, \ldots, M_{r+1}; \mathfrak{B})$, it follows that the function $\phi(., .)$ has continuous partial derivatives of orders, $j = 0, \ldots, r$; in addition, Lemma 3 in Lorentz (1966, p. 89) implies that $\omega_{\theta_1}(D^r_{\theta_1}\psi, h) \leq C\omega_u(\phi, h)$, where C is a generic constant.

Then, by virtue of Theorem 6 in Lorentz (1966, p.87), there are trigonometric polynomials, $T_{NM}(\theta_1, \theta_2)$, for which

$$\begin{aligned}
\max_{(x,y)\in\mathfrak{B}} \inf_{P_{NM}} |P_{NM}(x,y) - f(x,y)| &= \max_{(\theta_1,\theta_2)\in\mathbb{R}^2} \inf_{T_{NM}} |T_{NM}(\theta_1,\theta_2) - \psi(\theta_1,\theta_2)| \\
&\leq M_r^* \left(\frac{1}{N^r}\omega_u\left(\phi, \frac{1}{N}\right) + \frac{1}{M^r}\omega_v\left(\phi, \frac{1}{M}\right) \right).
\end{aligned}$$

Using the same method employed to prove Lemma 2 in Natanson (1964, p. 121), one can immediately verify that $\omega_u(\phi, h_1) \leq \omega_x(f, |b_1 - a_1|h_1/2)$ and $\omega_v(\phi, h_2) \leq \omega_y(f, |b_2 - a_2|h_2/2)$. Setting $h_1 = 1/N$ and $h_2 = 1/M$, Lemma 4 then follows. □

Table 1: GMM estimates of the correlation coefficients for the approximated bivariate Gamma density

ρ_{11}^{++}	ρ_{12}^{++}	ρ_{21}^{++}	ρ_{22}^{++}	ρ_{11}^{--}	ρ_{12}^{--}	ρ_{21}^{--}	ρ_{22}^{--}
0.6730637	0.1223867	0.1432347	0.6828758	0.9299874	0.4307273	0.2969893	0.5128842
0.4557293	0.1012662	0.4326108	0.3545097	0.5355086	-0.289183	1.00005	0.8956267
0.2106303	0.2465397	-0.248404	0.0754728	0.9241941	0.1088326	1	0.4574445
0.5749553	0.0099689	0.1304068	0.6057853	0.9102113	0.0488261	0.204786	-0.019437
0.5070421	0.1203055	0.0994527	0.1577376	0.7447884	0.001082	0.1388486	-0.004572
0.4057905	0.0637831	0.1571887	0.3725408	0.7876742	-0.015461	0.8533368	-0.403638
0.4976929	0.1123169	-0.215775	-0.047582	0.8426215	0.1688845	1	-0.146799
0.6118393	-0.140833	-0.866268	-0.069186	0.925677	0.2489968	0.9116853	0.9591395
0.3729886	0.1282258	0.1539372	0.2048856	0.586707	-0.232206	0.1135751	0.6836731
0.3140399	0.0463992	0.0650365	0.4953924	0.8095857	0.187269	0.9826298	-0.283015
0.513071	0.1155333	-0.279779	-0.089335	0.8936176	0.2219065	1	0.9805002
0.336247	0.2716652	0.0030007	0.6139369	0.6304902	-0.204807	0.2173592	0.9686705
0.3891339	0.2172356	0.0169506	0.6797232	0.6571301	0.1110787	0.5585681	0.8202901
0.5210298	0.2041331	0.1048178	0.6458012	0.8507025	0.0974844	0.8425314	-0.44332
0.6381421	-0.100557	-1	0.6012079	0.6365218	-0.183036	0.8913917	0.6096091
0.7172848	0.1224623	-0.367307	0.5833059	0.7586488	-0.024203	0.4972418	-0.191014
0.4220508	0.1251892	0.0579074	0.1245374	0.6099436	-0.207999	0.3214134	0.9163124
0.301063	0.2448463	-0.120842	0.7911762	0.9219299	0.233542	1	0.5609462
0.6371962	0.0175613	0.1178328	0.6433291	0.8050357	0.1557702	1	0.3288951
0.4893729	0.089287	0.4147181	0.33436	0.9634782	0.2103238	0.6938591	0.8230263
0.6933347	-0.097265	-0.894399	-0.446117	0.6513061	-0.237694	0.5662478	0.6403664
0.4896[m]	0.0921[m]	-0.099794[m]	0.34830[m]	0.77978[m]	0.03953[m]	0.67097[m]	0.41264[m]
0.141267[v]	0.113091[v]	0.3978[v]	0.33219[v]	0.13435[v]	0.2000[v]	0.33535[v]	0.49386[v]
0.39[t]	0.09[t]	-0.25[t]	0.52[t]	0.69[t]	0.08[t]	0.62[t]	0.51[t]

[m] Sample means of estimates.
[v] Sample variance of estimates.
[t] True values of estimates.

Table 1 continued							
ρ_{11}^{+-}	ρ_{12}^{+-}	ρ_{21}^{+-}	ρ_{22}^{+-}	ρ_{11}^{-+}	ρ_{12}^{-+}	ρ_{21}^{-+}	ρ_{22}^{-+}
0.9260262	0.1118412	0.2286519	0.4411966	0.5681918	-0.326645	0.8875549	-0.536944
-0.33234	0.087687	0.573756	-0.284318	-0.342045	-0.173939	0.3972483	-0.326392
0.16582	0.1437655	0.3006312	0.2999764	-0.231809	-0.144689	0.3460006	-0.064928
-0.273685	0.0657121	0.3003377	-0.121501	-0.285064	-0.151136	0.3339118	-0.270312
-0.264066	0.1010835	0.0076608	0.9518736	-0.237804	-0.20005	0.8075336	0.2885684
0.0125447	0.2109293	-0.006478	0.3668681	-0.308642	-0.123754	0.7476014	0.4872635
-0.224699	0.0628517	0.3639834	-0.160763	-0.275373	-0.176222	0.1109797	-0.146932
0.9834569	0.9216398	0.6013394	0.875493	-0.231336	-0.159749	0.6316131	-0.047345
-0.323284	0.0667034	0.2511512	-0.106199	-0.288612	-0.191036	0.4418834	-0.36108
-0.273869	0.087516	0.4135002	-0.159064	-0.152913	-0.160601	0.8479214	0.4339158
-0.284789	0.0779757	0.3636866	-0.144237	-0.215926	-0.242189	0.8870396	0.2122554
-0.043522	0.2578679	0.1071299	0.2923216	-0.267437	-0.192861	0.8706073	-0.02047
0.4192471	0.4530386	0.0892572	0.608358	-0.198461	-0.307013	0.5734237	0.5539023
-0.287876	0.0653155	-0.123094	0.1220608	-0.281743	-0.151958	0.4225094	-0.329487
-0.077189	0.190744	0.2632309	0.1502213	-0.213604	-0.156652	0.7565706	0.0769401
-0.161489	0.0888689	0.1432991	0.0253181	-0.213396	-0.175284	0.9500512	0.4301483
0.3247014	0.4326993	0.2787204	0.586907	0.2473671	-0.451846	0.5457911	-0.462346
-0.284522	0.0702579	0.1142013	-0.020286	-0.272232	-0.150437	0.9290078	0.379338
0.4309398	0.368555	-0.047874	0.6903701	-0.193553	-0.163183	0.7898284	0.1985597
-0.135778	0.1668892	0.1062807	0.4560351	-0.214194	-0.130621	0.8229974	0.4733741
-0.290543	0.0705196	0.2908847	-0.106976	-0.269851	-0.138472	0.4760661	-0.349936
0.000241[m]	0.19535[m]	0.22001[m]	0.22684[m]	-0.184468[m]	-0.19373[m]	0.646482[m]	0.029433[m]
0.3991[v]	0.20629[v]	0.189659[v]	0.36903[v]	0.20807[vv]	0.079021[v]	0.24004[v]	0.354438[v]
0.01[t]	0.11[t]	0.23[t]	0.29[t]	-0.18[t]	-0.29[t]	0.098[t]	0.42[t]

[m] Sample means of estimates.

[v] Sample variance of estimates.

[t] True values of estimates.

Table 2: MLE estimates of the correlation coefficients for the approximated bivariate Gamma density.

ρ_{11}^{++}	ρ_{12}^{++}	ρ_{21}^{++}	ρ_{22}^{++}	ρ_{11}^{--}	ρ_{12}^{--}	ρ_{21}^{--}	ρ_{22}^{--}
0.4607398	0.0644466	0.4211025	0.2547295	0.0280979	-0.137311	0.2338121	0.1211056
0.2258627	-0.047231	-0.183393	0	0.5508075	-0.346507	-0.76702	1
0.4011386	0.1387048	0.1843259	0	0.4541232	-0.395179	-0.826771	1
0.7700997	-0.04508	0.6235942	0.0317679	0.2685211	-0.092907	-0.415615	0.4099073
0.4974558	0.6751502	0.4090423	1	0.2987722	-0.20439	-0.188046	0.0956322
0.5866687	0.5584372	0.6452364	1	0.6342863	-0.491351	-0.778177	0.8700885
0.2847864	0.0968794	-0.201032	1.04E-56	0.4405474	-0.42776	-0.593584	0.8637688
0.7614751	0.809886	0.164107	0.2840303	0.4197586	-0.164767	-0.524091	0.3413737
0.521727	0.486447	0.0748803	0.1107439	0.7608856	-0.227846	-0.84362	0.4177064
0.4600954	0.3450186	0.3975359	0.4816216	0.6457732	-0.750145	-0.61858	1
0.6352237	0.6864784	0.5896205	1	0.3947875	-0.382323	-0.188913	0.2632566
0.2758658	0.2141674	-0.159259	0.1356528	0.5060383	-0.560728	-0.46626	1
0.2933962	0.236212	0.1733243	0.4470176	0.4386419	-0.505187	-0.435968	1
0.5919078	0.1925363	0.3977034	0	0.5745414	-0.356836	-0.639683	0.4320255
0.4924768	-0.125783	0.6179296	0.2748857	0.775738	-0.841606	-0.277038	0.6037128
0.3242453	0.2043094	0.0850034	0.0219106	0.1979255	-0.064461	-0.650024	1
0.2308542	0.0950268	-0.09589	0	0.4838811	-0.382961	-0.513645	0.5566687
0.2240084	-0.078468	0.10972	0	0.2596469	-0.22028	-0.049659	0.1831497
0.5290685	0.7102645	0.3322862	1	0.5381966	-0.533565	-0.713204	1
0.1939863	-0.008139	0.0800304	0.3632461	0.18187	0.0693153	-0.490022	0.413735
0.2444463	0.0548246	0.0049966	0.0323727	0.2782986	-0.085535	-0.27166	0.0881147
0.5900649	0.6295063	0.4021926	0.6158496	0.859559	-0.827611	-0.933004	1
0.3145514	0.6389762	0.1781695	0.9397804	0.7938697	-0.740086	-0.848555	1
0.0861852	-0.069242	-0.129151	0	0.5291788	-0.154138	-0.077584	0.0104949
0.5776924	-0.268346	0.8360624	0	0.3942675	-0.450462	-0.326114	0.6573463
0.2493126	-0.161985	0.1742394	0	0.4714314	-0.641221	-0.424911	0.883609
0.0017841	-0.301315	-0.059536	0	0.2570237	-0.220843	-0.410113	0.6773432
0.2324235	-0.171697	0.0662391	0.2790612	0.7365611	-0.803375	-0.766552	1
0.8018351	0.298521	0.4244802	0.201589	0.774632	-0.227086	-0.705571	0.2737713
0.6062305	0.7129064	0.6027052	1	0.348279	-0.575109	-0.39672	1
0.3684036	0.2442753	0.2593266	0.3909626	0.721836	-0.726049	-0.053728	9.656E-57
-0.031634	-0.390804	-0.192107	0	0.4731626	-0.549979	-0.621648	1
0.6013609	0.5048365	0.3871922	0.5062076	0.3408837	-0.224044	-0.554313	0.6188665
0.2885378	0.8015108	-0.07025	1	0.4588213	-0.44763	-0.212971	0.2394289
0.2834996	-0.787581	0.605897	1.04E-56	0.0890135	-0.093685	-0.031943	0.7398167
0.6240983	0.3511746	0.5842334	0.540459	0.4297839	-0.338546	-0.63785	1
0.0453896	0.013318	-0.248168	1.04E-56	0.3205413	-0.172252	-0.383051	0.4896278
0.3719561	0.383658	0.3717725	0.6181806	0.7091912	-0.685409	-0.459438	0.5108382
0.5072256	0.90715	0.1869558	1	0.5852738	-0.482649	-0.558238	0.5418521
0.3794262	0.4822563	0.3820422	1	0.1994026	-0.476244	-0.254841	1
0.4804846	0.0475961	-0.162319	0	0.430004	-0.736523	-0.246409	1
0.5220239	0.5496672	0.4788608	0.8724914	0.482516	-0.384237	-0.53465	0.6258008
0.132228	-0.071231	-0.084503	0	0.4181495	-0.130449	-0.688394	1
0.396247^{m}	0.223424^{m}	0.224772^{m}	7.404711^{m}	0.464059^{m}	-0.399766^{m}	-0.468474^{m}	2.998350^{m}
0.20561^{v}	0.37324^{v}	0.28728^{v}	25.93828	0.19875^{v}	0.23926^{v}	0.26114^{v}	15.30443^{v}
0.45^{t}	0.2450328^{t}	0.2450328^{t}	0.4542857^{t}	0.45^{t}	-0.345033^{t}	-0.345033^{t}	0.4542857^{t}

Table 2 continued							
ρ_{11}^{+-}	ρ_{12}^{+-}	ρ_{21}^{+-}	ρ_{22}^{+-}	ρ_{11}^{-+}	ρ_{12}^{-+}	ρ_{21}^{-+}	ρ_{22}^{-+}
-0.705045	0.7431995	-0.645447	1	-0.525542	-0.668617	0.5517036	1
-0.962715	0.156566	-0.798224	0.1955249	-0.172754	-0.252232	-0.273112	0
-0.53409	0.3749207	-0.175832	0.0187413	-0.261446	0.2220223	0.2185962	0
-0.514111	0.4872625	-0.440945	0.7089263	-0.243725	0.0954043	0.0881891	0
-0.185442	-0.028694	0.0175836	0	-0.701462	-0.548441	0.6190129	0.802483
-0.352638	0.4703795	-0.426141	1	-0.292949	0.1231165	0.1263402	0
-0.617529	0.5780983	0.1189296	9.658E-57	-0.557031	-0.720734	0.2800103	1
-0.45318	0.4389698	-0.565774	1	-0.383168	-0.77762	-0.160637	0.5858643
-0.3047	-0.015738	-0.191372	0.0039936	-0.754166	-0.792134	0.5397566	1
-0.734239	0.6338595	-0.199435	0.20735	-0.407514	-0.13154	0.357302	0.4972105
-0.208367	-0.004104	-0.047844	0	-0.120781	-0.113828	-0.13699	0.1274696
-0.490966	-0.205873	-0.618766	0.1313894	-0.247292	-0.025608	-0.108973	0
-0.845481	0.3842103	-0.958174	1	-0.18483	0.038339	0.0185968	0
-0.631977	0.2228924	-0.394311	0.0925887	-0.349066	-0.234666	0.1331822	0.2371404
-0.147441	0.8681121	0.4254843	1	-0.577308	-0.313897	0.8576076	0.9050508
-0.428523	0.2098789	0.2063592	0	-0.444945	-0.604043	0.3963558	1
-0.195115	0.0796259	-0.064716	0.3391465	-0.217772	0.0573444	-0.006842	0
-0.349041	0.3255628	-0.535979	1	-0.860591	-0.977145	0.6372728	1
-0.258603	0.0158704	0.0629071	0	-0.547418	-0.440004	0.7017435	1
-0.251898	-0.385538	-0.080279	0	-0.640278	-0.338996	0.7486358	1
-0.523897	0.4658516	-0.419458	0.6592378	-0.56806	-0.599613	0.574367	1
-0.52036	0.5278463	-0.580778	1	-0.474938	-0.166263	0.3056072	0.1001245
-0.390423	-0.019779	-0.080346	0	-0.131752	0.096296	0.0203089	0.1068036
-0.462914	0.1777521	-0.158341	0	-0.791983	-0.490404	0.41492	0.1324247
-0.257404	0.059505	-0.035735	0.4878197	-0.047859	0.2154999	-0.015896	0.0710316
-0.254791	0.2172544	-0.254799	0.6082698	-0.495463	-0.324794	0.0342687	0
-0.155532	0.458528	0.1735506	1	-0.495175	-0.472272	0.2107987	0.5336349
-0.361664	0.235348	-0.104689	0.0413851	-0.750559	-0.863926	0.6242197	1
-0.36796	0.4330418	-0.459109	1	-0.41783	-0.437001	0.4827321	1
-0.616083	0.6446127	-0.668016	1	-0.215951	0.145282	-0.004191	0
-0.207841	0.0825237	-0.258387	0.4734065	-0.487068	-0.381247	0.48589	1
-0.089511	-0.105263	0.1667338	0.1356506	-0.086739	0.0290682	-0.083497	0.1011624
-0.393798	-0.031707	-0.296334	0	-0.446715	-0.621007	0.2144225	0.8405274
-0.208137	0.0406794	0.0875552	0	-0.57997	-0.556281	0.3991365	0.6028199
-0.589002	0.49648	0.1459097	0	-0.728803	-0.382795	0.3391212	0.2609004
-0.499208	-0.100913	-0.424072	0	-0.408569	-0.472739	0.5403495	1
-0.928418	0.7831349	-0.892352	1	-0.126388	0.1413302	0.0794324	1.039E-56
-0.434092	0.0159417	-0.447964	0.2881554	-0.046625	-0.32437	-0.293003	0.2374036
-0.602489	-0.1798	-0.703628	0	-0.603892	-0.546673	-0.344696	0
-0.252773	0.0338561	0.0011723	0	-0.524503	-0.491964	0.6072495	1
-0.127825	-0.024011	0.1644927	0	-0.806799	-0.549551	0.9490256	1
-0.192625	0.1116957	0.2233106	0	-0.288536	-0.286661	-0.277324	1.039E-56
-0.835541	0.4649461	-0.849201	0.7650062	-0.524678	-0.513344	0.0247115	1.039E-56
-0.428916[m]	0.235744[m]	-0.255406[m]	2.747828[m]	-0.431137[m]	-0.331551[m]	0.252923[m]	7.584699[m]
0.22478[v]	0.29339[v]	0.34645[v]	15.50192[v]	0.22297[v]	0.31708[v]	0.33396[v]	26.16612[v]
-0.45[t]	0.245033[t]	-0.2450328[t]	0.380082[illegible][t]	-0.45[t]	-0.2450328[t]	0.245033[t]	0.4542857[t]

[m] Sample means of estimates.

[v] Sample variances of estimates.

Table 3: Bias of empirical means of estimates from their true values ($|\rho - \widehat{\rho}|$)

Coefficients of correlations[1]	ρ_{11}^{++}	ρ_{12}^{++}	ρ_{21}^{++}	ρ_{22}^{++}	ρ_{11}^{--}	ρ_{12}^{--}	ρ_{21}^{--}	ρ_{22}^{--}
GMM	0.0996	6.21^{-3}	0.1502	0.1716	0.0897	0.0404	0.0509	0.0973
	0.1412	0.1130	0.3978	0.3321	0.1343	0.2001	0.3353	0.4938
MLE	0.0537	0.0216	0.0202	6.9504	0.014059	0.054733	0.1234	2.544
	0.20561	0.37324	0.28728	25.93828	0.19875	0.23926	0.26114	15.30443

Coefficients of correlations[1]	ρ_{11}^{+-}	ρ_{12}^{+-}	ρ_{21}^{+-}	ρ_{22}^{+-}	ρ_{11}^{-+}	ρ_{12}^{-+}	ρ_{21}^{-+}	ρ_{22}^{-+}
GMM	9.76^{-3}	0.0853	9.99^{-3}	0.0631	4.687^{-3}	0.0962	0.3335	0.39056
	0.3991	0.2062	0.1896	0.3690	0.2080	0.07902	0.2400	0.35443
MLE	0.02108	9.289^{-3}	0.01037	2.36677	0.018863	0.08651	7.89^{-3}	7.1304
	0.22478	0.29339	0.34645	15.34948	0.22297	0.31708	0.33396	25.90839

[1] Small numbers under the parenthesis are variances of the estimates.