Conceptual Knowledge of Counting: How Relevant is Order Irrelevance?

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Abstract

How important is the principle of order irrelevance in the development of children's conceptual knowledge of counting? In Study 1 (N=168), high-skilled kindergarten children were more likely to apply a principle of order *relevance* to another's counting than were less-skilled children. In Study 2 (N=239), only the oldest children (Grade 5) showed evidence of applying the principle of order irrelevance. The shift towards acquisition of this principle was unrelated to children's numeration skills after Grade 2. Thus, children's judgments about counts with unconventional orders appear to represent practical rather than logical considerations.

Keywords: early numeracy; counting; children; cognitive development; conceptual knowledge; procedural knowledge

Introduction

Counting is a fundamental mathematical skill. It is one of the earliest numerical procedures that children learn and forms the basis for the development of arithmetic skills, especially addition and subtraction. Procedural competence at counting is acquired over several years (approximately between the ages of three and seven). Conceptual understanding also improves across age, such that kindergarten children understand cardinality (i.e, that the final number word in a count represents the numerosity). Gelman and Gallistel (1978) described other fundamental counting principles: (a) one-to-one correspondence between number words and objects being counted, (b) stable order of those number words, and (c) order irrelevance. The last principle, that the order in which objects are counted is irrelevant, was of interest in the current research. We explored the question of *when* children consistently apply the order irrelevance principle to counting. We also attempted to determine whether knowledge of order irrelevance is related to other aspects of numerical knowledge.

Order irrelevance, in contrast to the three other principles, is not an essential feature of a correct count. Instead, it reflects a conventional rather than logical feature of counting (Briars & Siegler, 1982; Laupa & Becker, 2004). Nevertheless, researchers have assumed that conceptual understanding of counting includes awareness that the order in which items in a set are counted is irrelevant to determining numerosity.

Briars and Siegler (1982) developed a methodology for testing conceptual knowledge of counting. Children judged whether a puppet's counts were acceptable. Preschoolers detected more violations of the essential counting principles across age (i.e., ages 2 through 5). However, they were also less likely to accept counts that were correct, but performed in an unconventional order, suggesting that order was considered by at least some kindergarteners as a relevant feature of counting.

In LeFevre et al. (2006), we used the counting judgment task with older children (i.e., 5- to 8-year-olds). An animated frog counted objects on a computer screen and children were asked to judge whether or not each count was acceptable. Older children more often correctly rejected trials where the principles of one-to-one correspondence were violated than younger children, but most of these children also rejected correct trials where the order of counting was varied. In particular, counts on which the frog counted from the end of an array to the beginning, started in the middle of the array, or counted alternate items were deemed unacceptable by the majority of children. Even at the end of Grade 2, children rarely agreed that these correct-but-unconventional counts were acceptable even though in all cases the final stated value was correct and all the essential counting principles were followed (LeFevre et al., 2006). These results raised the question of whether the principle of order irrelevance becomes part of children's conceptual understanding at all. In the current research, we further explored the question of *when* children acquire the principle of order irrelevance. In the first study, we assessed children in Kindergarten, Grade 2, and Grade 4. In the second study, we assessed children in Grades 2 through 5.

Geary and colleagues (2004) found that in Grade 1, children with mathematical disabilities were less able to detect violations of one-to-one correspondence than children with more typical mathematical skill. These children also rejected correct counts that were ordered from right-to-left. Geary et al. proposed that these children with math disability were conceptually weaker in this domain than more typically developing children. Our results, however, suggested that although detecting errors might develop more slowly for less-able children, what is remarkable is how few typically-developing children with mathematical skills accepted the correct-but-unconventional counts (LeFevre et al. 2006). Thus, our second question in the present research was whether there is in fact an inverse relation, or none at all, between the principle of order irrelevance and other aspects of children's mathematical knowledge.

In the two studies described in this paper, we used the same computerized version of the counting judgment task developed by LeFevre et al. (2006). However, we increased the number of trials and systematically varied set size. All of the correct-but-unconventional counting trials used by LeFevre et al. were of arrays of 9 or more objects. Children may have become confused about the counting process on the unconventional trials and been unsure whether the counting was procedurally correct. In the present task, half of the trials were on small sets of 3, 4, or 5 items. Here, even if the children found the unconventional counts more confusing, the cardinal values of the sets were obvious. Thus, rejection of these correct-but-unconventional counts would not likely reflect confusion about whether the counting procedure produced the correct answer.

Method

The data described in this paper were collected during the third and fourth years of a four-year longitudinal project on children's early mathematical development. The almost 600 participants involved in the project, from pre-kindergarten through Grade 5, completed a range of measures each year. These tests included measures of literacy skills (e.g., vocabulary, phonemic awareness, and word reading), cognitive skills (e.g., fine motor ability, spatial skills, and processing speed), and many math-related skills (e.g., counting, digit recognition, addition, subtraction, multiplication and place value). The current study is focused on a subset of tasks completed as a part of the larger study.

Materials and Procedure

Counting principles. Children viewed an array of alternatively coloured (red and blue) squares, arranged by rows of four squares per row on a computer screen. An animated frog moved from one square to another and verbalized the count in a child's voice. The corresponding numbers appeared on the screen beside each object as it is counted and remained there until the child responded. Instructions were very similar to those used by Briars and Siegler (1984). The child was told by the experimenter that "Hoppy knows his numbers but sometimes when he counts the squares, he does it wrong. Hoppy is going to do some counting for you. I want you to watch him very carefully to see if he makes a mistake." After Hoppy completed the counting sequence, the experimenter read the prompt displayed at the top of the screen "Did Hoppy make a mistake, or no, he made no mistakes?" The responses available are 'Yes' and 'No'. We found in pilot testing that children occasionally were confused by the question "Did Hoppy make a mistake?" – they responded yes as if the question asked if Hoppy had counted correctly. To clarify, we added the phrase "or no, he made no mistakes" to emphasize that 'no' meant 'no mistakes'.

Children saw Hoppy demonstrate three kinds of counts (correct and conventional, incorrect, and unconventional-but-correct) across 16 trials. On half of the trials, the number of squares visible was 3, 4, or 5 (small counts) and on the other half the number of visible squares was 11, 12, or 13 (large counts). Children saw a total of four correct counts, six counts where Hoppy made an error, and six unconventional counts that were correct but violated the left-to-right, top-to-bottom convention in three different ways (LeFevre et al., 2006).

On two of the unconventional counts (one small and one large), Hoppy started counting in the middle of the array, counted to the end, then started again in the top-left corner and finished counting. On two other counts, Hoppy counted the red squares first in the conventional order, and then went back and counted the blue squares (top-to-bottom and left-to-right). On two other counts, Hoppy started counting with the last square (at the end of the bottom row) and counted backward (right-to-left and bottom-to-top). In all cases, the final cardinal value of the count was correct, all squares had been counted, and the numerical value of the final count was displayed correctly.

A pseudo-random order of the 16 trials was created, such that type (i.e., correct, incorrect, and unconventional-but-correct) and magnitude of count (small, large) was represented equally in each half of the problem set. A second order was created by reversing the order of the first list. Children were assigned randomly to one of the two orders. For both orders, the first trial was a correct count and the second was an incorrect count.

Numerical skills test. Children completed the Numeration subtest of the KeyMath Test - Revised (Connolly, 2000), Form A (Study 2) or Form B (Study 1). The Numeration test is individually administered and includes questions on quantity, order, and place value (appropriate to the instructional level of the children). Children are stopped after they make three consecutive errors. We used standardized (z-scores) to determine skill. All of the scores available for a given grade in that year were used in the z-score calculation, even if the child did not participate in the current study. Thus, the z-scores are representative of the children's performance in a particular grade for this sample. We did not use scaled scores because many children in this project performed at ceiling, based on the norms.

Study 1

Participants

The 168 children who completed the counting principles task in the third year of the project were included in the present study: 38, 66, and 64 in Kindergarten, Grade 2, and Grade 4, respectively. Mean ages were 5:10, 7:10, and 9:8 in years:months. Children were tested at their schools near the end of the school year between April and June, with the majority participating in May. All of the children in Grades 2 and 4 were participants in LeFevre et al. (2006), at which point they would have been in Kindergarten and Grade 2, respectively. The children in this project came from middle-class homes, for the most part. Virtually all of the parents had finished high school and tfhe majority had at least a technical college or undergraduate university degree. The majority of families were intact and at least one of the parents worked full time outside the home.

Results

Children's performance on the counting principles task is described first, followed by an examination of the relation between numeration skill and conceptual knowledge.

Counting Principles. Recall that there were three kinds of trials on the counting principles task: (a) correct and conventional counts, (b) correct-but-unconventional counts, and (c) incorrect counts. Children responded either yes or no to each count, where a *no* response indicated that they thought that count was acceptable (i.e., the animated frog had made no mistakes), and a *yes* response indicated that they thought the count was not acceptable (i.e., yes, the frog made a mistake). If children can differentiate between essential and inessential features of counting, then they should accept correct and unconventional-but-correct counts and reject the incorrect counts.

Children's responses on correct, conventional counts showed almost no variance. Of the 168 children, 162 identified all of the four conventional counts as acceptable. Five children rejected one of the four counts, and one child rejected two counts. Thus, children's responses on these conventional counts indicate that they recognize and accepted as valid a conventional sequence of left-to-right and top-to-bottom counting. Otherwise, these data do not provide much information and were not considered further.

Children's responses to the unconventional-but-correct and incorrect counts, together, presumably indicate whether they can discriminate between essential and inessential-but-conventional features of counting. There were six of each count, or 12 trials in total. Examination of the patterns of responses indicated that children were often quite consistent in their responses within a type of trial. For example, 119 children (71%) correctly rejected five or six of the incorrect counts. For the unconventional counts, children again were individually quite consistent, but fell into two distinct groups: approximately the same number of children either accepted five or six of the unconventional-but-correct counts (35%) or *rejected* five or six of these counts (40%).

Overall, four response patterns were identified. These are described below, and the percentage of children in each grade who fell into each category is shown in Figure 1.

We assumed that, to be classified as understanding, a child should reject the incorrect counts and accept the unconventional-but-correct counts, thus showing both knowledge of essential principles and discrimination between essential principles and inessential but typical features of counting. We categorized children as showing ***understanding*** if they scored 10, 11, or 12 on the combination of incorrect and unconventional counts ($p < .05$). Thus, these children correctly rejected 4 to 6 of the incorrect counts and correctly accepted 4 to 6 of the unconventional counts, such that their total score was 10 or greater.

Three other response patterns were identified. First, some children competently detected violations of the essential counting principles (i.e., they correctly rejected incorrect counts), but also applied inessential, but conventional rules (i.e., they also rejected unconventional counts). Children who rejected 10 or more trials were categorized as ***rule-bound*** ($p < .05$). Second, in contrast to the rule-bound children, who rejected most trials, a few children accepted most trials, even the incorrect counts. Children who accepted 10 or more trials were categorized as having a ***positive bias***. They may not have understood the task. Third, the remaining children were categorized as ***inconsistent***, in that they accepted some or all of the unconventional counts but failed to reject many of the incorrect counts, or vice-versa, they rejected some of the error counts but failed to accept many of the unconventional counts. Thus, all of the children in the inconsistent group scored between 5 and 9 on the 12 trials. Notably, there were no children who both accepted incorrect counts and rejected correct-but-unconventional counts, a response pattern that would have been difficult to characterize.

As shown in Figure 1, the pattern of children's responses varied substantially with grade, $\chi^2(6, N=168) = 24.58$, $p < .001$. In all three grades, children were most often classified as rule-bound: they consistently and correctly rejected error trials, but also consistently rejected correct-but-unconventional counts. In Grade 4, the percentage of children classified as understanding approached that of rule-bound children, however, suggesting that by age 10 (approximately), at least some children were distinguishing between essential and conventional rules.

Comparison of performance across small and large sets indicated that children were not sensitive to set size in their counting judgments. Thus, LeFevre et al.'s results could not be attributed to children's potential confusion related to set size. In general, what was more noticeable was the consistency with which children applied the rule that unconventional counting orders were unacceptable.

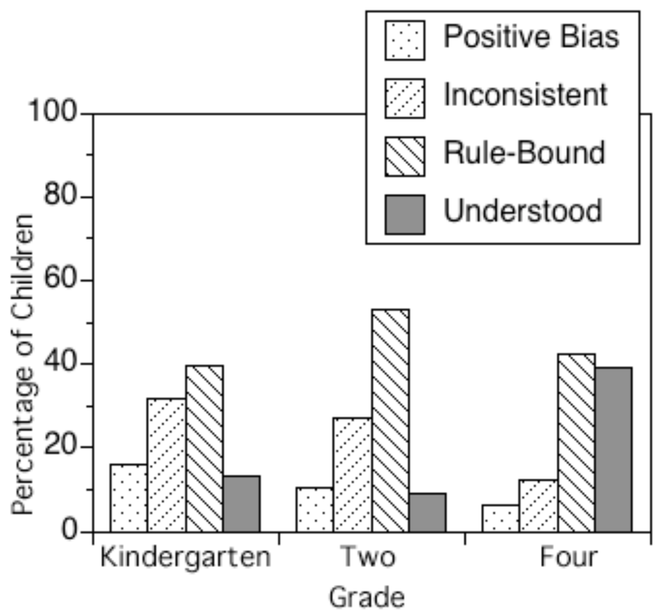

Figure 1. Percentage of children in each grade who responded to the counting principles trials according to each classification.

Numeration skill. To evaluate the relation between numeration skill and knowledge of the counting principles, children were categorized as low, average, or high in skill according to their performance (within grade) on the KeyMath numeration test. Low-skill children scored below the 25th percentile and high-skill children scored above the 75th percentile. We examined the relation between skill and the conceptual knowledge categories separately by grade. LeFevre et al. (2006) found that, in Kindergarten, more-skilled children were more likely to reject correct-but-unconventional counts than average- or less-skilled children. Similarly, among the kindergarten children in the present sample, more-skilled children were more likely to be classified as rule-bound than average or less-skilled children, $\chi^2(2, N=38)=6.54$, $p = .038$. Seventy percent of the high-skilled children were classified as rule-bound, as compared to 38% and 17% of the average and low-skill children, respectively. As shown in Figure 1, very few children in any skill group were classified as understanding. In general, most children appeared to be correctly rejecting some, but not all of the error counts and responding inconsistently on the correct-but-unusual counts. Thus, in Kindergarten, children who strictly adhere to conventional counting rules are also more likely to show evidence of other forms of numerical knowledge**.** Hence, order is relevant.

In Grade 2, LeFevre et al. (2006) found some evidence that high-skill children were more likely than children in the other two groups to accept the correct-but-unconventional counts. However, even among the high-skill children, on average, more than half of these trials were rejected. In the present study, there was scant evidence that children in Grade 2 had acquired the principle of order irrelevance. As shown in Figure 1, only 10% of the children in Grade 2 were classified as understanding. The high- and average-skill children were most often classified as rule-bound (50% and 61%, respectively). As a group, the low-skill children looked like kindergarteners, with 38% classified as rule-bound and most of the others responding inconsistently. Overall, there was no significant relation between skill and classification for children in Grade 2.

In general, these results replicate those of LeFevre et al. in that even more children in Grade 2 than in Kindergarten applied conventional rules to their evaluation of the counting procedure. In contrast to their findings, there was little evidence of a shift in conceptual understanding in relation to skill. Very few of the children could be classified as understanding. Note, however, that the classification scheme used in the present study was more stringent than that applied by LeFevre et al. and because there were more trials (12 vs. 7), classification is probably more reliable. In both studies, the majority of children from kindergarten to Grade 2 were unwilling to accept correct-but-unconventional counts and this adherence to an order relevance rule increased with age.

As is evident from Figure 1, by Grade 4, the tide had started to turn. Many more children (40%) were classified as understanding than in the earlier grades. Nevertheless, 42% were still classified as rule-bound. Critically, however, conceptual classification and numeration skill were completely unrelated in grade 4. Thus, although these older children were more likely to accept the unconventional counts, it was not related to their numeration skill.

Study 2

Participants

The 239 children who completed the counting principles task in the fourth year of the project were included in the present study: 112, 60, 35, and 32 in Grades 2 through 5, respectively. Mean ages (and standard deviations) were 7:10, 8:10, 9:10, and 10:9 in years:months. Children were tested at their schools near the end of the school year between April and June, with the majority participating in May. Many of the children in Grades 3 and 5 were participants in LeFevre et al. (2006; 52 and 22), at which point they would have been in Kindergarten and Grade 2, respectively. Thus, these children were task veterans, completing the counting principles task for the third time in this study. In contrast, the children in Grade 2 had never done the task; the children in grade 4 would have completed the task once, in Grade 1.

Results

As in Study 1, performance on the conventional and correct counts (top-to bottom and left-to-right) was excellent, averaging over 98% in each grade.

Children's counting judgments were categorized using the same criteria as described for Study 1. As shown in Figure 2, the percentage of children in each of the four categories varied with grade, $\chi^2(9, N=238) = 33.14$, $p < .001$. In Grades 2, 3, and 4, the majority of children were categorized as rule-bound. They correctly rejected incorrect counts in which essential counting principles are violated, but were also unwilling to accept correct-but-unconventional counts. Only a few children were categorized as inconsistent, having a positive bias, or understanding, and they were approximately equally distributed across grades.

By Grade 5, again, the tide had turned. The majority of children were categorized as understanding, although some still were rule-bound or inconsistent, such that about 40% of children presumably had not gained knowledge of order irrelevance. Thus, in this group of children, only the oldest showed consistent evidence for a complete understanding of the counting principles. The majority continued to assert that counting order was a relevant feature of good counts.

Cross-Year Comparisons, Most of the children in Grades 3 and 5 (i.e., 51 and 21, respectively) had also participated in Study 1 (i.e., the previous year, in Grades 2 and 4). For these children, we compared their conceptual knowledge categories across years. We asked, first, whether children's classification stayed stable, and second, whether there were similar patterns of change for the younger and older groups. In comparing across age groups, more older children showed stability (73%) than younger children (42%), $\chi^2(1, N=72) = 6.12$, $p = .013$. Among children who did not change classifications across years, 56% of the older children were classified as understanding in both years and 25% as rule-bound. In contrast, of the younger children who remained stable, most were classified as rule-bound (81%). Examination of the data for the children who changed categories does not show any clear pattern. More young children changed in each of the categories than older children, but the pattern of change was very similar over age, such that approximately equal numbers of children changed, for example, to understanding, to inconsistent, and to rule-bound proportionally across grades. These findings suggest that for these children (all of whom were performing this task for the third time), a shift from rule-bound performance to understanding was likely to occur in Grade 4 and persist in Grade 5. However, a substantial minority of children continued to reject correct-but-unconventional counts.

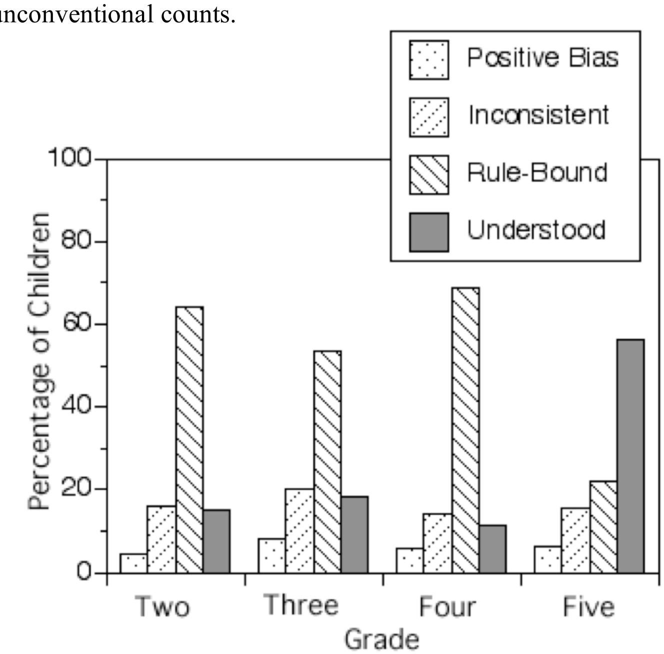

Figure 2. Percentage of children in each grade who responded to the counting principles trials according to each classification.

In contrast, the data for the children who were participating for the first time (i.e., those children who were first tested in the final year of the study), suggests that understanding was more likely to emerge in Grade 5. Of the new children in the study in Grade 5, 70% were classified as understanding. In contrast, only 10% of the children in Grade 4 (participating in this task for the second time, after a three year gap) were classified as understanding. In summary, although many more children are likely to show full conceptual understanding of all of the principles in Grade 5 than in the earlier grades, including order irrelevance, even by age 11 a minority continued to claim that correct-but-unconventional counts are not acceptable.

Numeration skill. As in Study 1, patterns of skill on the numeration test were analyzed in relation to conceptual knowledge. There were no significant relations between these variables in any grade. In Grade 2, there was a trend for more high-skill children to be classified as rule-bound (78%) than average (68%) and low-skill children (52%), $\chi^2(2) = 4.46$, $p = .108$, a pattern of performance that resembles that of the kindergarten children in Study 1. Clearly, however, children at all skill levels were most likely to consistently adhere to an order *relevance* rule. As grade increased, the distributions of skill and conceptual understanding become more and more uniform. Thus, although early application of conventional rules may be related to the development of numeration knowledge, the principle of order irrelevance does not seem to be crucial to continued development of other number skills.

Discussion

In this paper we explored the question of *when* children reliably use the principle of order irrelevance to judge the acceptability of another's counts. Our results suggest that, among children who have not experienced this task before, application of order irrelevance is common only in Grade 5 (that is, by age 11 or 12). About half of the children who had considerable prior experience with the judgment task started to consistently apply this principle in Grade 4. Nevertheless, even among these experienced children, application of the principle was not universal. About 25% of children in Grade 5 consistently rejected counts that violate a top-to-bottom, left-to-right counting sequence, but were correct in all other respects. Thus, these data indicate that consistent and reliable application of the order irrelevance principle is uncommon until children are about 11 years of age.

Our second question was whether acquisition of the principle of order irrelevance was related to the development of children's numeration knowledge more generally. The answer appears to be yes, but the relation was not in the predicted direction. For younger children (from kindergarten through about Grade 2) application of conventional counting rules in a rigid way (in our study, the so-called rule-bound children), was related to higher numeration skill. From Grade 3 onward, however, children's numeration knowledge was unrelated to whether they had acquired order irrelevance. Thus, for middle-class children, knowledge of order irrelevance does not appear to be a crucial factor in their conceptual development.

One implication of these findings is that for young children who must exert considerable effort to count accurately, application of an order *relevance* rule is associated with better numeration skill. Counting engages working memory (Rasmussen & Bisanz, 2005; LeFevre, DeStefano, Coleman, & Shanahan, 2005), especially for large sets that are diverse or not arranged in a straight line. Children who develop counting procedures that incorporate stricter guidelines may experience greater success. This counting success would actually strengthen the principle of order *relevance*. Only when counting becomes automatic might there be the opportunity for children to accept that it is not necessary to count a set in a strict order. Thus, for children who are still developing their counting skills and using them in the service of developing other skills (such as addition and subtraction), the principle of order irrelevance might be logical, but it is not practical. In the words of a nine-year-old child who was pilot testing this task: "Well, I *guess* it's OK to count that way", she said, as she raised an eyebrow and snorted, "but I wouldn't. And I don't think that Hoppy should either!"

Acknowledgements

This research was funded by the Social Sciences and Humanities Research Council of Canada through two grants to J. LeFevre, J. Bisanz, S. Skwarchuk, B. Smith-Chant, and D. Kamawar. We gratefully acknowledge the cooperation of the children, schools, parents, and others who made this work possible.

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