Just the facts: On the Representation and Solution of Basic Arithmetic Facts in the Adult Mind/Brain[1]

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Abstract

The current paper reviews models of arithmetic fact retrieval in adults, as well as evidence from empirical studies and neuropsychological findings.

The focus is on the form of mental representations used to access and store arithmetic facts. Additionally, the solution methods used by adults are reviewed, as are individual differences in proficiency and solution methods. Finally, criteria for future models of arithmetic fact retrieval are suggested with respect to mental representations and solution methods.

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Work in the area of numerical cognition is concerned with questions including: *How is arithmetic information represented in the mind?* (i.e., what are the symbols), *How do people solve arithmetic tasks?* (i.e., how are the symbols processed), *What are the factors affecting problem solution?*(e.g., culture, skill level, anxiety, problem format), and *What memory structures are involved?* (e.g., working memory, long-term memory). The purpose of the current paper is to review models of arithmetic fact solution in adults, primarily with a concern for the form of mental representations of facts but also for the solution methods used. The issue of representation is central and controversial within cognitive science more generally, as well as within the area of numerical cognition. The focus in the current paper is on basic arithmetic facts, the single-digit facts for addition and multiplication and the corresponding subtraction and division facts, because they are learned (to some extent) by all members of our society and serve as a basis for more advanced calculations.

The most robust finding in the study of arithmetic fact solution is the problem-size effect, the finding that calculation times and errors increase with the product of the operands (Ashcraft, 1992). Thus, larger problems (generally defined as problems where the product of the operands is greater than 25, e.g., 8 + 7) take longer to solve than smaller problems (generally defined as problems where the product of the operands is less than or equal to 25, e.g., 4 + 3). The introduction of the problem-size effect is made in the current paper because, as a cognitive phenomena, it can be assumed to arise either due to the representations and/or solutions. The explanation for the problem-size effect varies across models.

In the current paper an exploration of the mental representations used in the solution of basic arithmetic facts will be undertaken using evidence in the form of empirical studies, computational modeling, and neuropsychological findings. The framework for the investigation will be current models of numerical cognition. A number of models of numerical cognition have been proposed. The models differ in the representations they posit, the solution methods they assume (such as direct retrieval of a fact from memory or other procedural methods), and the tasks they model. Dehaene's Triple-code model (1992, 1997) is based on neuropsychological evidence, as is the modular model of McCloskey, Caramazza, and Basili (1985). Ashcraft's (1982) Network Retrieval model and Siegler and Shipley 's (1995) Adaptive Strategy Choice model both have a developmental influence, and Campbell's (1995) Network Interference model and Anderson's Brain-state-in-a-box model are based on adult empirical evidence. Most of the models have been implemented computationally.

<u>Models of Numerical Cognition</u>

Dehaene (1992, 1997) outlines the Triple Code Model of numerical cognition. Within the model, numerical information is represented in Arabic numerals, verbal (i.e., phonological), and analogue-magnitude codes. The form of mental representation used depends on the task being performed: simple calculation (i.e., single digit), complex calculation (i.e., multi-digit), quantification, or approximation. The analogue magnitude code is posited to take the form of a compressed number line (following Weber's Law) wherein larger numbers are less discriminable than smaller numbers. The analogue codes correspond to the semantic representation of number. Along with the three proposed code formats, Dehaene also proposes two separate calculation pathways, one for exact arithmetic (which uses Arabic or verbal codes) and one for approximate calculation (which uses analogue-magnitude codes).

Calculation of simple arithmetic facts within the Triple Code Model relies solely on verbal (phonological) representations and makes use of the exact arithmetic pathway. The arithmetic facts are stored as a “learned lexicon of verbal associations” (p.34, Dehaene,1992) that makes use of more general language areas not specialized for arithmetic. Dehaene's model predicts the “concurrent breakdown of language and calculation”.

Campbell (1995) outlines a computational simulation of the network-interference model. The scope of Campbell's model is smaller than that of Dehaene's, in that it only models the retrieval of simple addition and multiplication facts. Within the network interference model, numerical information is represented in both magnitude codes and physical codes (visual and verbal). The magnitude code is compressed using the same scaling method as proposed in Dehaene's model, such that larger magnitudes are less discriminable than smaller magnitudes. Visual codes again correspond to Arabic numerals. Thus, the actual codes used agree with Dehaene's. The difference, however, lies in the combined versus selective use of magnitude and physical codes for the solution of basic arithmetic facts. In Campbell's model the operand pair and operation are represented in physical codes, the approximate size of the answer is represented in a magnitude code. Each fact is stored as a combination of operand pairs, operation sign, and answer (e.g., 3 5 + 8). Arithmetic facts are stored in an interactive network wherein similarity of physical and magnitude codes leads to parallel activation of associated facts.

Within the Network Interference model, retrieval consists of a number of processing cycles during which facts receive excitatory input based on their similarity to the fact in question and inhibitory input based on the activation of other facts. Once a fact exceeds a threshold, its answer is given. As the magnitudes of larger problems are less discriminable, they activate more facts and thus result in more inhibition producing the problem-size effect. One of the largest criticisms leveled against the network interference model is that it is unfalsifiable.

Ashcraft (1997) outlines a computer simulation of the Network Retrieval model. The scope of Ashcraft's model is larger than that of Campbell's in that it models both production and verification of basic addition and multiplication facts over the course of arithmetic development to adulthood. Within the network retrieval model, facts are stored in an interconnected network. The network can be viewed as an arithmetic table with operands as row and column labels with the answer at the intersecting cell. Ashcraft's model also has dual-route calculation pathways, one for retrieval of facts and another for procedure use. The two pathways are activated in parallel and compete in horse race fashion.

In the Network Retrieval model, retrieval consists of spreading activation from the operands to the answer and is always faster than procedure use. The strength of activation for a given fact is a function of experience with the fact based on evidence of exposure through textbooks and other medium for the appropriate level of development. Given this definition, larger problems have lower activation rates based on less frequent exposure, leading to the problem-size effect in retrieval trials.

Procedure use only “wins the horse race” when no answer exceeds the threshold during retrieval. The only procedure built into the simulation was counting, but the use of other procedures was slated for future versions. The problem-size effect in procedure trials is a result of longer counting times for larger problems. Ashcraft, however, does not see much of a role for procedures in adult's solution of basic facts. Indeed, he states that procedure use is not expected to extend much beyond fourth grade. Ashcraft's model does not make specific the format or contents of arithmetic fact representations. Ashcraft does state, however, that he believes “the domain of arithmetic is, in principle, similar to other long-term memory knowledge, both in it's representational format and in the processes used to access this knowledge.” (1997, p. 302).

McCloskey, Caramazza, and Basili (1985) outline a modular model of cognitive arithmetic. The scope of the modular model encompasses calculation of basic facts, as well as multi-digit calculations, and the encoding and transcoding between multiple forms of numerical representation. Although the modular model posits roles for multiple representational forms including Arabic and verbal codes, “number-semantic representations constitute the principal numerical “language of thought,” and other internal representations serve principally to interface the semantic representations with external numerical formats”(p. 357, McCloskey & Macaruso, 1995). Number-semantic representations are magnitude representations akin to those posited by Dehaene and Campbell. McCloskey posits the content to be the quantities and associated power of ten for the numeral (e.g., 5030 as {5}10 EXP 3, {3}10 EXP 1).

Retrieval in the modular model consists of converting the problem to number-semantic representation which activates the stored fact in memory, allowing access to the number-semantic answer representation . The answer can then be converted to an external format for response. McCloskey states that it is unclear whether arithmetic performance uses specific processing areas but expects that at least the use of verbal codes for production and comprehension is expected to make use “general lexical processing mechanisms” (p. 152, 1992).

Seigler and Shipley (1995) outline a computational simulation of the adaptive strategy choice model (ASCM). The scope of ASCM is developmental, looking at the strategy choices children make in simple addition over the course of development from age 4 – 12. The model has also been used, however, by researchers interested in understanding adult performance, as ASCM provides a role for procedure use. ASCM is a further development of the distributions of associations (DOA) model of Siegler and Shrager (1984). In both the ASCM and DOA models, associations are formed between operands and answers based on experience. The associations may be peaked – associated strongly with one answer, or flat – weakly associated with multiple answers. One new feature of ASCM is the strategy database that stores information on all an individuals known strategies including the speed and accuracy (based on data and inference) for each strategy overall, for problems sharing a specific feature (e.g., 2's facts), and for specific problems (e.g., 2 + 3), along with an index of the strategy's novelty.

In ASCM, when a fact is presented for solution the data is used to predict how well each strategy will be at solving it. The choice of strategy to implement is made based on the predicted strength of each strategy relative to all other strategies. If the strategy chosen is a procedure it is implemented and completed and the arrived at answer is stated. If the strategy chosen is retrieval a retrieval attempt is made and evaluated based on a confidence criterion (i.e., how confident the individual is that the answer is correct) if the confidence criterion is not met then another retrieval attempt is made, assuming a set search length (i.e., the maximum number of search attempts permitted) has not been reached. Retrieval attempts continue until either the confidence criterion is met – and the answer is stated, or the search limit has been reached – at which time another strategy is chosen and completed. The problem-size effect arises in the ASCM model from two sources: weaker associations for larger problems and more lengthy procedures for larger problems. The form of the representations in ASCM is not explicit. It does, however, provide for non-retrieval strategy choices lacking in most other models of numerical cognition.

Anderson (1998) outlines a neural network model that makes use of the more general Brain-state-in-a-box model (BSB) of Anderson, Silverstein, Ritz, and Jones (1977). The BSB model is an auto-associative network (i.e., all nodes are connected to all other nodes, as in a Hopfield network), which gives the network a completion property such that it can regenerate missing parts of a pattern. The BSB model is psychologically based and is versatile; it has been used to model a variety of tasks. The focus in BSB models is on representations versus complex learning algorithms. The scope of the Anderson model was primarily multiplication fact retrieval, although number comparison, verification, and priming tasks were also completed by the model, demonstrating its versatility. Anderson (1998) posits a hybrid representation of number with an arbitrary symbolic component (i.e., number name) and an analog sensory component. The analog sensory component corresponds to a mental representation of magnitude akin to that seen in both the Dehaene and Campbell models. The hybrid representation allows movement between the magnitude and abstract components with the abstract component connecting up to other cognitive domains. Facts are represented as a concatenation of the three number representations - operands and answer.

The network was trained on facts (e.g., 2 x 4 = 8) and then given a fact to solve (e.g., 2 x 4 =).The network then generated the answer via its completion property. The problem-size effect arises from the magnitude component with answers to larger problems being less discriminable than answers to smaller problems. Errors in the network were similar to human errors in that they were close to the correct answer and associated, and in that non-product answers were rare. Anderson's findings led him to posit a human multiplication algorithm: the answer to a multiplication product is a product (i.e., familiar) and about the right size. Thus, he concluded that arithmetic fact retrieval is a process that combines memory with estimation.

Representations

So, given the models outlined in the previous section what can we conclude about the form of mental representations of arithmetic facts? First, are mental representations of arithmetic facts necessarily phonological? This question is one of the most controversial questions regarding representation of arithmetic facts. Dehaene posits that arithmetic facts are stored and accessed solely using phonological representations regardless of the external format the problem is presented in. Thus, a problem presented in Arabic format must first be translated to a phonological representation, which is then used to access the stored fact. The stored fact, also a phonological representation, can then be transformed to the appropriate external response code.

The position that arithmetic facts are represented phonologically is based on evidence including the finding that bilinguals solve arithmetic facts faster in the language they were acquired in and that multiplication impairment is often associated with language disorders (Butterworth, 1999; Dehaene, 1997; Whalen, McCloskey, Lindemann & Bouton, 2002). The evidence, however, is not entirely convincing in that differences in bilingual solution times may reflect comprehension and production differences across the two languages. Moreover, intact calculation is found in cases of severe aphasia and impaired calculation is found in cases with intact language abilities (Butterworth, 1999; Whalen, McCloskey, Lindemann & Bouton, 2002).

Whalen et al. (2002) further evaluated the position that facts are accessed and retrieved using phonological representations. Patients KSR and JM, who suffered cerebro-vascular accidents that left their ability to comprehend and produce spoken language severely impaired, performed experimental tasks designed to test the hypotheses. Both KSR and JM were able to produce and comprehend Arabic numerals and perform calculations on them. They were both severely impaired in producing and comprehending spoken numerals. To test whether facts are accessed using a phonological representation, simple arithmetic facts in Arabic numerals were presented (e.g., 2 + 6 =). The task was to say the problem aloud and then write the answer. KSR and JM were both impaired in their ability to correctly state the problem (12 % and 50 % accuracy respectively), however both were successful at writing the correct answer to the displayed problem (98 % and 91 % accuracy respectively). The results show that arithmetic facts were not accessed using the phonological representation. To test whether facts were retrieved in phonological representations, simple arithmetic facts were again presented in Arabic numerals, but the task now was to say the problem aloud, say the answer aloud, and finally to write the answer. The spoken and written answers were compared. KSR and JM were both impaired in their ability to correctly state the answer (35 % and 73 % accuracy respectively), however both were successful at writing the correct answer to the displayed problem (99 % for KSR, JM's value not given). The results show that arithmetic facts were not retrieved in phonological representations. Whalen et al. concluded that “arithmetic facts are neither stored nor retrieved exclusively in phonological form” (p. 516). If we conclude that arithmetic facts need not be represented phonologically other controversies remain.

Are arithmetic facts accessed and stored in a single representational code – whatever form it might be – or in multiple codes? This question constitutes another major controversy. Taking up the single code position are both McCloskey and Dehaene. McCloskey posits that the code is an abstract number-semantic code, in contrast to Dehaene's position of a phonological code. The single-code position requires that, regardless of the external format presented, the problem is transcoded into the internal representational code that is used to access and store facts. Taking up the multiple code position are both Campbell and Anderson. Campbell posits interacting verbal, visual (Arabic), and magnitude representations. Anderson's position seems less explicit than Campbell's. As movement occurs between the two components of his hybrid representation (symbolic and sensory), I have chosen to place it in the multiple code camp. The multiple code position “implies representational redundancy” (p. 293, Whetstone, 1998). Facts may be stored and accessed using different non-matching codes but solution times will be faster when the codes match (Whetstone, 1998).

Whetstone (1998) investigated the single versus multiple code positions, looking at transfer of training. The single code models predict complete transfer of training across Arabic, written, and spoken word formats. That is, with the exception of differences in encoding times across format types, response times and error rates should be the same regardless of whether the training format matches the testing format. The multiple code models predict partial transfer of training across formats. That is, response times, and possibly errors, should be greater when the training format does not match the testing format. Patient MC suffered selectively impaired multiplication fact retrieval following removal of a brain tumor. Three subsets of multiplication facts were created and MC relearned one subset of multiplication facts in each of the three different formats (Arabic numerals, written, and spoken words) to a level of 97 % accuracy. Each fact was learned in only one format. Testing of each fact was then done in each of the three different formats. Accuracy in testing was 97 % for both cases when training and testing conditions matched and did not match. The findings were consistent with the single code models and not inconsistent with multiple code models. Response times did show an effect of whether training and testing formats matched. Response times were indeed longer when the testing and training formats did not match. Thus, the response time data showed evidence for the multiple code models; “simple multiplication can be affected by the format in which problems are trained and tested” (p. 306, Whetstone, 1998).

If arithmetic facts are accessed and stored in multiple representational codes, then specifically which codes are they represented in? Campbell's (1995) model posits a role for magnitude codes along with physical codes (visual and verbal), and Anderson posits magnitude codes and an arbitrary symbolic component, which corresponds to number name but could be an Arabic numeral, verbal, or written word code. Which codes are internal representations fundamental to access and storage of facts and which are simply used as external codes? Empirical work is needed to determine the correct number and forms of internal representations of arithmetic facts. In Anderson's model, the magnitude component is responsible for the production of all major effects. Campbell, McCloskey, and Dehaene all posit a role for magnitude, though Dehaene does not view it as being used for fact retrieval, and all agree on how it should be represented. The representation of magnitude, as it occurs in all presented models, is biologically plausible (Anderson, 1998). Magnitude representations are activated when necessary for a task and even when not obviously so (McCloskey & Macaruso, 1995). Moreover, magnitude representations contain the semantic information about numbers and number facts and thus seem central to numerical cognition.

Both Butterworth (1999) and Dehaene (1997) posit an innate ability to make use of magnitude information that surfaces in newborn children and even in other animal species. This ability to discriminate between small numbers of objects (up to about four) is posited to provide the basis for more abstract representations of number and number facts. According to this evidence it seems reasonable that magnitude representations are central to our numerical abilities. Polk et al. (2001) provide evidence from a brain-damaged individual that the innate abilities proposed by

Butterworth and Dehaene are only one component of magnitude information and propose a dissociation between magnitude information and symbolic knowledge. As well, studies like that of Whetstone (1998) and studies with normal populations are outlining what additional representations are necessary. Both computational modeling and further experimentation should illuminate the representations used in fact storage and retrieval.

Another issue involves the organization of fact representations in memory. Are operations interconnected? Errors to facts are often made that are correct for other operations (e.g., 4 + 2 = 8) suggesting that they may be (Campbell, 1995). Neuropsychological evidence from brain-damage patients provides evidence for the opposite position, as single operations may be selectively impaired (Whetstone, 1998). Mauro and LeFevre (2003) argue that division facts are mediated by multiplication facts for some individuals, and similar arguments have been made for subtraction. This may also reflect the lack of representations for division and subtraction facts.

Solution methods

Do adults retrieve the answers to all simple facts from memory? Many of the models outlined either make this assumption, or restrict the model to only this method of solution. Ashcraft's (1997) model does has a poorly defined non-retrieval route that allows the answer to be counted, but this route is not implicated in adults' solutions. Siegler and Shipley's (1995) model has a large role for procedural solutions, again primarily in children. The ASCM model has been adapted as a model of adult solution by researchers who posit that adults use procedural strategies, in concert with retrieval, yet lack an agreeable model of adult performance.

Adults have been found to use a variety of methods in the solution of basic arithmetic facts (Campbell & Timm, 2000; Campbell & Xue, 2001; Geary, Frensch, & Wiley, 1993; Geary & Wiley, 1991; Hecht, 1999; LeFevre, Sadesky, & Bisanz, 1996; LeFevre, Bisanz, et al., 1996). Self-reports of solution methods have provided the bulk of evidence for use of procedural strategies, however, objective sources of converging evidence support the position (Penner-Wilger, 2003; Penner-Wilger, Leth-Steensen, & LeFevre, 2002). Adults report use of a variety of procedures including: counting based procedures (e.g., 3 + 4 = 3, 4, 5, 6, 7), use of derived facts, where another known fact is used to aid solution (e.g., 3 + 4 = [3 + 3 = 6] + 1 = 7), and repeated addition (e.g., 3 x 4 = 3 + 3 + 3 + 3 =12). Procedure use increases with problem size. Campbell and Xue (2001) found that procedure use rose from 12 % for small addition problems to 36 % for large problems, and these findings are consistent with others in the literature. Researchers have found it useful to classify individual participants into three distinct groups: frequent procedure users, occasional procedure users, and retrievers (LeFevre et al., 2003; Penner-Wilger, 2003). Separate patterns of performance can be seen across groups in terms of the problem-size effect and other phenomena. Neuropsychological research has determined a double-dissociation between retrieval of facts and use of procedural strategies based on studies of brain-damaged patients (Butterworth,1999; Dehaene, 1997). The incorrect assumption that adults retrieve the solution to all basic arithmetic facts is problematic for the Campbell (1995), McCloskey et al. (1985), and Anderson (1998) models, and the restricted variety and use of procedures is problematic for Ashcraft's (1997) model. Currently, no sufficient model of adults' solution of arithmetic facts exists. A sufficient model would need to both include explicit representational codes supported by empirical evidence and acknowledge and provide mechanisms for the variety and proportion of procedure use found in empirical studies of adult performance.

The problem-size effect results from both representations and solution methods. The magnitude representation plays a role via the decreasing discriminablilty of answers to larger problems. Solution methods also play a role as procedures including counting, repeated addition, and decomposition take longer to implement for large problems then small problems. LeFevre et al. (1996) found that the problem-size effect was reduced but not removed when procedure trials were omitted from the analysis. Penner-Wilger et al. (2003) also found a role for both magnitude representations and procedure use in the explanation of the problem-size effect.

Presentation format (Arabic numerals versus written words) has recently been found to affect the percentage of and choice of procedures used (Campbell & Timm, 2000). Campbell, Parker, and Doetzel (in press) examined the effects of presentation format on the solution of single-digit addition and multiplication facts. Participants were presented facts in either Arabic digits (e.g., 3 + 4 =) or written words (three + four =) and gave the response verbally. Participants then stated the solution method used from the following list: Transform, Count, Remember, and Other. Remember corresponds to retrieval and transform corresponds to the use of a derived fact. Campbell et al. found that participants used procedures more often for facts presented in word format than in Arabic digit format (45 % versus 29 % procedure use respectively). Moreover, this difference was due to an increase in counting rather than transformations. This is surprising because generally an increase in procedure use is fueled by an increase in transformations (derived facts). Campbell et al. explain this result by positing that the word format interrupts retrieval, both for the presented fact and for derived facts, thus participants are forced to use well known but less efficient counting procedures. The results of this research provide further support for a multiple code model and show that processing is also affected at the level of solution method.

Individual differences

Numerical cognition is an area “rife with individual differences”(Penner-Wilger, 2002). Large differences in proficiency are found even at the level of basic facts in adults (LeFevre, Sadesky & Bisanz, 1996; LeFevre, Bisanz et al., 1996). Within the described models, individual differences in proficiency are explained primarily in terms of representational strength. Within Ashcraft's Network Retrieval model (1997), representational strength influences whether retrieval is successful or the answer is calculated using a counting procedure. Campbell states that within his network interference model “adjustments, for example, in the efficiency of inhibition or in the relative contribution of physical and magnitude codes to retrieval represent a means for modeling individual differences”(p. 157, Campbell, 1995). Within the ASCM model (Siegler & Shipley, 1995) individual cognitive styles have been modeled based on empirical findings that participants can be divided into good students, not-so-good students, and perfectionists (Siegler, 1988). Good students have strong (peaked) association between facts and answers, they use retrieval and perform quickly and accurately. Not-so-good students have weak (flat) associations between facts and answers, they use a mix of retrieval and other procedures. Not-so-good students set a low confidence criterion, thus they are more likely to state an incorrect answer.

Perfectionists have strong (peaked) associations yet use a great deal of procedures. Their performance is fairly fast and very accurate. The difference between good students and perfectionists is the confidence criterion they set; the criterion is higher in perfectionist students leading them to use procedures more often. In the ASCM model these differences are modeled by controlling the distribution of associations between facts and answers and the confidence criterion.

In the described models, individual differences are explored at the level of strength of associations between facts and answers, and differences in solution strategies. It is important to note that though termed individual differences in the literature these are best viewed as group differences. But what about the possibility of differences, not just in the strength of the representations, but in the form of the representations themselves across individuals? Noel and Seron (1993) posit the preferred entry code hypothesis, which states that certain representations may be more suitable to certain tasks (in agreement with the Triple-code model) and, moreover, that individuals may prefer certain representations for idiosyncratic reasons. The possibility of individuals using different representations has not been taken very seriously in the numerical cognition literature. If correct, however, researchers would be required to analyze data at the individual rather than group level. The possibility of individual differences at the level of representational form is an interesting question for further investigation.

Conclusion

The issue of mental representation is controversial and hotly debated in numerical cognition, as well as in other areas of cognitive science. The current paper suggests that representations used to access and store arithmetic facts are not solely phonologically based. Instead, representations of multiple codes are used in fact solution including a core magnitude component along with some or all of the following codes: phonological, Arabic/visual, and written word codes.

Adults use a variety of solution strategies for solving basic arithmetic facts. Though controversial in the recent past, this finding is gaining acceptance within numerical cognition. Current models of numerical cognition are incomplete, in that no model has both explicit in it a group of representations (as described above), as well as mechanisms that allow for the pattern of strategy use found in adult performance. Future models will have to meet these criteria.

All models have mechanisms to account for individual differences seen in arithmetic proficiency and some have mechanisms for describing individual differences in solution methods. The preferred entry hypothesis suggests that at least some of these effects may be at the level of representational form. If correct, the hypothesis has huge implications for the area of numerical cognition.

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