Running head: INDIVIDUAL DIFFERENCES IN MATHEMATICS

Determining the Locus of Individual Differences in Mathematical Skill: A Tri-level Hypothesis Approach[1]

Marcie Penner-Wilger

Carleton University

Address correspondence to Marcie Penner-Wilger, Department of Psychology, Carleton University, Ottawa, Ontario, Canada K1S 5B6. E-mail: wilger@sympatico.ca.

---

[1] Carleton University Cognitive Science Technical Report 2002-09
URL http://www.carleton.ca/iis/TechReports

Abstract

In this paper, I address the pervasive phenomenon of individual differences in basic mathematical skill among adults. Within the framework outlined by Marr (1982), differences in mathematical skill must reflect differences at one or more of the following levels: computational theory (mapping), choice of representation and algorithm (solution procedure), or hardware implementation (brain). Research from the areas of neuroscience and cognitive psychology is evaluated with the goal of exploring which levels are indicative of individual differences in mathematical skill.

Determining the Locus of Individual Differences in Mathematical Skill:

A Tri-level Hypothesis Approach

"The greatest unsolved theorem in mathematics is why some people are better at it than others." (Eves, 1998).

Mathematics is an area rife with individual differences. The skill level of elementary school children varies by seven years on average within a single classroom (Cockcroft, 1982). However, the widespread individual differences in mathematics ability are not simply a developmental phenomenon. Educated adults have been found to show large differences in skill, even for single-digit facts (LeFevre, Sadesky & Bisanz, 1996; LeFevre, Bisanz et al., 1996; Geary, 1996).

In the current paper, I employ a novel approach to the phenomenon of individual differences in basic arithmetic skill among adults. Empirical results are explored in a framework developed by Marr (1982). Marr outlines three levels at which any information processing task must be understood: 1) the most abstract level of what a processing device does and why, 2) how the device solves the problem, and 3) the hardware implementation of the device. Marr used his framework to explore human vision, however many cognitive scientists have adopted the tri-level hypothesis in various areas of research. Dawson (1998) posits that adherence to this framework is a defining feature of cognitive science.

In applying the framework to basic arithmetic, what the devise does, at an abstract level, involves choosing a representation of the problem. Taking as our information problem the mapping of two operands to a single answer (e.g. 2 and 2 to 4), the representation could take many forms. Some researchers have posited a number-line representation of arithmetic facts, others have posited a number-chart representation, whereas still others have posited a network representation (Campbell, 1995). The second level involves choosing an algorithm to solve the problem. Educated adults have been shown to use a variety of solution algorithms to solve basic problems (LeFevre, Sadesky & Bisanz, 1996; LeFevre, Bisanz et al., 1996; Geary, 1996). The third level involves the hardware implementation, or human brain. Marr asserts that understanding any information processing task requires attention to each of these three levels of description.

Of the three, the level that has received the greatest amount of empirical research is individual differences in solution procedures, or algorithms. Experimental participants are asked to self report their solution procedures (i.e., asked "how did you solve the problem?"). Such self-reports have shown that adults use a variety of procedures, and that many adults use multiple procedures to solve single-digit problems (LeFevre, Sadesky & Bisanz, 1996; LeFevre, Bisanz et al., 1996). Although early results were controversial, researchers generally now accept that adults do not solely retrieve the answers to single-digit problems (Campbell & Timm, 2001). Some of the procedures commonly reported include:

Retrieval from memory- participants often report simply knowing the answer to a problem, having memorized it, or that it simply popped into their head.

Derived fact- participants frequently report using a known fact to solve another problem; for example, 5 x 6 = (5 x 5 = 25) + 5 = 30.

Repeated addition- for multiplication, participants report adding one operand the number of times of the second operand; for example, 3 x 4 = 3 + 3 + 3 + 3 = 12.

Counting based procedures- participants report counting to find the answer; for example, 5 + 4 = 5, 6, 7, 8, 9.

Rules- zero rules(e.g., anything times zero is zero, anything plus zero is itself, anything minus zero is itself), nines rules, etc.

Some of the reported algorithms are obviously more efficient than others. Retrieval and application of a rule are generally faster (REF). These algorithms are also more robust. Repeated addition, derived fact and counting strategies all require additional steps and memory, and thus are more time consuming and allow more opportunity for error. The solution procedures used are well documented, what is not clear is *why* certain algorithms are selected. High skilled individuals have been found to use a greater amount of retrieval, one of the most efficient, robust solution methods (REF), whereas low skilled performance is more likely to include alternative solution methods.

Marr (1982) states that the choice of algorithm is influenced both by the representation, and by the hardware. Algorithm choice is based on the efficiency and robustness, given the representation and hardware constraints. Strategy reports show that solution procedure choices are often neither efficient nor robust (e.g. repeated addition for large operands), especially for low skilled individuals. Siegler (1988) posits that in problem solution, a confidence criterion is chosen and then retrieval is attempted. If the retrieved answer exceeds the criterion, the answer is given. However, if the criterion is not met a backup procedure is applied. Thus, use of a non-retrieval solution procedure reflects either a high criterion, or lack of a problem-answer representation (i.e., the problem-answer representation is not stored in memory). As Siegler could be understood in holding the hardware constant since he was studying human cognition, and therefore the brain, these two approaches are not so different. However, as the purpose of this paper is to provide a thorough description of mathematical performance within Marr's framework, both representational and hardware influences on algorithm choice will be considered (see Figure 1).

At the most abstract level, basic arithmetic performance is the mapping of two operands to a single answer (e.g., 2 and 2 to 4). Both Marr and Siegler posit a role for representations in influencing the solution algorithm chosen. Siegler's (1988) model explains why a non-retrieval solution procedure would be chosen; the answer arrived at through retrieval does not exceed a set confidence criterion. Thus, low skilled individuals may use non-retrieval solution procedures as backup because they have not formed strong representations. Marr contends that the choice of representation impacts the choice of algorithm significantly. A chosen representation may make certain information explicit while cloaking other useful information (e.g., the relation between operations). Thus, the choice of representation effects how easy it is to perform a given operation. If low skilled individuals have formed representations lacking useful information, their ease of computation is compromised.

Mauro et al. (2001) have posited a hybrid-representation model of multiplication and division, but which could be equally expected to apply to addition and subtraction. According to Mauro et al., skilled individuals have an integrated representation of multiplication and division problems. A single representation consists of a problem family (e.g., 5, 7, 35) through which one can find the answer to multiple problems (e.g., 5 x 7, 7 x 5, 35 / 5, 35 / 7). In contrast, low skilled individuals are posited to lack division representations all together, having only multiplication representations through which the solution of division problems must be mediated (e.g., 35 / 7 = ?, ? x 7 = 35, 5 x 7 = 35, 35 / 7 = 5). Therefore, because high skilled individuals have formed superior representations, the use of efficient, robust solution is supported. However, low skilled individuals are forced to make do with inferior representations resulting in the use of less efficient algorithms, prone to error due to multiple steps.

The remaining level, that of hardware implementation, is garnering an increasing amount of investigation with the use of positron emission tomography (PET) and functional magnetic resonance imaging (fMRI) technology. Marr noted that the same algorithm may be implemented on different hardware and that some algorithms are better suited to different hardware implementations. Thus, the hardware implementation influences the choice of algorithm in that it creates a set of constraints on how an algorithm can be performed. Certain algorithms are better suited to different physical substrates (e.g., the brain's parallel processing vs. a digital computer's serial processing). The current paper discusses only human performance, and thus the hardware implementation is necessarily the brain. PET and fMRI scans have identified the areas involved in mental calculation (Gruber et al., 2001; Zago et al. 2001). Presenti et al. (2001) investigated the different patterns of activation for an expert and non-expert calculators using PET scans. The results showed that different brain regions were activated based on skill level with increased use of right prefrontal and medial temporal regions for the expert calculator. Menon et al. (2000), used fMRI to investigate differences in activation among perfect and imperfect performers. The results showed that perfect performers, those who achieved 100% accuracy, had less activation of the left angular gyrus.

The finding of less activation, and activation within different areas in skilled individuals is not specific to the mathematics domain. More generally, skilled participants have been found to show the same patterns on a variety of tasks (Jaušovec, 1998). However, the difficulty in understanding the neurological evidence within this framework is that the different patterns of activation for low and high skilled individuals do not necessarily imply that the structures themselves are different, but may instead reflect the use of different solution algorithms. Alternative neurological evidence is necessary to determine if the brain structures (hardware) are in fact different among high and low skilled individuals.

The current paper investigated individual differences in mathematical skill within Marr's tri-level framework. This framework set the stage for an investigation of the influences on algorithm choice, a well-documented phenomenon lacking satisfying explanation. An interdisciplinary literature review provided insight into two influences on solution procedure choice 1) differences in representations formed, and 2) possible neural differences. An interdisciplinary assault with attention to multiple levels of description would seem to be the best approach to fully understanding the phenomenon of individual differences in mathematical skill. More specifically, further research into the neurological substrates for low and high skilled individuals, and psychological research into the representations formed by low and high skilled individuals (in the vein of Mauro et al., 2001) may ultimately solve the "greatest unsolved theorem in mathematics"(Eves, 1998) and lead to methods of improving mathematics performance.

References

Campbell, J.I.D. (1995). Mechanisms of simple addition and multiplication: A modified network-interference theory and simulation. *Mathematical Cognition, 1,* 121-164.

Campbell, J.I.D., & Timm, J.C. (2000). Adults' strategy choices for simple addition: Effects of retrieval interference. *Psychonomic Bulletin and Review, 7,* 692-699.

Cockcroft, W. H. (1982). *Mathematics counts*. London: HMSO.

Dawson, M. R. W. (1998). *Understanding cognitive science*. Malden: Blackwell.

Eves, H. (1988). *Return to mathematical circles*. Boston: Prindle, Weber & Schmidt.

Geary, D.C. (1996). The problem-size effect in mental addition: Developmental and cross-national trends. *Mathematical Cognition, 2,* 63-93.

Gruber, O., Indefrey, P., Steinmetz, H., & Kleinschmidt, A. (2001). Dissociating neural correlates of cognitive components in mental calculation. *Cerebral Cortex, 11,* 350-359.

Jaušovec, N. (1998). Are gifted individuals less chaotic thinkers? *Personality and Individual Differences, 25,* 253-267.

LeFevre, J., Bisanz, J., Daley, K.E., Buffone, L., Greenham, S.L., & Sadesky, G.S. (1996). Multiple routes to solution of single-digit multiplication problems. *Journal of Experimental Psychology: General, 125,* 284-306.

LeFevre, J., Sadesky, G.S., & Bisanz, J. (1996). Selection of procedures in mental addition: reassessing the problem size effect in adults. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 22,* 216-230.

Marr, D. (1982). *Vision*. San Francisco: W.H. Freeman & Company.

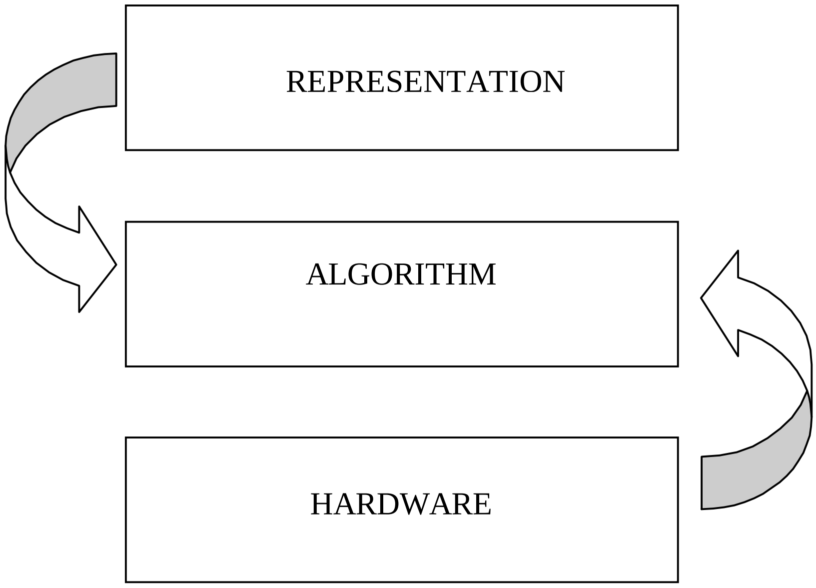
Menon, V., Rivera, S.M., White, C.D., Eliez, S., Glover, G.H., & Reiss, A.L. (2000) Functional optimization of arithmetic processing in perfect performers. *Cognitive Brain Research, 9*, 343-345.

Mauro, D. G., LeFevre, J., & Morris J. (2001). *Effects of problem format on division and multiplication performance: Evidence for a hybrid-representational model.* Manuscript submitted for publication.

Presenti, M., Zago, L., Crivello, F., Mellet, E., Samson, D., Duroux, B., Seron, X., Maxoyer, B., & Tzourio-Mazoyer, N. (2001). Mental calculation in a prodigy is sustained by right prefrontal and medial temporal areas. *Nature Neuroscience, 4,* 103-107.

Zago, L., Pesenti, M., Mellet, E., Crivello, F., Mazoyer, B., & Tzourio-Mazoyer, N. (2001). Neural correlates of simple and complex mental calculation. *NeuroImage*, *13*, 314-327.

Figure 1. Marr's (1982) Levels of description with direction of influence shown (arrows).



Representation level influences algorithm choice by making certain information explicit, making some computations easier.

Hardware level influences algorithm choice as hardware constraints make certain algorithms more efficient.