More Ambiguity Aversion or More Risk Aversion?

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Abstract

Using a pair of preferences $\succeq^{\min}$ and $\succeq^{\max}$ with utility functions $\min_{s\in S} f(s)$ and $\max_{s\in S} f(s)$ over the Ellsberg three color acts, this paper argues that the existing notions of attitudes toward ambiguity either mix ambiguity and risk or still involve attitudes toward risk even though ambiguity is exogenously given. Based on those observations, a new approach to defining attitudes toward ambiguity is suggested. Moreover, some relevant notions such as certainty equivalent for ambiguity and ambiguity premium are also proposed.

Keywords: Ellsberg Paradox, Ambiguity, Ambiguity Aversion, Risk, Risk Aversion, Unambiguous Event, Unambiguous Act, Probabilistic Sophistication, Convex Capacity, Choquet Expected Utility, Multiple-Priors Model, Expected Utility.

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1 Introduction

Consider two preferences $\succeq^{\min}$ and $\succeq^{\max}$ with utility functions $\min_{s\in S} f(s)$ and $\max_{s\in S} f(s)$ over the Ellsberg three color acts f. An Ellsberg three color act f is defined as the following: The Ellsberg urn contains 90 balls, 30 red ones and the rest to be either black or yellow with unknown proportions. One ball is to be drawn at random from the urn and the payoffs of f would depend on the color of the ball. For example, the payoff would be x_R if the ball is red. Thus, a typical Ellsberg three color act has the following form

$$f = \begin{pmatrix} x_R & \text{if } s = R \\ x_B & \text{if } s = B \\ x_Y & \text{if } s = Y \end{pmatrix}.$$

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The *key point* of the Ellsberg three colour set-up is that there are two kinds of events: unambiguous and ambiguous. Intuitively, an event is unambiguous if its probability is precisely known. Based on the available objective information, the set of unambiguous events is

$$\mathcal{A}^{ua} = \{\{R\}, \{B, Y\}, \varnothing, S\} \tag{1}$$

with unambiguous probabilities

$$P^{ua}(\varnothing) = 0,\ P^{ua}(\{R\}) = 1/3,\ P^{ua}(\{B, Y\}) = 2/3, \text{ and } P^{ua}(S) = 1 \tag{2}$$

where $S = \{R, B, Y\}$. As a result, any event outside $\mathcal{A}^{ua}$ is ambiguous. For example, $\{B\}$ is ambiguous since the chance of a ball to be black could be any number between 0 and 2/3. Similarly, an act f is unambiguous if it is measurable with respect to $\mathcal{A}^{ua}$,[1] otherwise it is ambiguous.

Based on the behavior exhibited in the Ellsberg paradox

$$f = \begin{pmatrix} \$100 & \text{if } s \in R \\ \$0 & \text{if } s \in B \\ \$0 & \text{if } s \in Y \end{pmatrix} \succ \begin{pmatrix} \$0 & \text{if } s \in R \\ \$100 & \text{if } s \in B \\ \$0 & \text{if } s \in Y \end{pmatrix} = g \text{ and}$$

$$f' = \begin{pmatrix} \$100 & \text{if } s \in R \\ \$0 & \text{if } s \in B \\ \$100 & \text{if } s \in Y \end{pmatrix} \prec \begin{pmatrix} \$0 & \text{if } s \in R \\ \$100 & \text{if } s \in B \\ \$100 & \text{if } s \in Y \end{pmatrix} = g',$$

we can see that a typical agent would treat unambiguous event $\{R\}$ ($\{B, Y\}$) and ambiguous event $\{B\}$ ($\{R, Y\}$) much differently.

To see the attitudes toward ambiguity of preferences $\succeq^{\min}$ and $\succeq^{\max}$, express them as the following Multiple-prior Expected Utility (MEU) and Choquet Expected Utility (CEU) forms, respectively

$$\min_{s \in S} f(s) = \min_{\delta_s \in C} \int f d\delta_s \text{ (Multiple-prior Expected Utility)} \tag{3}$$

$$= \int f dv_{CO(C)} \text{ (Choquet Expected Utility)} \tag{4}$$

and

$$\max_{\delta_s \in C} f(s) = \max_{\delta_s \in C} \int f d\delta_s \text{ (Multiple-prior Expected Utility)} \tag{5}$$

$$= \int f d\bar{v}_{CO(C)} \text{ (Choquet Expected Utility)} \tag{6}$$

where

- δ_s is the Dirac distribution which assigns all the probability mass to state s only;

[1] That is, $f^{-1}(x) = \{s \in S : f(s) = x\} \in \mathcal{A}^{ua}$ for any outcome x.

- $CO(C)$ is the convex hull generated from set $C = \{\delta_s : s \in \{R, B, Y\}\}$;
- $\upsilon_{CO(C)}$ is the capacity satisfying

$$\upsilon_{CO(C)}(A) = \min\{P(A) : P \in CO(C)\}, \; A \in 2^S;$$

- $\bar{\upsilon}_{CO(C)}\left(\bar{\upsilon}_{CO(C)}(A) = \upsilon_{CO(C)}(S) - \upsilon_{CO(C)}(A^c)\right)$ is the conjugate of $\upsilon_{CO(C)}$; and
- $\int f d\upsilon_{CO(C)}$ and $\int f d\bar{\upsilon}_{CO(C)}$ are the Choquet integrals with respect to capacities, $\upsilon_{CO(C)}$ and $\bar{\upsilon}_{CO(C)}$, respectively.

For ambiguous act

$$f = \begin{pmatrix} 100 & \text{if } s = R \\ 0 & \text{if } s = B \\ 1 & \text{if } s = Y \end{pmatrix},$$

$$\begin{aligned} \min_{s \in S} f(s) &= \min_{\delta_s \in C} \int f d\delta_s = \int f \delta_B = 100 \times 0 + 0 \times 1 + 1 \times 0 = 0 \text{ and} \\ \max_{s \in S} f(s) &= \max_{\delta_s \in C} \int f d\delta_s = \int f \delta_R = 100 \times 1 + 0 \times 0 + 1 \times 0 = 100. \end{aligned}$$

Though event $\{B\}$ is ambiguous, agents $\succeq^{\min}$ and $\succeq^{\max}$ put weights 100% and 0% on it, respectively. That is, $\succeq^{\min}$ is much more pessimistic than $\succeq^{\max}$ at ambiguous event $\{B\}$. Those observations seem to imply that agent $\succeq^{\min}$ is much more ambiguity averse than agent $\succeq^{\max}$ if they were not ambiguity neutral.

Some notions of attitudes toward ambiguity have been proposed since Ellsberg's seminal 1961 paper. The three typical representatives are the Schmeidler (1989), the Gilboa and Schmeidler (1989), the Epstein (1999). Though the intention of those notions is to focus on attitudes toward ambiguity exclusively, however, it seems to me no one to be satisfactory in the following sense:

> *Risk is risk, ambiguity is ambiguity, they are different and should not be mixed together.*

In other words, those notions still either confound the concepts of ambiguity and risk or involve attitudes toward risk even though ambiguity is exogenously given. Those observations may shed some light on a new approach to defining attitudes toward ambiguity.

Roughly, risk refers to the situations where the likelihoods of relevant events could be represented by a probability measure, while ambiguity refers to the situations where information available for the decision-maker is too insufficient to be represented by a probability measure. Ellsberg demonstrates that such a distinction is empirically meaningful. That is, the Ellsberg paradox is due not only to the ambiguity embodied in it, but more importantly to the decision maker's aversion to the ambiguity.

There are, at least, two different points of view to look at the Ellsberg paradox. One is that the existing models such as the subjective expected utility model (Savage (1954)) and the probabilistic sophistication model (Machina and Schmeidler (1992)) are not general enough to accommodate it. Following this line, some more general models such as the Choquet expected utility theory (Schmeidler (1989)) and the multiple-priors model (Gilboa and Schmeidler (1989)) have been proposed. The other is that to have a more *accurate* explanation of the paradox, we should identify the underlying ambiguity first and then recognize the decision maker's attitudes toward it. Following this line, two types of events, unambiguous and ambiguous, have been suggested by Epstein and Zhang (2001) and Zhang (2002).

Based on the two points of view, there are also two approaches to defining attitudes toward ambiguity. One is to use a nonadditive convex capacity rather than an additive probability, or a set of additive probabilities rather than a single additive probability to represent decision makers' aversion to ambiguity. The other is to focus on decision makers' preferences over ambiguous acts and unambiguous ones. For example, if agent 2 prefers unambiguous act h to ambiguous act f more than agent 1 does, then agent 2 is more ambiguity averse than agent 1.

The main purpose of this paper is to examine if the existing notions of attitudes toward ambiguity can accommodate the above observations on preferences $\succeq^{\min}$ and $\succeq^{\max}$. Since Epstein's notion of attitudes toward ambiguity, based on my knowledge, is the only one trying *explicitly* to separate attitudes toward ambiguity and risk, we mainly focus on it in this paper. The other two notions mentioned above obviously confound the concepts of ambiguity and risk. This can be seen from the following simple preference with utility function

$$\int_S f(s)\, dg(P), \tag{7}$$

where

- P is a probability measure over (S, Σ); and
- g is strictly increasing and concave with $g(0) = 1$ and $g(1) = 1$. For example, $g(x) = \sqrt{x}$.

Accordingly, the preference represented by (7) is a both Choquet expected utility and multiple-prior expected utility model. Based on the two notions, the agent is strictly ambiguity averse. However, there is no any ambiguity embodied here since (7) is probabilistically sophisticated (Machina and Schmeidler (1992)).

The paper is organized as follows. Epstein's notion of attitudes toward ambiguity is examined in Section 2. A new approach to defining attitudes toward ambiguity and some relevant notions such as the certainty equivalent for ambiguity and ambiguity premium are presented in Section 3. Section 4 provides an example in which attitudes toward ambiguity and risk are separated completely.

2 The Epstein Notion of Attitudes toward Ambiguity

Following the approach of defining risk aversion by using both comparative risk aversion and absolute risk aversion introduced by Yaari (1969), Epstein suggests to use both comparative ambiguity aversion and absolute ambiguity aversion to define ambiguity aversion.

Yaari uses the Savage set-up (S, Σ) to define comparative risk aversion as follows:

Agent $\succeq^2$ is more risk averse than agent $\succeq^1$ if

$$w \succeq^1 x \Longrightarrow w \succeq^2 x$$

for all constant acts w and acts x.

The intuition of this notion is that if agent 2 prefers riskfree act w to risky act x more than agent 1 does, then agent 2 is more risk averse than agent 1.[2]

However, if the uncertain payoff x contains not only risk, but also ambiguity at the same time, the Yaari notion has to be reformulated as follows:

Agent $\succeq^2$ is more uncertain (risk and ambiguity) averse than agent $\succeq^1$ if

$$w \succeq^1 x \Longrightarrow w \succeq^2 x$$

for all constants w and payoffs x.

Epstein provides this reformulation and then goes further one more step. He replaces constants w with unambiguous acts which are measurable with respect to an exogenously given set $\mathcal{A}^{ua}$ of unambiguous events (Epstein and Zhang, 2001) to define comparative ambiguity aversion as follows:

Say agent $\succeq^2$ is more ambiguity averse than agent $\succeq^1$ if

$$h \succeq^1 \left(\succ^1\right) f \Longrightarrow h \succeq^2 \left(\succ^2\right) f$$

for all unambiguous acts h and all acts f.

The intuition of this notion is that if agent 2 prefers unambiguous act h to ambiguous act f more than agent 1 does, then agent 2 is more ambiguity averse than agent 1.[3]

[2] It is noted here that both agents 1 and 2 must view w as the same as riskfree, otherwise the choices from w and x do not necessarily reflect their attitudes toward risk. That is, the risk is exogenously given. This assumption is necessary for defining comparative risk aversion.

[3] Similar to risk case, both agents 1 and 2 must view h as the same as unambiguous, otherwise the choices from h and f do not necessarily reflect their attitudes toward ambiguity. That is, the set of unambiguous event $\mathcal{A}^{ua}$ must be exogenously given. This assumption is necessary for defining comparative ambiguity aversion.

To define absolute risk aversion, we have to adopt a "normalization" for risk neutrality. For the von Neumann-Morgenstern (vNM) set-up, the risk neutral normalization for each lottery is unique since the underlying probability is given. That is, the standard normalization is the "expected value function" with respect to the given probability. For the Savage set-up, however, risk neutral normalizations are not necessary to be unique (Yaari, 1969). Thus, risk neutral normalizations $\succeq^{rnn}$ are those satisfying

$$f \succeq^{rnn} g \Longleftrightarrow E^{m}(f) \geq E^{m}(g)$$

for some probability measures m on Σ. This leads to the following definition of risk aversion:

> *Say that $\succeq$ is risk averse if there exists a risk neutral normalization $\succeq^{rnn}$ such that $\succeq$ is more risk averse than $\succeq^{rnn}$.* [4]

Similarly,

> *Say that $\succeq$ is risk loving if there exists a risk neutral normalization $\succeq^{rnn}$ such that $\succeq^{rnn}$ is more risk averse than $\succeq$.*

Combining the above two notions, we can define:

> *Say that $\succeq$ is risk neutral if there exist two risk neutral normalizations $\succeq_1^{rnn}$* and *$\succeq_2^{rnn}$ such that $\succeq$ is more risk averse than $\succeq_1^{rnn}$* and $\succeq_2^{rnn}$ is more risk averse than $\succeq$.[5]

Epstein then uses probabilistic sophistications[6] as ambiguity neutral normalizations $\succeq^{ann}$ in the following sense:

$$f \succeq^{ann} g \Longleftrightarrow f \succeq^{ps} g$$

for *some probabilistic sophistications $\succeq^{ps}$ with respect to some probability measures m* which are extensions of unambiguous probability P^{ua} from $\mathcal{A}^{ua}$ to Σ.

Similar to risk aversion mentioned above, Epstein finally defines ambiguity aversion as follows:

> *Say agent $\succeq$ is ambiguity averse if there exists an ambiguity neutral normalization $\succeq^{ann}$ such that $\succeq$ is more ambiguity averse than $\succeq^{ann}$.*

Similarly,

[4] Here we use risk neutral normalization $\succeq^{rnn}$ rather than risk neutral preference $\succeq^{rn}$ since risk neutral preferences and risk neutral normalizations are slightly different.

[5] The two risk neutral normalizations $\succeq_1^{rnn}$ and $\succeq_2^{rnn}$ could be the same.

[6] Roughly, agent $\succeq^{ps}$ is probabilistically sophisticated with respect to a probability measure m if she is indifferent between act f and the induced lottery $\left(x_1, m\left(f^{-1}(x_1)\right); \ldots; x_n, m\left(f^{-1}(x_n)\right)\right)$.

Say that $\succeq$ is ambiguity loving if there exists an ambiguity neutral normalization $\succeq^{ann}$ such that $\succeq^{ann}$ is more ambiguity averse than $\succeq$.

Combining the above two notions, we can define:

Say that $\succeq$ is ambiguity neutral if there exist two ambiguity neutral normalizations $\succeq_1^{ann}$ and *$\succeq_2^{ann}$ such that $\succeq$ is more ambiguity averse than $\succeq_1^{ann}$* and $\succeq_2^{ann}$ is more ambiguity averse than $\succeq$.[7]

The Epstein notion seems making sense since it tries to separate ambiguity and risk and treats them differently.

In order to avoid unnecessary repetition, the notion of attitudes toward ambiguity in this section refers only to the Epstein notion if not specified clearly. Since the two terms, ambiguity neutral normalization and probabilistic sophistication, are the same, we will use the latter one in the rest of this section.

The first question naturally raised is:

Question 1 *Is $\succeq^{\min}$ more ambiguity averse than $\succeq^{\max}$?*

The answer is no from the following example:

$$h = \left(\begin{array}{ll} \$100 & \text{if } s = R \\ \$50 & \text{if } s = B \\ \$50 & \text{if } s = Y \end{array} \right) \text{ and } f = \left(\begin{array}{ll} \$90 & \text{if } s = R \\ \$70 & \text{if } s = B \\ \$60 & \text{if } s = Y \end{array} \right).$$

Obviously, f is ambiguous and h is unambiguous since the set of unambiguous events is $\mathcal{A}^{ua} = \{\varnothing, S, \{R\}, \{B, Y\}\}$ in the Ellsberg three color set-up.

From

$$\max_{s \in S} h(s) = 100 > 90 = \max_{s \in S} f(s),$$

and

$$\min_{s \in S} h(s) = 50 < 60 = \min_{s \in S} f(s),$$

we have

$$h \succ^{\max} f,$$

but

$$h \prec^{\min} f.$$

That is,

$$h \succ^{\max} f \nRightarrow h \succ^{\min} f.$$

Accordingly, $\succeq^{\min}$ is not more ambiguity averse than $\succeq^{\max}$.

Next, we ask two even simpler questions:

Question 2 *Is*

[7] Similarly, the two ambiguity neutral normalizations $\succeq_1^{ann}$ and $\succeq_2^{ann}$ could be the same.

1. *$\succeq^{\min}$ ambiguity averse?*

2. *$\succeq^{\max}$ ambiguity loving?*

The answers are also no. To this end, for any probabilistically sophisticated preference $\succeq^{ps}$ as long as the underlying probability is an extension of P^{ua} from $\mathcal{A}^{ua}$ to 2^S, what we need is to prove that $\succeq^{\min}$ is not more ambiguity averse than $\succeq^{ps}$ and $\succeq^{\max}$ is not more ambiguity loving than $\succeq^{ps}$.

It is noted that

1. no state s is null for preferences $\succeq^{\min}$ and $\succeq^{\max}$; and

2. the notion of probabilistic sophistication adopted by Epstein is based on the Machina and Schmeidler (1992), any probabilistically sophisticated preference $\succeq^{ps}$ must satisfy $P3$ (Eventwise Monotonicity).

Thus, for any $x > y > 0$, and any probabilistically sophisticated $\succeq^{ps}$,

$$h = \begin{pmatrix} x & \text{if } s = B \\ x & \text{if } s = Y \\ 0 & \text{if } s = R \end{pmatrix} \succ^{ps} f = \begin{pmatrix} y & \text{if } s = B \\ x & \text{if } s = Y \\ 0 & \text{if } s = R \end{pmatrix} \Longleftrightarrow x > y.$$

From

$$\min_{s \in S} f(s) = 0 = \min_{s \in S} h(s)$$

we have

$$h \sim^{\min} f.$$

Therefore,

$$h \succ^{ps} f \nRightarrow h \succ^{\min} f.$$

That is, $\succeq^{\min}$ is not ambiguity averse.

Similarly, we can prove that $\succeq^{\max}$ is not ambiguity loving.

Since $\succeq^{\min}$ is not ambiguity averse and $\succeq^{\max}$ is not ambiguity loving, the next question naturally raised is:

Question 3 *Are both $\succeq^{\min}$ and $\succeq^{\max}$ ambiguity neutral?*

The answers are still no. Based on the Epstein notion, $\succeq^{\min}$ is ambiguity neutral if and only if there exist two probabilistically sophisticated preferences $\succeq_1^{ps}$ and $\succeq_2^{ps}$ such that $\succeq^{\min}$ is more ambiguity averse than $\succeq_1^{ps}$ and $\succeq_2^{ps}$ is more risk averse than $\succeq^{\min}$. However, this is impossible since we have proved that $\succeq^{\min}$ is not more ambiguity averse than any probabilistically sophisticated preference $\succeq^{ps}$. Similarly, $\succeq^{\max}$ is also not ambiguity neutral.

Thus, a contradiction seems having been produced:

- On the one hand, since $\succeq^{\min}$ and $\succeq^{\max}$ are very extreme, $\succeq^{\min}$ and $\succeq^{\max}$ should be ambiguity averse and loving, respectively. However, we have proved that this is not true;

- On the other hand, if $\succeq^{\min}$ is not ambiguity averse, and $\succeq^{\max}$ is not ambiguity loving, then both should be ambiguity neutral. However, we have also proved that this is not true.

The question is: The Epstein approach seems fine and making sense, why does it lead to the above contradictory results? Based on the above discussion, at least, it has the following two issues:

1. For the comparative ambiguity aversion, attitudes toward risk is involved from unambiguous act h to ambiguous act f. In general, not only the ambiguity contained in acts h and f is different, but also the risk contained in acts h and f is also different. Therefore, the preferences revealed from unambiguous act h to ambiguous act f do not only reflect attitudes toward ambiguity, but also reflect attitudes toward risk. For example, if both acts h and f are unambiguous, then preferring h to f must be totally due to attitudes toward risk since there is no ambiguity in both h and f at all. Thus, in order to use comparative ambiguity aversion, the (degree of) risk contained in unambiguous act h and (ambiguous) act f must be the same, otherwise the comparative ambiguity would involve comparative risk.

2. For the absolute ambiguity aversion, the choice of a probabilistically sophisticated preference $\succeq^{ps}$ could be too arbitrary to be reasonable. For example, let m be the following probability measure

$$m(\{R\}) = \frac{1}{3},\ m(\{B\}) = 0,\ \text{and}\ m(\{Y\}) = \frac{2}{3} \tag{8}$$

   which is an extension of the unambiguous probability P^{ua} from $\mathcal{A}^{ua}$ to 2^S. Consider the following probabilistically sophisticated preferences based on m

$$h = \begin{pmatrix} \$100 & \text{if } s = R \\ \$0 & \text{if } s = B \\ \$0 & \text{if } s = Y \end{pmatrix} \sim^{ps} \begin{pmatrix} \$100 & \text{if } s = R \\ \$100 & \text{if } s = B \\ \$0 & \text{if } s = Y \end{pmatrix} = f.$$

   Since she believes black color impossible, therefore she would be indifferent between h and f.

   Now consider another agent with preferences

$$h = \begin{pmatrix} \$100 & \text{if } s = R \\ \$0 & \text{if } s = B \\ \$0 & \text{if } s = Y \end{pmatrix} \prec \begin{pmatrix} \$100 & \text{if } s = R \\ \$100 & \text{if } s = B \\ \$0 & \text{if } s = Y \end{pmatrix} = f.$$

   From

$$f \sim^{ps} h \Longrightarrow f \succ h,$$

   we cannot conclude that agent $\succeq$ is ambiguity loving at acts f and h since h is weakly dominated by f.[8] Thus, in order to use absolute ambiguity aversion, the choice of a probabilistically sophisticated preference

[8] The weak domination can be seen from $f(s) \geq h(s)$ for all s and $f(B) > h(B)$.

$\succeq^{ps}$ must be based on some principles. For example, the probability measure supporting $\succeq^{ps}$ should be not only an extension of the unambiguous probability P^{ua} from $\mathcal{A}^{ua}$ to 2^S, but also consistent with the principle of insufficient reason and equal ambiguity described in next section.

A final question we are very interested in is:

Question 4 *Based on common sense, what should $\succeq^{\min}$ and $\succeq^{\max}$ be regarding attitudes toward ambiguity?*

We strongly believe that both $\succeq^{\min}$ and $\succeq^{\max}$ would be ambiguity neutral. Although there may exist some ambiguity in some acts f, but the two agents $\succeq^{\min}$ and $\succeq^{\max}$ do not pay any attention to it. This could be seen from the following two facts:

Fact I: All events, based on the Epstein and Zhang notion of unambiguous event, are (subjectively) unambiguous to $\succeq^{\min}$ and $\succeq^{\max}$ no matter how little objective information is embodied in the model.[9] That is, they don't pay any attention to the underlying ambiguity.

Fact II: The two preferences $\succeq^{\min}$ and $\succeq^{\max}$ exactly follow maxmin and maxmax criteria, respectively, based on the expressions (3) and (5). Thus, they totally ignore not only ambiguous information, but also unambiguous information. For example, though ambiguous event $\{B\}$ with chance $0 \leq P(\{B\}) \leq \frac{2}{3}$, but they still could view the chance as 1 or 0 sometimes. Similarly, though unambiguous event $\{R\}$ with chance $P(\{R\}) = \frac{1}{3}$ for sure, but they still could view the chance as 1 or 0 sometimes. That is, they just ignore the actual chances of events to happen.

3 A New Approach to Defining Attitudes toward Ambiguity

Based on our discussions, the existing notions of attitudes toward ambiguity seem not consistent with the pair of preferences $\succeq^{\min}$ and $\succeq^{\max}$. Thus, a new notion about attitudes toward ambiguity is needed. We will provide some idea

[9] This can be seen from the two acts

$$f = \begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ w(s) & \text{if } s \in T^c \backslash (A \cup B) \\ z & \text{if } s \in T \end{pmatrix} \text{ and } g = \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ w(s) & \text{if } s \in T^c \backslash (A \cup B) \\ z & \text{if } s \in T \end{pmatrix}$$

used in Epstein and Zhang (2001). Since the two acts f and g have the same outcomes

$$\{x, x^*, z, \text{ and } w(s), s \in T^c \backslash (A \cup B)\},$$

therefore,

$$\min_{s \in S} f(s) = \min_{s \in S} g(s) \text{ and } \max_{s \in S} f(s) = \max_{s \in S} g(s).$$

Accordingly, based on the Epstein and Zhang notion, any event T is unambiguous.

for the new notion in this section next, more details will be provided in Zhang (2019).

Though attitudes toward risk can be defined by comparative approach *indirectly*, however, it seems to me this approach would not be very useful for attitudes toward ambiguity. This is because there must exist risk if there exists any ambiguity in any situation. Thus, the choice from ambiguous acts and unambiguous ones reflects not only attitudes toward ambiguity, but also attitudes toward risk.

Fortunately, attitudes toward risk can also be defined by absolute approach *directly*. The key to this approach is to use "expected value" with respect to the underlying probability measure as a "normalization" for risk neutrality (risk neutral normalization). Thus, the only difference between risk x and its expected value $E(x)$ is to have different risk, nothing else such as income, ambiguity, etc. The choice from risk x and $E(x)$ reflects the decision maker's attitudes toward risk exclusively.

A similar question naturally raised for ambiguity is: For any ambiguity f, can we identify a "normalization" for ambiguity neutrality $\Psi_{f,P^{an}}$ (to be specified shortly) such that the only difference between ambiguity f and the normalization $\Psi_{f,P^{an}}$ is to have different ambiguity, nothing else such as income, risk, etc. In other words, both ambiguity f and its ambiguity neutral normalization $\Psi_{f,P^{an}}$ contain the 'same risk'. As a result, the choice from ambiguity f and its ambiguity neutral normalization $\Psi_{f,P^{an}}$ reflects the decision maker's attitudes toward ambiguity exclusively. We will argue next that this is possible and the key to this is to identify an ambiguity neutral probability P^{an}.

To this end, based on the underlying "objective" information, what we should do is to identify a set of unambiguous events $\mathcal{A}^{ua}$ and the corresponding unambiguous probability P^{ua} on $\mathcal{A}^{ua}$; Following the principle of insufficient reason introduced by Bernoulli (1738) and equally ambiguous probabilities introduced by Epstein and Zhang (2001), derive the ambiguity neutral probability P^{au} which is an extension of the unambiguous probability P^{ua} from $\mathcal{A}^{ua}$ to all the relevant events; Using the ambiguity neutral probability P^{au}, transform each act f into an ambiguity free lottery[10]

$$\Psi_{f,P^{an}} = \left(x_1, P^{an}\left(f^{-1}(x_1)\right); x_2, P^{an}\left(f^{-1}(x_2)\right); \ldots, x_n, P^{an}\left(f^{-1}(x_n)\right)\right)$$

such that 1. both act f and the induced lottery $\Psi_{f,P^{an}}$ contain the same risk, and 2. both the induced lottery $\Psi_{f,P^{an}}$ and the expected value $E(\Psi_{f,P^{an}})$ contain the same ambiguity (actually no ambiguity).

Obviously, for unambiguous act f,

$$\Psi_{f,P^{an}} = \Psi_{f,P^{ua}}.$$

In this case, we assume

$$f \sim \Psi_{f,P^{ua}}$$

since there is no ambiguity contained in act f.

Finally, we define the following: An agent is ambiguity

[10] For simplicity, we consider simple acts only in this paper.

1. averse at (ambiguous) act f if she prefers the induced lottery $\Psi_{f,P^{an}}$ to the act f;

2. loving at (ambiguous) act f if she prefers the act f to the induced lottery $\Psi_{f,P^{an}}$;

3. neutral at (ambiguous) act f if she is indifferent between the induced lottery $\Psi_{f,P^{an}}$ and the act f.

Similarly, an agent is risk

1. averse at act f if she prefers the expected value of the induced lottery $E\left(\Psi_{f,P^{an}}\right)$ to the induced lottery $\Psi_{f,P^{an}}$;

2. loving at act f if she prefers the induced lottery $\Psi_{f,P^{an}}$ to the expected value of the induced lottery $E\left(\Psi_{f,P^{an}}\right)$; and

3. neutral at act f if she is indifferent between the induced lottery $\Psi_{f,P^{an}}$ and the expected value of the induced lottery $E\left(\Psi_{f,P^{an}}\right)$.

It is noted that an ambiguity averse person still could be risk loving at (ambiguous) act f. Thus, there are total $3 \times 3 = 9$ possibilities regarding attitudes toward ambiguity and risk at each act f. Accordingly, a (complete) separation of attitudes toward ambiguity and risk has been derived.

Next, we will provide some concrete ideas for finding such a reasonable ambiguity neutral probability measure P^{au}. To the end, we consider the following two pairs of ambiguous acts in the three color Ellsberg model

$$f_1 = \begin{pmatrix} \$100 & \text{if } s = R \\ \$100 & \text{if } s = B \\ \$0 & \text{if } s = Y \end{pmatrix} \quad \text{and} \quad g_1 = \begin{pmatrix} \$100 & \text{if } s = R \\ \$0 & \text{if } s = B \\ \$100 & \text{if } s = Y \end{pmatrix}$$

and

$$f_2 = \begin{pmatrix} \$60 & \text{if } s = R \\ \$0 & \text{if } s = B \\ \$50 & \text{if } s = Y \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} \$60 & \text{if } s = R \\ \$50 & \text{if } s = B \\ \$0 & \text{if } s = Y \end{pmatrix}.$$

Of course, both events $\{B\}$ and $\{Y\}$ are ambiguous, however, we may learn more about them from the decision maker's typical behaviors.

For the pair acts f_1 and g_1, since the only difference is the payoffs on events $\{B\}$ and $\{Y\}$ and she has no any information about which one of the two events $\{B\}$ and $\{Y\}$ would occur more likely, therefore the following behavior

$$f_1 \sim g_1$$

is reasonable regardless her attitudes toward risk and ambiguity, respectively. Similarly,

$$f_2 \sim g_2.$$

Epstein and Zhang (2001) call $\{B\}$ and $\{Y\}$ equally ambiguous.

To assign reasonable ambiguity neutral probabilities on ambiguous events $\{B\}$ and $\{Y\}$, we borrow the *principle of insufficient reason* from Bernoulli (1738). It states that in situations where there is no logical or empirical reason to favor any one of a set of mutually exclusive events, we should assign them all equal probability. Since this implies a uniform probability distribution over the events, it accordingly may qualify as a probabilistically sophisticated model of beliefs for ambiguity neutral agents.

Thus, based on both equal ambiguity and the principle of insufficient reason, if an agent is ambiguity neutral and wants to use probability to represent her beliefs, then the natural choice should be the extension of P^{ua} such that

$$P^{an}(\{B\}) = P^{an}(\{Y\}).$$

That is, the reasonable ambiguity neutral probability measure should be

$$P^{an}(\{R\}) = \frac{1}{3},\ P^{an}(\{B\}) = P^{an}(\{Y\}) = \frac{1}{3} \tag{9}$$

Accordingly, if she is ambiguity neutral, she should be probabilistically sophisticated with this particular ambiguity neutral probability (9) rather than any other probability measure m as long as satisfying $m(\{R\}) = 1/3$.

Based on the ambiguity neutral probability P^{an}, any act

$$f = \begin{pmatrix} x_1 & \text{if } s = R \\ x_2 & \text{if } s = B \\ x_3 & \text{if } s = Y \end{pmatrix}$$

can be transformed into a lottery $\Psi_{f,P^{an}}$ as follows

$$\Psi_{f,P^{an}} = (x_1, P^{an}(\{R\}); x_2, P^{an}(\{B\}); x_3, P^{an}(\{Y\})) = (x_1, 1/3; x_2, 1/3; x_3, 1/3)$$

and she is indifferent between ambiguous act f and lottery $\Psi_{f,P^{an}}$ if she is ambiguity neutral.

Obviously, based on our new approach, both $\succeq^{\min}$ and $\succeq^{\max}$ are ambiguity neutral since

$$\min_{s\in S} f(s) = \min_{s\in S} \Psi_{f,P^{an}} \text{ and } \max_{s\in S} f(s) = \max_{s\in S} \Psi_{f,P^{an}}$$

from the fact that both act f and the induced lottery $\Psi_{f,P^{an}}$ have the same outcomes.

For any other non ambiguity neutral agents, they may view the two types of probabilities differently:

- Unambiguous probability

  $$P^{ua}(\{R\}) = 1/3,\ P^{ua}(\{B, Y\}) = 2/3;$$

  and

- Ambiguity neutral probability

$$P^{an}(\{R\}) = 1/3,\ P^{an}(\{B\}) = P^{an}(\{Y\}) = 1/3.$$

Based on the two types of probabilities, it seems reasonable to say agent $\succeq$ to be ambiguity averse if she prefers lottery $\Psi_{f,P^{an}}$ to ambiguous act f.

After separating attitudes toward ambiguity and risk completely, two certainty equivalents for (ambiguous) act f: risk, ambiguity and risk, can be defined as usual:

- Constant lottery $c^R(f)$ is called a certainty equivalent for the risk contained in f if agent $\succeq$ is indifferent between $c^R(f)$ and the induced lottery $\Psi_{f,P^{an}}$;
- Constant act $c^{AR}(f)$ is called a certainty equivalent for the ambiguity and risk contained in f if agent $\succeq$ is indifferent between $c^{AR}(f)$ and act f.

Similarly, three premiums can be defined as follows

- $E(\Psi_{f,P^{an}}) - c^R(f)$ is called the risk premium for act f;
- $c^R(f) - c^{AR}(f)$ is called the ambiguity premium for act f; and
- $E(\Psi_{f,P^{an}}) - c^{AR}(f)$ is called the ambiguity and the risk premium for (ambiguous) act f.

Obviously, agent $\succeq$ is

- risk averse at act f if and only if the risk premium $E(\Psi_{f,P^{an}}) - c^R(f)$ is non negative;
- ambiguity averse at (ambiguous) act f if and only if the ambiguity premium $c^R(f) - c^{AR}(f)$ is non negative; and
- risk averse or ambiguity averse or both at (ambiguous) act f if and only if the ambiguity and risk premium

  $$E(\Psi_{f,P^{an}}) - c^{AR}(f) = \left[E(\Psi_{f,P^{an}}) - c^R(f)\right] + \left[c^R(f) - c^{AR}(f)\right]$$

  is non negative.

Based on the same logic, we can define comparative risk and ambiguity between two agents as follows:

Say agent 1 is

- more risk averse than agent 2 at act f, if agent 1's risk premium $E(\Psi_{f,P^{an}}) - c_1^R(f)$ is not smaller than agent 2's risk premium $E(\Psi_{f,P^{an}}) - c_2^R(f)$; and
- more ambiguity averse than agent 2 at (ambiguous) act f, if agent 1's ambiguity premium $c_1^R(f) - c_1^{AR}(f)$ is not smaller than agent 2's ambiguity premium $c_2^R(f) - c_2^{AR}(f)$.

Of course, a person may become more risk averse or more ambiguity averse or both after becoming richer. This can be defined as the same as above.

4 An Example

We will end this paper with an example to have all the nine possibilities regarding attitudes toward ambiguity and risk, respectively.

Using the three colour Ellsberg set-up, we define two capacities: Inner measure P_* and Outer measure P^* derived from unambiguous probability P^{ua} as follows: For any event $C \in 2^S$,

$$\begin{aligned} P_*(C) &= \sup\{P^{ua}(A) : A \subseteq C \text{ and } A \in \mathcal{A}^{ua}\} \text{ and} \\ P^*(C) &= \inf\{P^{ua}(A) : A \supseteq C \text{ and } A \in \mathcal{A}^{ua}\} \end{aligned}$$

where $\mathcal{A}^{ua}$ is the set of unambiguous events defined in (1) and P^{ua} is the unambiguous probability defined in (2).

Consider a preference $\succeq$ over all the acts f and the induced lotteries $\Psi_{f,P^{an}}$

$$\mathcal{H} = \bigcup_{\text{All acts } f} \{f, \Psi_{f,P^{an}}\}$$

by using the following utility function

$$U(h) = \begin{cases} \alpha \int u(f)\, dP_* + \beta \int u(f)\, dP^* + (1-\alpha-\beta)\, Eu(\Psi_{f,P^{an}}) & \text{if } h = f \\ Eu(\Psi_{f,P^{an}}) & \text{if } h = \Psi_{f,P^{an}} \end{cases}$$

where $0 \le \alpha, \beta \le 1$, and u is strictly increasing in R^1, $\int u(f)\, dP_*$ and $\int u(f)\, dP^*$ are the Choquet integrals.

Obviously, for unambiguous act f, we have

$$U(f) = Eu(\Psi_{f,P^{ua}})$$

and $\succeq$ is risk averse if and only if u is concave.

For ambiguous act f, $\succeq$ is

1. ambiguity averse and risk averse at act f if and only if $0 < \alpha \le 1$, $\beta = 0$, and u is concave;

2. ambiguity loving and risk averse at act f if and only if $0 < \beta \le 1$, $\alpha = 0$, and u is concave; and

3. ambiguity neutral and risk averse at act f if and only if $\alpha = \beta = 0$, and u is concave.

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