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Abstract

We construct a two period consumption-saving model with two agents where rising income inequality leads to declining equilibrium rates of interest, rising debt levels, and lower future aggregate demand. Importantly, our model does not rely on non-homothetic preferences to generate these outcomes. Instead, borrowers face a borrowing constraint which eases when income inequality increases. This feature is supported by the stylized fact that consumer credit and inequality have strongly co-moved and risen in the U.S. since the mid-1980s.

Key words: Income inequality, Borrowing Constraint, Interest rate, Indebted demand
JEL classification: E21, E43

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1 Introduction

The theory of *indebted demand* proposed by Mian, Straub and Sufi (2021) has attracted significant attention, in part due to its unifying explanation of several prominent secular trends which have occurred since the 1980s, such as rising income inequality, rising debt levels, financial deregulation, and declining rates of interest. One contentious point of their theory is the reliance on non-homothetic preferences, which generates a positive correlation between saving rates and permanent income. For example, Fagereng, Holm, Moll and Natvik (2019) study how saving rates vary with wealth in Norway and find that they are relatively flat, which is consistent with saving rates being flat with respect to permanent income.

In this paper we present a two period consumption-saving model in which rising income inequality reduces the equilibrium rate of interest, promotes debt accumulation by borrowers, and consequently reduces future aggregate demand. Importantly, and in contrast to Mian, Straub and Sufi (2021), our results do not rely on non-homothetic consumption-saving preferences. The advantage of our framework is its simplicity, which allows us to derive analytical expressions describing the impact of income inequality on consumption, savings, interest rates, and aggregate demand.

Our model features two agents, savers and borrowers, who are endowed with exogenous income and face no uncertainty. Savers derive utility from consumption in periods one and two, and have a warm-glow bequest motive, deriving utility from the level of assets remaining at the end of the second period. Borrowers derive utility from consumption in periods one and two and have no bequest motive.[1]

We show that in our framework increases in income inequality monotonically reduce equilibrium rates of interest. This happens because as the income of savers rises, they prefer to smooth additional income over period two consumption and their bequest, which shifts saving supply outwards. Since borrowing demand is downward sloping with respect to interest rates, these shifts lead to lower equilibrium rates.

A key aspect of our model is that borrowers face a borrowing constraint which is positively correlated with the level of income inequality. The stylized fact that consumer credit (relative to output) and inequality have strongly co-moved and risen in the U.S. since the mid-1980s provides support for this feature, along with empirical findings documented in the recent literature. As income inequality rises, borrowing constraints are relaxed, which

[1] Allowing savers to derive utility from bequests, while borrowers do not is reminiscent of Kumhof, Ranciére and Winant (2015) who assume that top income earners derive utility from financial wealth, while bottom income earners do not.

allows borrowers to increase debt financed consumption in the first period. However, rising debt financed consumption comes at the cost of reducing aggregate demand in the second period. Since rising debt levels increase borrowers' debt service costs, they must reduce their second period consumption, which is only partially offset by increases in savers' consumption. In this sense, our model generates what we refer to as *two period indebted demand* without appealing to non-homothetic preferences.

Lastly, we show that in our model when this particular feature of the borrowing constraint is present, an increase in leverage due to an exogenous increase in borrowing capacity unrelated to inequality, will lower future aggregate demand and amplify the two period indebted demand channel that we have highlighted. The reason is that, for any given level of inequality, borrowers can now accumulate more debt in the first period which results in less demand for consumption in period two.

In Section 2 we describe the model and Section 3 concludes.

2 Model

The model features two households, savers and borrowers. Time is discrete and households live for two periods. Income is exogenous and there is no uncertainty. Borrowers face a borrowing constraint in the first period which limits the amount of first period consumption that can be financed from second period income. Each household takes the interest rate, which is determined in equilibrium, as given.

2.1 Savers

Savers receive exogenous income y_1^s in period one and y_2^s in period two. It is assumed that savers have a warm-glow bequest motive. Utility from bequests is determined by the log level of assets remaining at the end of the second period. Savers' discount factor is given by β^s. Additionally, it is assumed that there exists a central authority that can redistribute resources from borrowers to savers given by t_1. Redistributing resources from borrowers to savers, in essence, increases income inequality.[2]

[2] This is a reduced-form way to capture a variety of reasons that may have contributed to an increase in inequality in the US documented in the literature.

The optimization problem of savers is given by

$$\underset{c_1^s, c_2^s, a_1^s, a_2^s}{\text{Max}} \quad \log c_1^s + \beta^s \log c_2^s + \beta^s \log a_2^s, \tag{1}$$

subject to

$$c_1^s + a_1^s = y_1^s + t_1, \tag{2}$$
$$c_2^s + a_2^s = y_2^s + (1+r)a_1^s. \tag{3}$$

The solution to the optimization problem yields the following consumption functions, saving demand in period one, and bequest function

$$c_1^{s,\star} = \left(\frac{1}{1+2\beta^s}\right)\left(y_1^s + t_1 + \frac{y_2^s}{1+r}\right), \tag{4}$$
$$c_2^{s,\star} = \left(\frac{\beta^s}{1+2\beta^s}\right)\left((y_1^s + t_1)(1+r) + y_2^s\right), \tag{5}$$
$$a_1^{s,\star} = \left(\frac{2\beta^s}{1+2\beta^s}\right)(y_1^s + t_1) - \left(\frac{1}{1+2\beta^s}\right)\frac{y_2^s}{1+r}, \tag{6}$$
$$a_2^{s,\star} = \left(\frac{\beta^s}{1+2\beta^s}\right)\left((y_1^s + t_1)(1+r) + y_2^s\right). \tag{7}$$

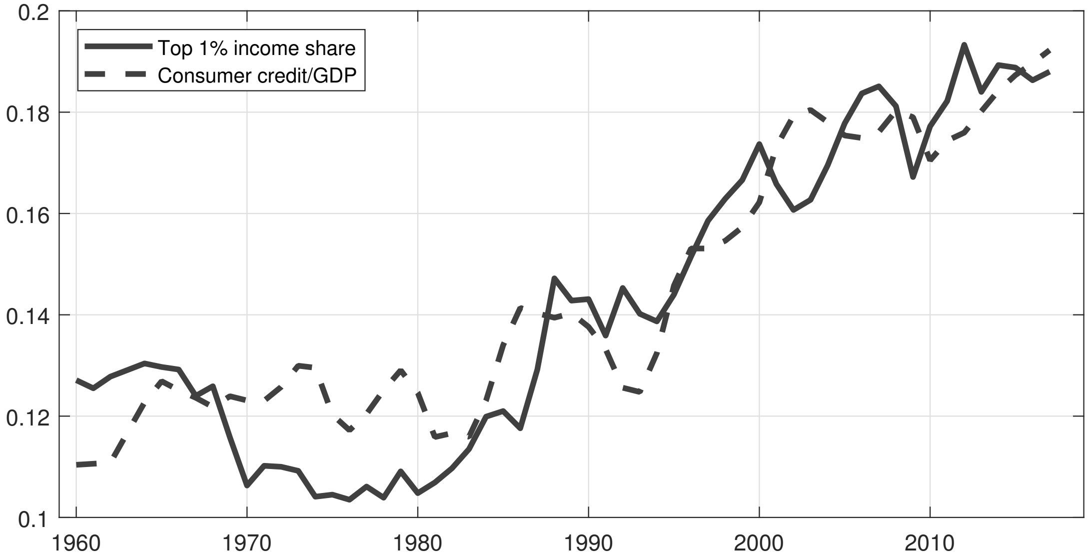
Equation (6) highlights that the marginal propensity to save out of transfers is constant, consistent with the evidence in Fagereng, Holm, Moll and Natvik (2019).

2.2 Borrowers

Borrowers receive exogenous income y_1^b in the first period and y_2^b in the second period. Borrowers derive no utility from bequests and their discount factor is denoted by β^b.

Borrowers face a constraint which puts an upper bound on the amount of first period consumption that can be financed by borrowing. The source of the constraint is left unspecified, but could be rationalized, for example, by limited commitment or asymmetric information problems in credit markets. Borrowers can finance a fraction up to $\vartheta(\gamma)$ of the present value of second period income, where $\gamma \equiv g(t_1; y_1^s, y_1^b)$ is a general function capturing income inequality. We make the following functional form assumption for γ:

Figure 1: The top 1% income share and credit/GDP in the U.S.



Notes: Consumer credit/GDP refers household and non-profit total liabilities relative to GDP. Consumer credit and GDP were downloaded by from the Federal Reserve Bank of St. Louis database. The corresponding data codes are `HCCSDODNS` and `GDP`. The top 1% income share data is pre-tax and was obtained from the World Inequality Database.

$$\gamma = \frac{y_1^s + t_1}{y_1^s + y_1^b} \quad \text{and} \quad \vartheta(\gamma) \equiv \phi \times \left(\frac{y_1^s + t_1}{y_1^s + y_1^b} \right) \quad \text{and} \quad \frac{\partial \vartheta(.)}{\partial t_1} > 0. \tag{8}$$

However, it is worth noting that our results do not hinge on the specific functional form of γ, and only require that $\frac{\partial \vartheta(\gamma)}{\partial t_1} > 0$. The property, $\frac{\partial \vartheta(.)}{\partial t_1} > 0$, implies that as income inequality rises, the borrowing constraint is relaxed and borrowers can feasibly finance a larger amount of first period consumption by accumulating more debt. ϕ is a constant and captures the Loan-to-Income (LTI) ratio, which is separate from income inequality. We assume that any perturbations to ϕ or t_1 are reasonable, such that $\vartheta(\gamma) \in (0, 1)$.

This constraint is motivated by and consistent with the stylized fact that U.S. credit as a fraction of GDP and income inequality (measured as the top 1% income share) comove strongly, with both rising since the mid-1980s. This can be seen in Figure 1.

Moreover, numerous arguments and evidence support a borrowing constraint with a positive correlation between credit supply and income inequality. For example, Rajan (2010) argues that in the lead up to the Great Recession in response to increasing pressure about rising income inequality, the U.S. government pursued policies which expanded credit to appease voters. In particular, there was a focus on expanding credit to low-income households. Kumhof, Ranciére and Winant (2015) document that prior to both the Great Depression and Great Recession top income shares and debt-to-income ratios of low and middle income households had risen dramatically. Coibion, Gorodnichenko, Kudlyak and Mondragon (2020) use U.S. county level data to study how debt accumulation varies with local measures of income inequality over the period 2000-2012. Their evidence favours an interpretation that debt accumulation over this period was primarily driven by supply side factors and credit expansion was tilted towards higher income individuals. At the state level they find that in states with higher inequality, households accumulated relatively more debt, consistent with the spirit of our constraint. Similarly, using state level data over the 1980-2008 period, Bertrand and Morse (2016) find that nonrich households have higher consumption shares in states with higher inequality, and that these households reported being financially worse off and have higher levels of bankruptcy.

The optimization problem of borrowers is then given by

$$\underset{c_1^b, c_2^b, a_1^b}{\text{Max}} \quad \log c_1^b + \beta^b \log c_2^b, \tag{9}$$

subject to the following constraints

$$c_1^b + a_1^b = y_1^b - t_1, \tag{10}$$

$$c_2^b = y_2^b + (1+r) a_1^b, \tag{11}$$

$$c_1^b \leq y_1^b - t_1 + \vartheta(\gamma) \frac{y_2^b}{1+r}. \tag{12}$$

In borrowers case, t_1 is subtracted from first period income as the transfer moves resources from borrowers to savers. (12) is the borrowing constraint which is assumed to be binding in the rest of the paper.[3] Then consumption functions and borrowing demand are given by

[3]This assumption is typically motivated by assuming that borrowers are more impatient than savers. This approach is common in two agent models (e.g., Iacoviello (2005)).

$$c_1^{b,\star} = y_1^b - t_1 + \vartheta(\gamma)\frac{y_2^b}{1+r}, \tag{13}$$

$$c_2^{b,\star} = y_2^b(1 - \vartheta(\gamma)), \tag{14}$$

$$a_1^{b,\star} = -\vartheta(\gamma)\frac{y_2^b}{1+r}. \tag{15}$$

2.3 Equilibrium and Comparative Statics

A competitive equilibrium in this model is given in Definition 1.

Definition 1. *Given endowments $\{y_1^s, y_2^s, y_1^b, y_2^b\}$ and a transfer $\{t_1\}$, a competitive equilibrium of the model is an interest rate $\{r\}$ and quantities $\{c_1^s, c_2^s, a_1^s, a_2^s, c_1^b, c_2^b, a_1^b\}$ such that savers and borrowers maximize utility subject to budget constraints in (2)-(3) and (10)-(11) and the borrowing constraint (12); the asset market clears such that $a_1^s + a_1^b = 0$. By Walras' Law the goods market clears in period one and period two.*

Given Definition 1, it is straightforward to solve for the equilibrium interest rate in this model using the asset market clearing condition. Imposing $a_1^s + a_1^b = 0$ and solving for the equilibrium rate of interest in terms of exogenous variables yields

$$1 + r^\star = \frac{1}{2\beta^s}\frac{y_2^s + \vartheta(\gamma)y_2^b(1 + 2\beta^s)}{y_1^s + t_1}. \tag{16}$$

We can now make the following statement about the relationship between income inequality and the equilibrium interest rate.

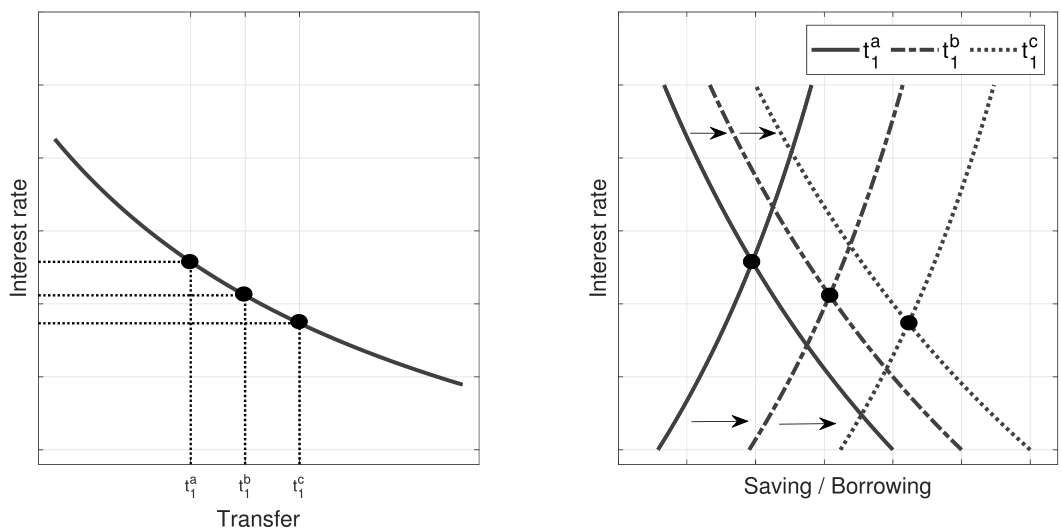
Proposition 1. *An increase in period one income inequality, due to a transfer from borrowers to savers, $t_1 > 0$, strictly decreases the equilibrium rate of interest.*

By totally differentiating (16) and substituting in the expression for $\frac{\partial\vartheta(\gamma)}{\partial t_1}$, while holding changes in incomes constant, we get

$$\frac{\mathrm{d}r^{\star}}{\mathrm{d}t_1} = -\frac{1}{2\beta^s}\frac{y_2^s}{(y_1^s + t_1)^2} < 0. \tag{17}$$

As long as incomes of savers are positive in period one and two, increasing transfers from borrowers to savers monotonically decreases the equilibrium rate of interest.

Figure 2: The impact of rising inequality on equilibrium rates, saving supply, and borrowing demand



Notes: The transfers are such that $t_1^c > t_1^b > t_1^a$. The equilibrium interest rate is computed using (16). Saving supply is calculated using (6). Borrowing demand is the negative of borrowers asset position, calculated using (15).

What is the intuition for this result? One way to understand this result is by examining savings supply (a_1^s) and borrowing demand (a_1^b) schedules. These schedules are pictured in Figure 2, along with the equilibrium interest rate. A transfer from borrowers to savers in period one (an increased inequality) increases savers' lifetime income. But since savers are optimizing over period one and two consumption, and their bequest, they only consume a fraction of the transfer in period one, preferring to smooth the remaining transfer amount over period two consumption and their bequest. This is done by increasing their desired saving in period one for any given level of the interest rate, resulting in a rightward shift in the saving supply curve.

Borrowers on the other hand are at the borrowing limit. Then a transfer from borrowers to savers has a direct effect on their period one consumption by lowering their resources in period one. However, this direct effect is partially offset by the fact that a transfer increases income inequality, which allows borrowers to finance more first period consumption for any given level of interest rates. This results in a rightward shift in the borrowing demand curve.

Importantly, however, the shifts in saving supply are always larger than the shifts in borrowing demand, resulting in a negative relationship between transfers and the equilibrium rate of interest.

One important feature to highlight with respect to Proposition 1 is that it does not depend on the borrowing constraint being a function of income inequality. Suppose instead that the borrowing limit was fixed at some constant LTI ratio ϕ. In this case, positive transfers do not shift borrowing demand to the right at all and only saving supply shifts to the right. This generates a rise in saving and borrowing, and a decline in equilibrium interest rates.

Next, define aggregate demand as follows.

Definition 2. *Let aggregate demand be defined by the sum of consumption by savers and borrowers in each period. That is, $AD_t \equiv c_t^s + c_t^b$ for $t = [1, 2]$.*

First, note that based on Definition 2, aggregate demand (AD) is fixed in period one. Any saving in period one equals debt financed consumption by borrowers, ensuring that all period one resources are consumed by either borrowers or savers. In period two, however, aggregate demand can vary. This allows us to assess the impact of debt financed consumption in period one on aggregate demand in period two, and leads to the following proposition.

Proposition 2 (Two Period Indebted Demand). *An increase in inequality in period one represented by transfer from borrowers to savers, $t_1 > 0$, increases borrowers' debt financed consumption in period one and reduces aggregate demand in period two.*

Rising inequality generates increases in borrowers' debt financed consumption. To see this result formally, totally differentiating (15) and holding changes in incomes constant, yields

$$\frac{\mathrm{d}a_1^{b,\star}}{\mathrm{d}t_1} = -\frac{\partial\vartheta(\gamma)}{\partial t_1}\frac{y_2^b}{1+r^\star} + \vartheta(\gamma)\frac{y_2^b}{(1+r^\star)^2}\frac{\mathrm{d}r^\star}{\mathrm{d}t_1} < 0. \tag{18}$$

The sign of the derivative follows from the fact that $\frac{\partial\vartheta(\gamma)}{\partial t_1} > 0$, given in (8), and $\frac{\mathrm{d}r^\star}{\mathrm{d}t_1} < 0$, as shown in (17). Further, since declining asset levels are increases in debt, and debt by borrowers is used for consumption, (18) says that borrowers increase debt financed consumption as the transfer t_1 increases.

Using Definition 2, aggregate demand in period two is then given by

$$AD_2 = \left(\frac{\beta^s}{1+2\beta^s}\right)\left((y_1^s + t_1)(1+r^\star) + y_2^s\right) + y_2^b\big(1-\vartheta(\gamma)\big). \tag{19}$$

Totally differentiating (19) and substituting expressions for $1+r^\star$, $\frac{\partial\vartheta(\gamma)}{\partial t_1}$, and $\frac{\mathrm{d}r^\star}{\mathrm{d}t_1}$, while holding changes in incomes constant, gives

$$\frac{\mathrm{d}AD_2}{\mathrm{d}t_1} = -\frac{1}{2}\frac{\phi y_2^b}{y_1^s + y_1^b} < 0, \tag{20}$$

which says that increasing transfers from borrowers to savers in period one leads to declines in aggregate demand in period two.

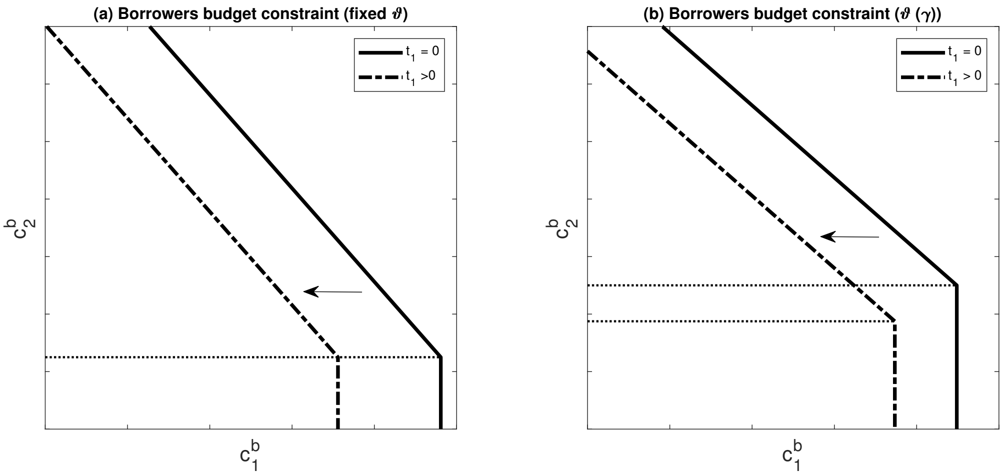
The intuition is as follows: As borrowers accumulate higher levels of debt in period one to finance consumption, their debt service costs rise. Rising debt service costs directly lower period two consumption of borrowers. Savers, who are the recipients of these debt service costs, only consume a fraction of these payments in period two and use the other fraction to increase their bequest. This results in borrowers' period two consumption declines only being partially offset by savers' increases in period two consumption, and consequently leads to a decline in period two aggregate demand.

To illustrate how transfers and rising debt service costs impact borrowers, Figure 3 plots the budget constraint of borrowers under a fixed ϑ and compares it to the constraint where $\vartheta(\gamma)$.

Under a fixed constraint in panel (a), an increase in inequality represented by $t_1 > 0$, shifts the budget constraint inwards as less resources are available in period one. However, the inward shift has no impact on the level where the kink in the budget constraint occurs. Another way of stating this is that an increase in transfers under a fixed constraint only impacts consumption of borrowers in period one, leaving period two consumption unchanged.

By contrast, the borrowing constraint which is a function of income inequality in panel (b) shows that an increase in transfers shifts the budget constraint inwards, but importantly the magnitude of the shift from kink point to kink point along the x-axis is not as large as under the fixed constraint. This is because the transfer leads to rising income inequality and allows

Figure 3: Borrowers budget constraint under fixed ϑ and $\vartheta(\gamma)$



Notes: Transfers change the equilibrium interest rate and the slope of the budget constraint, but the purpose of illustration we hold the rate fixed.

borrowers to partially offset the shift inwards by borrowing more. However, borrowing more leads to an increase in debt service costs, which is why the kink point is now at a lower level on the y-axis, as borrowers cannot obtain the same level of consumption in period two. Thus under the inequality constraint, transfers impact borrowers consumption in periods one and two.

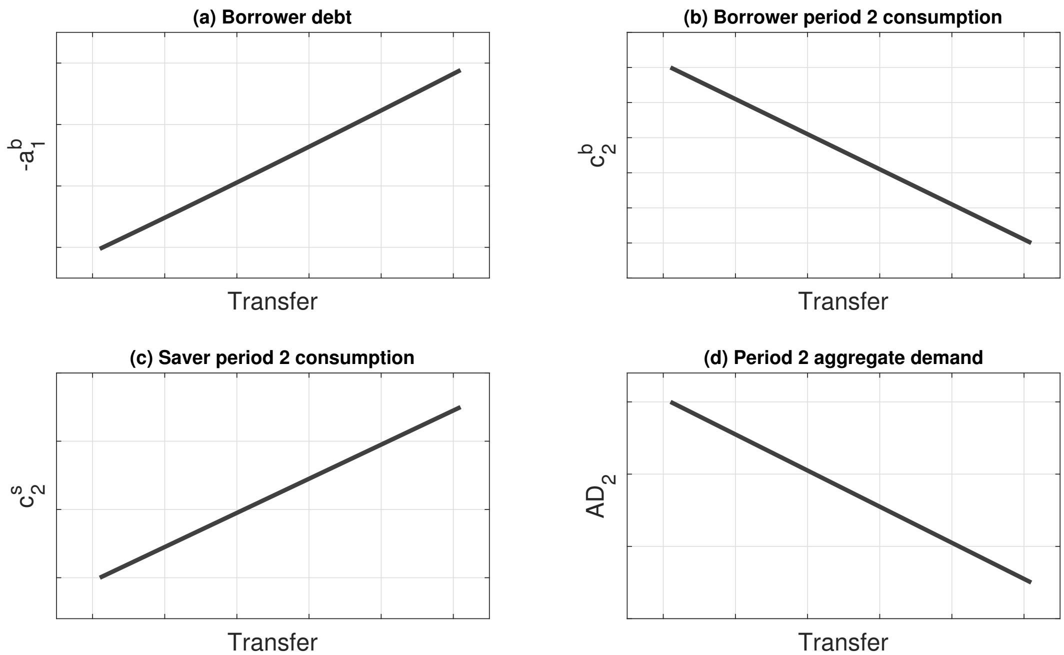
Figure 4 plots borrowers' debt in period one, borrowers' and savers' consumption in period two, and aggregate demand in period two as functions of the transfer, t_1. This figure shows that as transfers increase, borrowing debt rises, consumption of borrowers in period two falls, consumption of savers in period two rises, and aggregate demand in period two falls.

Lastly, we highlight that in our framework exogenous increases in the LTI ratio, due perhaps to financial deregulation, lower aggregate demand in period two and amplify the effects of income inequality.[4]

Proposition 3. *An increase in the loan-to-income parameter, ϕ, lowers period two aggregate demand and amplifies the effects of income inequality on period two aggregate demand.*

[4] Jensen, Petrella, Ravn and Santoro (2020) find that increasing skewness in the US business cycle since the onset of the Great Moderation can be explained by a rising loan-to-value ratios for households and firms.

Figure 4: The impact of rising income inequality on debt, consumption, and aggregate demand



Notes: In panel (a) we plot the negative of borrowers' assets, which is debt, and slopes upwards with the transfers. Since the utility function arguments are the same for savers' period two consumption and bequest, panel (c) also shows that the bequest is rising as a function of the transfer.

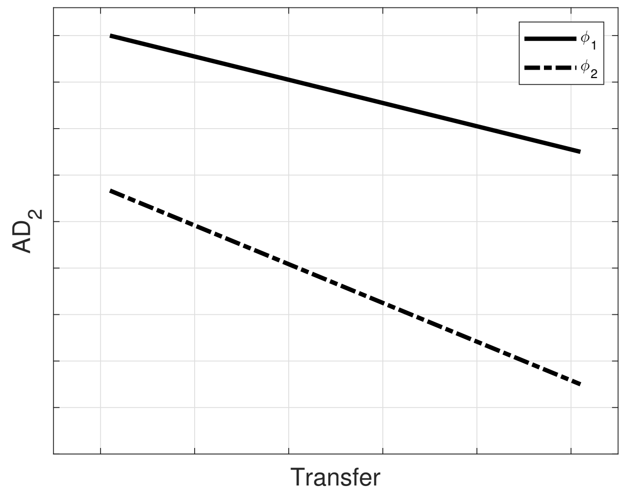
The first part of this proposition can be shown by totally differentiating (19) with respect to the LTI ratio parameter, holding changes in incomes constant, that

$$\frac{\mathrm{d}AD_2}{\mathrm{d}\phi} = -\frac{1}{2}\frac{\partial\vartheta(\gamma)}{\partial\phi}y_2^b < 0. \tag{21}$$

The second part of the proposition, that the effects of rising income inequality are amplified, follows directly from (20). The derivative of aggregate demand with respect to the transfer is increasingly negative as ϕ rises. This works through the same channel as that discussed for inequality. Rising debt levels weigh down future demand, and allow higher debt limits for any given inequality level.

Figure 5 plots aggregate demand in period two as a function of the transfer for two levels of the LTI ratio, ϕ_1 and ϕ_2, where $\phi_2 > \phi_1$. Notably, for a higher LTI ratio, period two

Figure 5: Rising LTI ratios and the effects of inequality on aggregate demand



aggregate demand is lower for any level of inequality. In addition, as income inequality increases the *two period indebted demand* effect is stronger for a higher LTI ratio, since the slope of the ϕ_2 line is steeper than the ϕ_1 line.

3 Conclusion

We presented a two period consumption-saving model with two agents where rising income inequality leads to lower equilibrium rates of interest, increased debt accumulation, and lower future aggregate demand. In contrast to the work by Mian, Straub and Sufi (2021), our results do not rely on non-homothetic preferences. Instead, we propose a borrowing constraint which is positively correlated with the level of income inequality. The stylized fact that credit (relative to output) and income inequality strongly comove in the US, with both rising since the mid-1980s, supports this feature of the borrowing constraint.

Rising income inequality in this framework was represented by within period transfers from borrowers to savers. However, one could alternatively interpret transfers as redistribution policy from savers to borrowers, which would decrease income inequality. In this case all the effects highlighted in our framework would be reversed. Redistribution could generate higher levels of equilibrium rates of interest, lower debt levels, and higher future aggregate demand.

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A Appendix

[Proposition 1]

To show that the equilibrium rate is monotonically declining with the transfer, totally differentiate (16) and hold changes in incomes constant. This gives

$$\mathrm{d}r^\star = \frac{1}{2\beta^s}\left(\frac{\frac{\partial\vartheta(\gamma)}{\partial t_1}y_2^b(1+2\beta^s)(y_1^s+t_1) - y_2^s - \vartheta(\gamma)y_2^b(1+2\beta^s)}{(y_1^s+t_1)^2}\right)\mathrm{d}t_1,$$

and noting that $\frac{\partial\vartheta(\gamma)}{\partial t_1}(y_1^s+t_1) \equiv \vartheta(\gamma)$, the expression simplifies to

$$\mathrm{d}r^\star = -\frac{1}{2\beta^s}\frac{y_2^s}{(y_1^s+t_1)^2}\mathrm{d}t_1.$$

Finally, dividing both sides of the equation by $\mathrm{d}t_1$ gives the expression in (17).

[Proposition 2]

The first part of proposition 2, to show that debt financed consumption rises with the transfer, follows directly from totally differentiating (15). To show that aggregate demand in period two falls when the transfer rises, start by totally differentiating (19) and hold changes in incomes constant. This gives

$$\mathrm{d}AD_2 = \left(\frac{\beta^s}{1+2\beta^s}\right)(y_1^s+t_1)\mathrm{d}r^\star + \left(\left(\frac{\beta^s}{1+2\beta^s}\right)(1+r^\star) - y_2^b\frac{\partial\vartheta(\gamma)}{\partial t_1}\right)\mathrm{d}t_1.$$

Dividing both sides by $\mathrm{d}t_1$ and substituting expressions for $\frac{\mathrm{d}r^\star}{\mathrm{d}t_1}$, $1+r^\star$, and $\frac{\partial\vartheta(\gamma)}{\partial t_1}$, gives

$$\frac{\mathrm{d}AD_2}{\mathrm{d}t_1} = -\left(\frac{\beta^s}{1+2\beta^s}\right)\frac{1}{2\beta^s}\frac{y_2^s}{y_1^s+t_1} + \left(\frac{\beta^s}{1+2\beta^s}\right)\frac{1}{2\beta^s}\frac{y_2^s+\phi\left(\frac{y_1^s+t_1}{y_1^s+y_1^b}\right)y_2^b(1+2\beta^s)}{y_1^s+t_1} - \frac{\phi y_2^b}{y_1^s+y_1^b}.$$

Noting that $\frac{\vartheta(\gamma)}{y_1^s+t_1} = \phi\frac{1}{y_1^s+y_1^b}$, the above simplifies to

$$\frac{\mathrm{d}AD_2}{\mathrm{d}t_1} = -\frac{1}{2}\frac{\phi y_2^b}{y_1^s + y_1^b},$$

as shown in (20).

[Proposition 3]

For proposition 3, start by totally differentiating (19), allowing for changes in the LTI parameter and holding incomes and transfers constant, which gives

$$\mathrm{d}AD_2 = \left(\frac{\beta^s}{1+2\beta^s}\right)(y_1^s + t_1)\mathrm{d}r^\star - y_2^b \frac{\partial \vartheta(\gamma)}{\partial \phi}\mathrm{d}\phi.$$

Dividing both sides by $\mathrm{d}\phi$ and noting that

$$\frac{\mathrm{d}r^\star}{\mathrm{d}\phi} = \left(\frac{1+2\beta^s}{2\beta^s}\right)\frac{\partial \vartheta(\gamma)}{\partial \phi}\frac{y_2^b}{y_1^s + t_1},$$

the above simplifies to

$$\frac{\mathrm{d}AD_2}{\mathrm{d}\phi} = -\frac{1}{2}\frac{\partial \vartheta(\gamma)}{\partial \phi}y_2^b.$$

The remaining part of the proposition follows directly from results in (20).