Identification of Trends and Cycles in Economic Time Series

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1. Introduction

In macroeconomics, it is accepted that economies undergo short-run variation apart from their long-run general behavior but the relation between the two has not yet been fully identified. The relationship between the short-run and long-run could be explained if the characteristics of short-run variation are fully known. Several hypotheses have been proposed to identify the characteristics of short-run variation. One of these hypotheses of short-run activity, as studied in conventional macroeconomic theory, is a shock originating from the imperfections in the market that causes the economy to deviate from its the long-run behavior (Keynes, 1936). Imperfections prevail because of staggered price adjustments, wage rigidities, price stickiness, and asymmetry of information (for details see, Romer, 2001). On the other hand, real business cycle theory claims that a short-run variation is also a short-run activity that causes the economy to deviate from its long-run behavior but is based on technological innovations instead of the presence of imperfections in the market[1]. Whether it is the conventional macroeconomic theory or the real business cycle approach, the economy is assumed to undergo shocks that have transitory effects on the long-run trend of the economy. An alternative perspective for identification of the characteristics of the short-run variation is to consider them as innovations that permanently effect the output (Nelson and Plosser, 1982). Under this perspective, the economy is assumed to undergo shocks that have permanent affects on the long-run trend of the economy.

All business cycle theories are based on time series concepts in identifying whether the shocks are permanent or transitory. If the shocks are permanent, the

[1] See Kydland and Prescott (1982); Long and Plosser (1983); Prescott (1986); and Black (1982).

macroeconomic series inherits a stochastic trend or else if the shocks are transitory, the macroeconomic series inherits a deterministic trend.

My analysis indicates that the short-run variations have permanent effects on the long-run trend for the U.S GDP series and this is supported by an ARIMA (1, 1, 2) process, which indicates that the series has a unit root. The analysis is pursued in two sections. In section 2, brief definitions, explanations and environment for the time series concepts of autocorrelations, stationarity, autoregressive moving average models, non-stationarity and unit root test are provided. In Section 3, a literature review on U.S GDP series is provided. Then, the dynamics of the U.S GDP series is analyzed by studying the autocorrelation (ACF) and partial autocorrelation function (PACF) of the series. This examination indicates that the GDP series is non-stationary. Afterwards, Augmented Dickey-Fuller (1984) and Phillips-Perron (1988) unit root tests are implemented and it is found that the series has a unit root. Next, ACF, PACF and residual diagnostic for the first differenced stationary series are analyzed and the diagnostics indicate towards an ARIMA (1, 1, 2) and ARIMA (2, 1, 2) model. Later, a forecast environment is designed to compare the performance of the ARIMA (1, 1, 2) and ARIMA (2, 1, 2) models. It is found that ARIMA (1, 1, 2) has more forecast accuracy. Next, the same forecast technique is implemented to compare ARIMA (1, 1, 2) with the random walk model of Nelson and Plosser (1982) and the trend-stationary model of Perron (1989). The comparison suggested that the ARIMA (1, 1, 2) model has the best forecast accuracy. Hence, the U.S GDP has a stochastic trend component as indicated by the ARIMA (1, 1, 2) model. I conclude with Section 4.

2. Concepts, Definitions and Environment

2.1 Definitions of time series

A serial data over an ordered time sequence is called time series data. A time series of an event may convey the possibility that a time pattern amongst observations is apparent. The time pattern is formed from dependence amongst observations. The dependence should be such that at least one of the lags is correlated to any other lag. This dependence characterizes the underlying time series process and allows an analyzer to specify a model to understand the implications and also to undertake future predictions.

Amongst all other time series, this research examines macroeconomic time series. Such time series may include quarterly consumption figures, seasonally adjusted GNP or GDP figures, money demand, annual investment and consumption. Time series analysis of such macroeconomic indicators would reveal the particular behavior of the economy by explaining whether the shocks to a particular economy are permanent or transitory and would allow the analyzer to undertake future predictions.

2.2 Autocorrelation Function (ACF)

The foundation of a time series analysis is that there are dependencies in the data and hence in pursing a time series analysis, the examiner should first examine whether there are autocorrelations visible in the data. This could simply be examined by calculating the correlation coefficient of ρ_j which measures the degree of correlation between any lag j. If the coefficients are statistically significant, then the series is not independently and identically distributed but correlations are apparent amongst observations.

There are two ways by which an examiner may verify the existence of autocorrelations in the data. First, one may construct a confidence interval for the sample ρ_j and test if ρ_j is off the boundaries or not, and second, one may design a hypothesis to test for the degree of the statistical significance.

In order to illustrate the former, consider the equation given below where $\{\ y_t\ \} = \{ ...y_{t-1}, y_t, y_{t+1}...\}$ is a time series as such and ρ_j is the correlation coefficient of any two lags

$$\rho_j = \frac{\text{cov}(y_t, y_{t-j})}{\sqrt{\text{var}(y_t)\,\text{var}(y_{t-j})}} = \frac{\text{cov}(y_t, y_{t-j})}{\text{var}(y_t)} = \frac{\gamma_j}{\gamma_0} \quad (2.2.1)$$

[2].

Equation (1.2.1) could be estimated from the sample as by,

$$\hat{\rho}_j = \frac{\sum_{t=j+1}^{T}(y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=1}^{T}(y_t - \bar{y})} \quad (2.2.2)$$

[2] Stationarity condition has been used where $\text{var}(y_t) = \text{var}(y_{t-j})$.

where T is the sample size and $0 \leq j \leq T-1$.

If y_t is independently and identically distributed, then the central limit theorem would imply that ρ_j has zero mean and variance of $1/T$. A 95% confidence interval can be used to verify if there are autocorrelations in the data. Based on the central limit theorem, actual confidence intervals are $\pm 1.96/\sqrt{T}$ (Zivot, 2002). If for instance, the sample ρ_j is outside such confidence intervals, the test would imply that y_t is not independently and identically distributed and hence, statistically significant correlation between those lags exists.

Alternatively, autocorrelations could be tested by the Ljung-Box (1978) test[3]. Under the null hypothesis, y_t is independently and identically distributed. The test is designed such that

$$H_0 : \rho_1 = \rho_2 = ... = \rho_m$$

$$H_a : \rho_b \neq 0$$

where *m* is the degrees of freedom and *b* is any time within the boundaries of *m*.

The test statistic is given below

$$Q(m) = T(T+2)\sum_{l=1}^{m} \frac{\hat{\rho}_j}{T-j} \qquad (2.2.3)$$

where *Q(m)* has an asymptotic chi-squared distribution with *m* degrees of freedom.

When pursuing this particular test the examiner should pay attention to two important issues. At first, the examiner should keep in mind that under the rejection of

[3] Ljung Box (1978) test is a modified Box and Pierce (1970), with a greater power than the other. See Tsay (2001), Section 2.2.

the null hypothesis, the test would not reveal which lags are correlated. It only indicates that there is autocorrelation. Secondly, the examiner should be careful in choosing the degrees of freedom. The literature indicates that the choice of the degrees of freedom is approximately equal to $\ln(T)$ for a reliable power (for details see Tsay, 2001).

Partial autocorrelation functions (PACF) demonstrate the statistical significance of the lags in the time series process. They investigate the dependence of each individual lag with the original series. Analogous to autocorrelations, partial autocorrelations can either be tested by confidence intervals or by hypothesis testing.

2.3 Definitions of Stationarity

Apart from autocorrelations in the series, the examiner should study whether the data is stationary through the necessary conditions for stationarity. These necessary conditions are certain restrictions on the moments of the underlying distribution.

There are two types of stationarity. The first type is strict stationarity, and the second type is weak stationarity. In order to prove under which conditions such properties are revealed, consider a time series $\{\ y_t\ \} = \{ \ldots y_{t-1}, y_t, y_{t+1} \ldots \}$ and $\{\ y_{t+k}\ \} = \{ \ldots y_{t+k-1}, y_{t+k}, y_{t+k+1} \ldots \}$ where k is any time shift from the origin. If the probability distribution of $\{\ y_t\ \}$ is identical to that of $\{\ y_{t+k}\ \}$, then $\{\ y_t\ \}$ is strictly stationary. That is a very strong condition, requiring that all moments of the two distributions be identical. A weaker condition is just to verify whether the first two moments of the distributions are the same. In other words, a weak stationarity criterion requires that the first two moments of the series be time invariant, so that they are identical to each other. That would imply the following:

$$E(y_t) = E(y_{t+k}) \qquad (2.3.1)$$

$$\text{var}(y_t) = \text{var}(y_{t+k}) \quad . \qquad (2.3.2)$$

Because of the difficulty of testing strict stationarity, weak stationarity is used commonly as a test for stationarity in economic time series.

2.4 Autoregressive Moving Average (ARMA) Models

2.4.1 General linear process

A time series represented by a weighted sum of past and present random shocks is called a general linear stochastic process. In such a process, all previous shocks have a specified long lasting effect on the process. The only shock that is not fully known is the present shock which is required to be estimated. The process is illustrated below, where a_t is a white noise[4],

$$y_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + ... \qquad (2.4.1)$$

$$y_t = \mu + a_t + \sum_{j=1}^{\infty} \psi_j a_{t-j} \text{ [5]}. \qquad (2.4.2)$$

2.4.2 Moving average processes

A moving average process of an order q is a linear process having a finite time domain. The process remembers the shocks that occurred in past periods down to lag q and assigns a value to each one. The only shock that the process fails to assign a certain

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[4] $E(a_t) = 0$ and $\text{var}(a_t) = \sigma_a^2$.

[5] It is worthwhile to note that in order to satisfy the stationary conditions, $\sum_{j=1}^{0} |\psi_j| < 0$.

value to is the shock that is coming from the present time. The present shock is a random shock that could attain any unpredicted value due to limited or insignificant information available. Therefore, an expectation of a current shock should be formed based on the information available along with the knowledge of previous shocks as determined by lag q in order to specify the model properly.

2.4.2.1 The first order moving average process (MA (1))

The first order moving average process utilizes the information gathered from the present and previous shocks. The process has information down to the first lag only at which some value is assigned to it as determined by θ_1. In order to understand the dynamics of the model properly, an expectation of the current shock should be formed along with the knowledge of the previous shocks. The process is shown in equation form below

$$y_t = \mu + a_t + \theta_1 a_{t-1}. \qquad (2.4.3)$$

Taking the first and the second order expectations and calculating the first auto-covariance would yield the following three identities[6]

$$E(y_t) = \mu, \qquad (2.4.4)$$

$$\text{var}(y_t) = (1 + \theta_1^2)\sigma^2, \qquad (2.4.5)$$

$$\text{cov}(y_t, y_{t-j}) = \theta_1\sigma^2. \qquad (2.4.6)$$

Considering the essentials of stationary conditions and using the above identities, we obtain the following from equation (2.2.1) for the first auto-correlation for an MA (1) process,

[6] For details see Hamilton (1994), Section 3.3.

$$\rho_1 = \frac{\theta_1\sigma^2}{(1+\theta_1^{\;2})\sigma^2} = \frac{\theta_1}{1+\theta_1^{\;2}} \;. \tag{2.4.7}$$

Autocorrelation function of MA (1) would depend on the sign of θ_1. Positive values of θ_1 indicate positive autocorrelation, whereas negative values of θ_1 indicate negative autocorrelation. In other words, a positive sign of θ_1 induces large values of y_t to be followed by larger values of y_{t+1} and a negative value of θ_1 induces large values of y_t to be followed by small values of y_t (Hamilton, 1994).

2.4.2.2 The q^{th} Order Moving Average Process (MA (q))

As mentioned previously, the q^{th} order moving average process remembers the previous shocks down to q^{th} lag and forms expectations for the current shock. It is shown in equation form below

$$y_t = \mu + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \ldots\ldots + \theta_q a_{t-q} \,. \tag{2.4.8}$$

Taking the first and the second order expectations and calculating the first auto-covariance would yield the following four identities[7]

$$E(y_t) = \mu \quad , \tag{2.4.9}$$

$$\text{var}(y_t) = (1+\theta_1^{\;2} + \theta_2^2 + \ldots\ldots\theta_q^2), \tag{2.4.10}$$

$$\text{cov}(y_t, y_{t-j}) = \{\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \ldots\ldots\theta_q\theta_{q-j}\}\sigma^2 \;, \quad \text{for } j = 1,2,\ldots\ldots q \tag{2.4.11}$$

$$\text{cov}(y_t, y_{t-j}) = 0 \qquad \text{for } j > q \,. \tag{2.4.12}$$

Considering the essentials of stationary conditions and using the above identities, we obtain the following from equation (2.2.1),

[7] For details see Hamilton (1994).

$$\rho_q = \frac{\left(\ \theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \ldots.. + \theta_q\theta_{q-j}\ \right)}{(1 + \theta_1^{\ 2} + \theta_2^2 + \ldots. + \theta_q^2)} \quad . \tag{2.4.13}$$

The dynamics of the autocorrelation function of *MA (q)* would rely on the order *q*. As explained previously, if the order is one, then the sign of θ_1 would be indicative of the autocorrelation function. If the order is two, then the autocorrelation function lies within the segments of three curves as determined by the same criterion for the correlation coefficients[8].

2.4.3 Autoregressive Processes

An autoregressive process of order *p* utilizes the information gathered from past events. All of the information about past events is known down to lag *p* and a proportion of each has been reflected in the process. Analogous to moving average processes, the only segment which remains to be forecasted is the current shock. Thus, in order to specify the model properly, an expectation of a current shock should be formed based on the information available along with the knowledge of previous events as determined by lag *q*.

2.4.3.1 The first order autoregressive process (AR (1))

The first order autoregressive process utilizes the information gathered from the previous event. The process assigns a value to the previous event only based on the information available. As mentioned earlier, in order to understand the dynamics of the

[8] For more details on this subject see Box and Jenkins (1994), Chapter 3.

model properly, an expectation of the current shock should be formed along with the knowledge of the previous event. The process is shown in equation form below

$$y_t = \phi_0 + \phi_1 y_{t-1} + a_t \qquad (2.4.14)$$

with the characteristic equation of

$$\phi(B) = 1 - \phi_1 B \qquad (2.4.15)$$

where B is the lag operator.

Taking the expectation of equation (2.4.14) and solving for $E(y_t)$

$$E(y_t) = \frac{\phi_0}{1-\phi_1} \quad . \qquad (2.4.16)$$

Taking the square of equation (2.4.14) and taking the expectation would yield the following

$$\operatorname{var}(y_t) = \frac{\sigma^2}{(1-\phi_1^2)} \qquad (2.4.17)$$

where $|\phi_1| < 1$.

Also, multiplying equation (2.4.14) by y_{t-k} and taking the expectation would yield the following

$$\gamma_j = \phi_1 \gamma_{j-1} \qquad (2.4.18)$$

and hence

$$\rho_j = \phi_1 \rho_{j-1} . \qquad (2.4.19)$$

It is also known that $\rho_0 = 1$, and hence $\rho_j = \phi_1^j$. Therefore, the autocorrelation function of an AR (1) process decays at an exponential rate of ϕ_1 (Tsay, 2001).

2.4.3.2 The second order autoregressive process (AR (2))

The second order autoregressive process utilizes the information gathered from the last two events as measured by the coefficients, ϕ_1 and ϕ_2. As mentioned earlier, in order to understand the dynamics of the model properly, an expectation of the current shock should be formed along with the knowledge of the past two events. The process is shown in equation form below

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t \qquad (2.4.20)$$

with the characteristic equation of

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 \qquad (2.4.21)$$

where B is the lag operator.

Taking the expectation of the above and solving for $E(y_t)$

$$E(y_t) = \frac{\phi_0}{1 - \phi_1 - \phi_2} \qquad (2.4.22)$$

where $\phi_1 + \phi_2 \neq 1$ for stationary conditions to satisfy.

Also, multiplying equation (2.4.20) by y_{t-k} and taking the expectation would yield the following where γ_j denotes the lag j covariance,

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \qquad (2.4.23)$$

and hence

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} \qquad (2.4.24)$$

with starting values of $\rho_0 = 1$ and $\rho_1 = \frac{\phi_1}{1 - \phi_2}$.

Equation (2.4.24) could also be written as $\rho_j = C_1 R_1^k + C_2 R_2^k$ where $1/R_1$ and $1/R_2$ are the roots of the characteristic equation (2.4.21) (Tsay, 2001),

$$x = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad . \tag{2.4.25}$$

The autocorrelation function either exhibits damped exponentials or pseudo periodic behavior depending on the sign of $\phi_1^2 + 4\phi_2$. If $\phi_1^2 + 4\phi_2 \geq 0$, then the ACF consists of a composition of damped exponentials. That is, it either decays smoothly or decays as alternating signs depending on the signs of ϕ_1 and ϕ_2 (Box and Jenkins, 1970). On the other hand, in case $\phi_1^2 + 4\phi_2 \leq 0$, ACF exhibits pseudo periodic behavior. That is, it exhibits damping sine and cosine waves (Tsay, 2001).

2.4.3.3 The p^{th} order autoregressive process (AR (p))

As mentioned earlier, the p^{th} order autoregressive process remembers the previous events down to the p^{th} lag and forms expectations for the current shock. It is shown in equation form below

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots\ldots + \phi_p y_{t-p} + a_t \tag{2.4.26}$$

with the characteristic equation of

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 + \ldots\ldots + \phi_p B^p \tag{2.4.27}$$

where B is the lag operator.

Taking the expectation of (2.4.26) and solving for $E(y_t)$

$$E(y_t) = \frac{\phi_0}{1 - \phi_1 - \ldots\ldots - \phi_p} \tag{2.4.28}$$

where $\phi_1 + ... + \phi_p \neq 1$ for stationary conditions to satisfy.

Also, multiplying equation (2.4.26) by y_{t-k} and taking the expectation would yield the following

$$\gamma_j = \phi_1 \gamma_{j-1} + ... + \phi_p \gamma_{j-p} \qquad (2.4.29)$$

and hence

$$\rho_j = \phi_1 \rho_{j-1} + ... + \phi_p \rho_{j-p}. \qquad (2.4.30)$$

Equation (2.4.30) cannot be written as root form unless *p* is known. Therefore, you may not come up with a firm conclusion regarding the shape of the ACF without knowing the order of AR process.

Fortunately, the order of AR may be checked by inspecting partial autocorrelation coefficients of the equations below

$$y_t = \phi_0 + \phi_{11} y_{t-1} + a_t$$

$$y_t = \phi_0 + \phi_{21} y_{t-1} + \phi_{22} y_{t-2} + a_{2t}$$

.

.

$$y_t = \phi_0 + \phi_{p1} y_{t-1} + \phi_{p2} y_{t-2} + ... + \phi_{pp} y_{t-p} + a_{pt}$$

where ϕ_{jj} represents the coefficients of partial autocorrelations. For instance, if the process is AR (1), then ϕ_{11} is nonzero and the rest of the coefficients are zero. Or if the process is AR (1), then ϕ_{11} is nonzero and the rest of the coefficients are zero.

2.4.4 Autoregressive Moving Average Processes

Autoregressive moving average processes are the mixture of autoregressive processes and the moving average process. That is, they remember the information gathered from past events as determined by the memory index of lag *p*, and past shocks as determined by a memory index of lag *q*. All of the information about past events and shocks is known and some proportion of each has been carried in the process. Analogous to moving average processes and autoregressive processes, the only part which remains to be forecasted is the current shock. Thus, in order to specify the model properly, an expectation of a current shock should be formed based on the information available along with the memory of index *p* of previous events and the memory of index *q* of the past shocks.

2.4.4.1 ARMA (1, 1)

ARMA (1, 1) utilizes the information gathered from the previous event and previous shock. The process has information down to the first lag on past events and past shocks carrying out a proportion of each. As mentioned earlier, in order to specify the model correctly, an expectation of the current shock should be formed along with the knowledge of the previous event and previous shock. The process is shown in equation form below,

$$y_t = \phi_0 + \phi_1 y_{t-1} + \theta_1 a_{t-1} + a_t \,. \qquad (2.4.31)$$

Normalizing y_t, multiplying above by y_{t-k} and taking expectations, the following three identities are obtained

$$\gamma_0 = \phi\gamma_1 + (1 - \theta_1(\phi_1 - \theta)\sigma_a^2 \qquad \text{for } j=0, \qquad (2.4.32)$$

$$\gamma_1 = \phi\gamma_0 - \theta\sigma_a^2 \qquad \text{for } j=1, \qquad (2.4.33)$$

$$\gamma_j = \phi_1\gamma_{j-1} \qquad \text{for } j \geq 2\,. \qquad (2.4.34)$$

Solving equations (2.4.32) and (2.4.33) for $\gamma_0(\text{var}(y_t))$ yields

$$\gamma_0 = \frac{(1 - 2\theta_1\phi_1 + \theta_1^2\ )}{1-\phi^2}\sigma_a^2 \quad . \qquad (2.4.35)$$

Each recursive substitution would yield[9]

$$\gamma_k = \frac{(1-\theta_1\varphi_1)(\phi_1 - \theta_1)}{1 - 2\theta_1\phi_1 + \theta^2}\phi^{k-1} \qquad \text{for } k \geq 1\,. \qquad (2.4.36)$$

Equation (2.4.36) illustrates that the autocorrelation function of ARMA (1, 1) decays at an exponential rate of φ_1 starting at an initial value of ρ_1 depending on the value θ_1 (Cryer, 1985).

2.4.4.2 ARMA (p, q)

As mentioned earlier, the ARMA (p, q) remembers the past events down to the p^{th} lag and remembers the past shocks down to the q^{th} lag and forms expectations for the current shock. It is shown in equation form below

$$y_t = \phi_0 + \phi_1 y_{t-1} + \ldots\ldots + \phi_p y_{t-p} + \theta_1 a_{t-1} + \ldots\ldots\ldots + \theta_p a_{t-p} \ldots\ldots + a_t\,. \qquad (2.4.37)$$

Normalizing y_t, multiplying above by y_{t-k} and taking expectations

$$\gamma_j = \phi_1\gamma_{j-1} + \phi_1\gamma_{j-2} + \ldots\ldots\ldots + \phi_p\gamma_{j-p} \qquad \text{for } j=q+1, q+2, \ldots \quad (2.4.38)$$

and hence

$$\gamma_j = \phi_1\rho_{j-1} + \phi_1 p_{j-2} + \ldots\ldots\ldots + \phi_p p_{j-p} \ . \qquad (2.4.39)$$

[9] For detailed explanation see Cryer (1985).

2.6 Trend Stationary versus Stochastic Trends

ARMA models are stationary, however many financial and economic data exhibit non-stationary dynamics. Examples include interest rates, exchange rates, gross domestic product (GDP), consumption and investment series.

There are two types of non-stationary models. The first type is trend-stationary; the second type is stochastic trend. They both have a trend component but trend-stationary models inherit trend reverting dynamics, whereas stochastic trends do not. The behavior of trend-stationary series could be anticipated, whereas stochastic trends could not.

2.5.1 Trend Stationary Models

A simple trend stationary model is illustrated below where α is a constant; ν is the rate of time and e_t is an error term that is independently and identically distributed with a mean of zero and a variance of σ_a^2.

$$y_t = TR + e_t, \tag{2.5.1}$$

$$TR = \alpha + vt. \tag{2.5.2}$$

Taking the expectation and variation of equation (2.5.1) would yield the following two equations

$$E(y_t) = \alpha + vt \tag{2.5.3}$$

$$\text{var}(y_t) = \sigma_a^2. \tag{2.5.4}$$

The above equations prevail because $E(e_t) = 0$, and the variation of y_t is that of e_t. Equation (2.5.3) is due to the assumption that e_t is independently and identically distributed with a zero mean. Equation (2.5.4) is due to the only random variable in the process, e_t having a variation of σ_a^2 .

The findings so far illustrate that a trend-stationary model passes on a smooth trend on average. Any movement around a trend is temporary and the series comes back to the trend. In a macroeconomic data such as the gross domestic product, a temporary movement is called short-run economic activity and can be treated as noise. The temporary short-run activity arises because there are rigidities in the economy. The rigidities prevail because there are market imperfections such as, missing markets, asymmetric information, and staggered price adjustments. On the other hand, in the long-run the series always comes back to its trend because all of the imperfections in the market cannot last forever[10].

Also, the series should be transformed to a stationary series if further analysis is desired, such as an examination of autocorrelation function. In order to transform models into stationary time series, de-trending is required. That simply means taking the trend out from the original series as illustrated below so that the series reverts to constant α .

$$y_t - vt = \alpha + e_t . \qquad (2.5.4)$$

2.5.2 Stochastic Trends

In order to understand the dynamics of a stochastic trend, a simple trend stochastic model is illustrated below where the notations are identical to the previous section.

$$y_t = TR + b_t \qquad (2.5.5)$$

[10] For more information, see Chapter 4 and Chapter 5 in Romer (2001).

where

$$b_t = b_{t-1} + e_t \quad (2.5.6)$$

and

$$TR = \alpha + vt\,. \quad (2.5.7)$$

By using the properties of AR (1), equation (2.5.5) could be re-written as,

$$y_1 = \alpha + v + b_0 + e_1 \quad (2.5.8)$$

$$y_2 = \alpha + 2v + b_1 + e_2 = \alpha + 2v + b_0 + e_1 + e_2 \quad (2.5.9)$$

$$y_t = TR + b_0 + e_t + e_{t-1} + e_{t-2} .... + e_1\,. \quad (2.5.10)$$

Taking the first and the second order expectations of equation (2.5.10) would yield the following,

$$E(y_t) = \alpha + vt + b_0 \quad (2.5.11)$$

$$\mathrm{var}(y_t) = t\sigma_a^{\ 2}\,. \quad (2.5.12)$$

Equation (2.5.11) is due to the fact that $E(e_t) = 0$ and the expectation of b_0 is itself ($E(b_0) = b_0$). Therefore the expectation of y_t is TR plus b_0 ($E(y_t) = TR + b_0$).

Equation (2.5.12) is because the only random variable in the process is e_t having a variation of σ_a^2. Since there are t numbers of e_t in the process, the variation of y_t is t times σ_a^2 ($\mathrm{var}(y_t) = t\sigma_a^{\ 2}$).

The result of equation 1.5.12 illustrates that the second moment is not well defined due to the presence of the time component. This points out that the process is unpredictable. In macroeconomic literature, this situation arises when the shocks are permanent so that the series would not revert to the initial trend disabling the analyzer to observe the general dynamics of the model. In macroeconomic literature it is accepted that such shocks are not simple demand and supply shocks but complicated shocks like technology[11]

Furthermore, the series should be converted to stationary if further analysis is desired. In order to convert such models to a stationary time series, the examiner should take the differences of lags until the series is stationary.

2.6 Unit Root Tests

Unit root tests exploit the mean-reverting dynamics of the time series data. Under the null hypothesis, the series has no mean-reverting dynamics and hence the process has a unit root. Conversely, in the alternative hypotheses, the series has a mean-reverting dynamic and hence the process has no unit root.

2.6.1 A test for AR (1) without any constant (Dickey-Fuller)

A design of a unit root t-test for a possible AR (1) model without any constant is called the Dickey-Fuller test and is shown below

$$y_t = \phi_1 y_{t-1} + a_t \qquad (2.6.1)$$

$$H_0 : \phi_1 = 1 \Rightarrow I(1) \qquad (2.6.2)$$

$$H_1 : \phi_1 < 1 \Rightarrow I(0) \qquad (2.6.3)$$

where $\hat{\phi}_1$ is the least square estimate and $SE(\hat{\phi}_1)$ is the standard error. Dickey and Fuller (1979) showed that under the null hypothesis, the process is non-stationary with a

[11] See Romer (2001), Chapter 5.

variance of $t\sigma^2$ and first differencing is appropriate to convert the series to a stationary series. The test is shown below

$$t_{\phi_1=1} = \frac{\hat{\phi}_1 - 1}{SE(\hat{\phi}_1)} \quad . \qquad (2.6.4)$$

However, the above equation is not sustainable since under the null hypothesis sample moments converge to fixed constants. Instead, Phillips (1987) showed that the correct test statistic is

$$t_{\phi_1=1} \to \frac{\int_0^1 W(r)dW(r)}{\sqrt{\int_0^1 W(r)^2 dr}} \qquad (2.6.5)$$

at which sample moments converge to Brownian Motion (Zivot and Wang, 2002).

2.6.2 A test for AR (1) with time components

A design of a unit root t-test for a possible AR (1) with trend is shown below

$$y_t = \phi_0 + \phi_1 y_{t-1} + a_t \qquad (2.6.6)$$

where ϕ_0 exhibits a time component.

$$H_0 : \phi_1 = 1 \Rightarrow I(1) \text{ with trend} \qquad (2.6.7)$$

$$H_1 : \phi_1 < 1 \Rightarrow I(0) \text{ with deterministic trend.} \qquad (2.6.8)$$

Under the null hypothesis, the process is non-stationary with a time trend component[12] and first differencing is appropriate to convert the series to a stationary series. The t-test

[12] However, the trend component is not deterministic. There is no trend reverting dynamics. In Econometrics literature such a trend component is named as drift.

design differs from equation (2.6.5) mainly in how it accommodates for the constant term[13].

2.6.3 Augmented Dickey-Fuller Unit Root Tests

So far, the unit root tests explained were for simple AR (1) models but were not suitable for ARMA (p, q) models. The model's tests are modified by Said and Dickey (1984) to capture the dynamics of ARMA (p, q) for an unknown order of p and *q*. Two types of test regression are considered,

$$y_t = \omega_t + \phi y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta y_{t-j} + e_t \qquad (2.6.9)$$

$$\Delta y_t = \omega_t + \delta y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta y_{t-j} + e_t \qquad (2.6.10)$$

where ω_t is a vector that captures the constant or trend component and e_t is an error term that is independently and identically distributed.

Under regression (2.6.9), the null hypothesis is $\phi_1 = 1$ which implies that the series has a unit root. The test statistic is that of (2.6.2). The normalized bias statistic is shown below

$$t_n = \frac{T(\hat{\phi} - 1)}{1 - \lambda_1 - \dots\dots - \lambda_p} \quad . \qquad (2.6.12)$$

Under regression (2.6.10), the null hypothesis is $\delta = 0$ which tests the significance of each lag. The test statistic is the usual test statistic when $\delta = 0$. The normalized bias statistic is shown

[13] For details see Hamilton (1994), Section 17.3 Case 4.

$$t_n = \frac{T\delta}{1-\lambda_1-......-\lambda_p} \quad .$$

The subtle part in this analysis is the selection of lag *p*. Dickey and Fuller (1984) suggested that a possible lag *p* is chosen and a model is specified by using regressions (2.6.9) and (2.6.10). After, another model is specified with lag *p*-1. If the two models specified exhibit similar dynamics, then lag p chosen may be correct. (Hamilton, 1994)

However, the selection of *p* described above is arbitrary and may bias the test if *p* is chosen to be very large or very small. For that reason, Zivot and Wang (2002) considered another type of procedure. First choose[14]

$$p_{\max} = \left[12(\frac{T}{100})^{1/4} \right] \quad . \tag{2.6.11}$$

Afterwards, an ADF test regression is performed as in regression (2.6.10) testing for the significance of each lag up to $p_{\max}$. If the last lag is statistically significant as granted by the test statistics greater than the absolute value of 1.6, then choose $p = p_{\max}$ and perform a regression as in (2.6.9). Otherwise, reduce the lag length by one and start the procedure again.

2.6.4 Phillips-Perron Unit Root Tests

The design of Phillips-Perron (1988)'s unit root tests are similar to that of Augmented Dickey-Fuller (1984)'s unit root tests but different in accommodating moving average terms and heteroskedasticity in the errors. They both have the same type of test statistic asymptotic distributions under the null and they both deal with the deterministic terms in the same way. Augmented Dickey-Fuller (ADF) tests

[14] For details see Schwert (1989).

accommodate moving average terms but disregard heteroskedasticity in the error terms. Conversely, Phillips-Perron (PP) tests accommodate heteroskedasticity in the error terms but disregard moving average terms. Furthermore, lag specification is necessary for ADF but not necessary for the PP test. The test regression for the PP is shown below

$$\Delta y_t = \omega_t + \delta y_{t-1} + u_t \tag{2.6.12}$$

where u_t is an error component that may be heteroskedastic. Under the null hypothesis, $\delta = 0$ and the first difference of y_t is a stationary process. Zivot and Wang (2002) shows that the test statistic and normalized bias statistics are modifications of $t_{\delta=0}$ and $T\hat{\delta}$. They are shown below

$$Z_t = \sqrt{\frac{\hat{\gamma}_0}{\hat{\tau}^2}} * t_{\delta=0} - \frac{1(\hat{\tau}^2 - \hat{\gamma}_0)}{2\hat{\tau}^2} * \left(\frac{T * SE(\hat{\delta})}{\hat{\gamma}_0} \right) \tag{2.6.13}$$

$$Z_\delta = T\hat{\delta} - \frac{T^2 * SE(\hat{\delta})}{2\hat{\gamma}_0}(\hat{\lambda} - \hat{\gamma}_0^2) \tag{2.6.14}$$

where $\hat{\gamma}_0$ and $\hat{\tau}^2$ denote the least square estimate sample variance of u_t on a different basis[15] .

[15] For details, see Hamilton (1994), Section 17.6.

3. An Application to Macroeconomic Time Series

3.1 Literature Review

The applications of time series concepts to macroeconomic series in literature focus on identification of trends and cycles in the series. Nelson and Plosser (1982) performs Dickey Fuller's (1979) unit root test on the logarithm of real quarterly U.S GNP from 1909 to 1970. They formally accept the null that the GNP series has a unit root and hence the process is similar to a random walk. Even though they accept the fact the economy experiences monetary and fiscal shocks that have transitory effects on the trend, they claim that they are dominated by real shocks that have permanent effects.

Perron (1989) designs a unit root test for U.S real quarterly GNP[16] with the null hypothesis that the series has a unit root with possibly a non-zero drift for U.S GNP. He allows for a structural change[17] in his regression by adding dummy variables that would become zero in times of that change. Unlike Nelson and Plosser (1982), he formally rejects the null and concludes that the trend is stationary but there is a break in slope when the structural changes occur, postulating that those shocks were not a realization of the series. Thus, he concludes that the shocks that the economy experiences are transitory.

Zivot and Andrews (1992) modify Perron's (1989) unit root test accommodating a structural change that is unknown for U.S GNP. This accommodation makes his result

[16] He uses the data set of the Nelson and Poller (1982) and postwar quarterly GNP series.
[17] He refers to the Great crash of 1929 and the oil price shock of 1973.

robust to any structural change. Zivot and Andrews (1992) conclude that the trend is deterministic, finding even more evidence than Perron (1989).

Leybourne and McCabe (1994) develop a unit root test in which the null is that of the stationary ARIMA (p, 0, 0)[18] process against the alternative hypothesis of a non-stationary ARIMA (p, 1, 1) for U.S GNP series. Unlike the Dickey Fuller test, they claim that the selection of p would affect the test. Nelson and Murray (1997) choose *p=2*, and conduct the Leybourne and McCabe test for the postwar U.S real GDP. They reject the null and conclude that the model is that of ARIMA (2, 1, 1). Thus, the series contains a unit root.

3.2 Dynamics of U.S GDP series.

The dynamics of the U.S.GDP series are analyzed by conducting a Ljung Box (1978) test, inspecting the plot of the series and analyzing the plot of ACF and PACF.

For the U.S GDP series, the Ljung Box (1978) test is implemented for 5 degrees of freedom[19]. The p-value for this test is zero which strongly suggests that the null hypothesis cannot be rejected. Thus, there are autocorrelations present in the data.

As shown in Figure 1, the U.S GDP series clearly exhibits non-stationarity. The non-stationarity dynamics may result from the existence of a time component in its mean or the presence of a time component in its variance or both. Since it is observed in the series that the data is increasing over time, the mean embodies a time component. Thus,

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[18] It is worthwhile to note that under the null, MA component has a unit root.
[19] As explained in section 1.2 the degrees of freedom should be selected so that it is approximately equal to *ln(m).* ln a sample size of 228 that corresponds to five.

the series has a trend component. However, no further details could be revealed regarding the dynamics of the series.

The ACF on the top of Figure 2 not only suggests that there are autocorrelations apparent in the data, but that the series is non-stationary. The autocorrelations are apparent because autocorrelation coefficients are outside of confidence bands. Also, the data is non-stationary because there is persistence in autocorrelations. However, the figures do not reveal any further information regarding the characteristics of non-stationarity.

The above findings illustrate that there are dependencies in the observations and that the series has a non-stationary dynamics. It seems that the dependencies and non-stationarity are caused by the trend, however it may also be caused by the variance. The findings do not reveal any further information regarding the characteristics of the variance.

3.3 Unit Root tests

The Augmented Dickey Fuller (1984) and Phillips-Perron (1988) (PP) tests are implemented for the U.S GDP series[20]. Under the null hypotheses in both tests, the series inherits a stochastic trend component, and under the alternative hypothesis the series inherits a deterministic trend component.

The PP test yields a t-value of -2.491 through equations (2.6.13). That corresponds to a p-value of 0.3326 representing moderate evidence to accept the null.

[20] All calculations are carried out in S-Plus.

Thus, the series contains a unit root and first differencing is appropriate to transform the series into a stationary series.

As mentioned earlier, the specification of lag p in the Dickey Fuller (1984) test requires recursive steps. Based on equation (2.6.11), $p_{\max}$ is chosen to be 15 for a sample size of 228 and the degree of significance of it is calculated by (2.6.4) which results from regression (2.6.10). The results when p is 15 are reported in Table 1. A t-value of 0.3422 indicates that the last lag is insignificant since it is lower than the critical value of absolute value of 1.6. Thus, lag 15 should be reduced by 1 to 14 and the process should be repeated. The results when p is 14 are reported in Table 2. A t-value of -0.6007 indicates that the last lag is insignificant since it is lower than the critical value of absolute value of 1.6. Thus, lag 14 should be reduced by 1 to 13 and the process should be repeated. The results when p is 13 are reported in Table 3. A t-value of -2.8270 indicates that the last lag is significant since it is higher than the critical value of absolute value of 1.6. Therefore, lag p should be set to 13.

As shown in Table 3, the Dickey Fuller (1984) test, when p=13 yields a t-value of -2.372 through equation (2.6.14). This corresponds to a p-value of 0.3326 representing moderate evidence to accept the null. Thus, the series contains a unit root and requires first differencing to exhibit stationarity.

3.4 Possible Model Specifications

3.4.1 Inspection of ACF and PACF

The autocorrelation function of the first differenced stationary U.S GDP series is shown on top of Figure 4. It exhibits damping cosine and sine waves. This is a characteristic of an AR (2) process. Thus, the series may contain an autoregressive component of order 2.

The partial autocorrelation function of the first differenced stationary U.S GDP series is shown in the bottom panel of Figure 4. The partial autocorrelation function of the underlying GDP series is similar to its autocorrelation function in exhibiting damping waves like cosine and sine. The duality between the moving average process and the autoregressive process indicates that a moving average component of order 2 may exist.

The partial autocorrelation function of the G.D.P series also indicates that the 1st and the 12th lags are statistically significant. The significance of the 12th lag may be captured by either MA (1) or MA (2) added to AR (1) or AR (2) keeping in mind that either AR (2) or MA (2) should exist in the process as indicated by ACF and PACF.

In brief, the inspection of ACF and PACF of the first differenced U.S GDP series may suggest that the series is possibly an ARMA (2, 1), ARMA (2, 2) or ARMA (1, 2). For the original U.S GDP series; it is possibly an ARIMA (2, 1, 1), ARIMA (2, 1, 2) or ARIMA (1, 1, 2).

3.4.2 Residual Diagnostics

Residual diagnostic indicates whether the model is correctly specified or misspecified. If the residuals are independently and identically distributed, then the model is correctly specified. Otherwise, there are autocorrelations in the residuals and the model is misspecified.

In this analysis, three types of techniques are discussed to test whether residuals behave as an independently and identically distributed process. The first technique involves detecting ACF for autocorrelations amongst residuals; the second technique involves inspecting PACF for the degree significance of the lags; the third technique involves conducting the Ljung-Box test for testing autocorrelations amongst residuals. In order to conclude that the residuals are independently and identically distributed, both techniques should yield similar results.

The residual diagnostic is conducted for the ARIMA (2, 1, 2) and ARIMA (1, 1, 2) models[21]. Figure 6 illustrates the results for ARIMA (2, 1, 2) and Figure 5 illustrates the results for ARIMA (1, 1, 2). On the second panel of Figure 6, ACF of the residuals indicates that the residuals are independently and identically distributed for ARIMA (2, 1, 2) since all the correlation coefficients lie within confidence intervals. On the contrary, on the second panel of Figure 5, ACF of the residuals indicates that the residuals may not be independently and identically distributed for ARIMA (1, 1, 2) since the correlation coefficient for lag 2 is outside of the confidence interval. On the third panel of Figure 6, the PACF of residuals indicates the significance of 4th, 5th and 12th lag for ARIMA (2, 1, 2). On the third panel of Figure 5, the PACF of residuals indicate the

[21] The residual Diagnostic for ARIMA (2, 1, 1) is omitted because when estimated, the coefficients becomes greater than one suggesting that the process becomes explosive.

significance of the 2^{th} lag and 12^{th} only for ARIMA (1, 1, 2). On the last panel of Figure 6, the Ljung Box test indicates that the residuals of ARIMA (2, 1, 2) have no autocorrelations as supported by a p-value lower than 0.05 for the 10^{th} lag. On the last panel of Figure 5, the Ljung Box test indicates that the residuals of ARIMA (1, 1, 2) have no autocorrelations as supported by a p-value that is almost zero for 10 consecutive lags.

The findings so far illustrate that each model has its strengths and weaknesses in comparison to one another. ARIMA (1, 1, 2) has its strength in the PACF and p-values for the Ljung Box test but its weakness in the ACF of autocorrelations. On the other hand, ARIMA (2, 1, 2) has its strength in the ACF of residuals but its weakness in the PACF. Thus, the findings of residual analysis fail to distinguish one model from the other and further analysis is required for model selection.

3.5 Forecasts

A useful method of comparing different models is to forecast the actual data from the in-sample series. The comparison is finalized in three steps. First, an in-sample data is generated from the original series by excluding the last few observations. Second, the out-of sample observations are estimated by the original models proposed. Third, a model that has the most accurate prediction is chosen.

3.5.1 Forecast for ARIMA (1, 1, 2) and ARIMA (2, 1, 2)

A comparison of ARIMA (2, 1, 2) and ARIMA (1, 1, 2) is also conducted in three steps. First, the last 20 observations, which correspond to the last 5 quarters, are excluded from the series and a new set of observations is generated ending in the 1st quarter of 1999. Second, the quarters that are between the first of 1999 and the last of 2003 are forecasted. Third, the model that has the most accurate prediction is chosen.

Figure 7 represents a prediction of ARIMA (2, 1, 2). Even though the actual data appears to lie within the distribution confined by the standard deviation, the model fails to capture the general dynamics of the series. It seems that the trend in the actual data is larger than the forecasted trend. Thus, the model fails to forecast the quarters accurately.

Figure 8 represents a prediction of ARIMA (1, 1, 2). The predicted values are around the predicted trend indicating that there are no significant deviations apart from the data. Thus, the model succeeds in forecasting quarters accurately and is chosen over ARIMA (2, 1, 2).

3.5.2 Forecast for trend-stationary and stochastic trend models

In this analysis, the model proposed for the U.S GDP series is ARIMA (1, 1, 2), in addition to the ARIMA (2, 1, 1) model proposed by Nelson and Murray (1997) and the random walk model proposed by Nelson and Plosser (1982). Both of these models have a unit root in their representation and require first differentiation to exhibit stationary dynamics. However, there are also other models proposed by Perron (1989) and Zivot (1992) for U.S GNP[22] which contain a deterministic trend in their representation and require de-trending to exhibit stationary dynamics. To further explore which model

[22] The dynamics of gross domestic national product series should be almost identical to that of gross national product because gross national product differs from the gross domestic product in separating nation income from foreign income.

correctly accommodates the dynamics of the particular series, a method of forecast as described earlier may be implemented[23].

Figure 9 represents a Perron (1989) type of model which contains a deterministic trend with an autoregressive component of order 1. Even though the model seems to capture the trend in the data, there are some deviations visible apart from the actual data. Thus, the model fails to forecast the GNP series accurately.

Figure 10 represents a Nelson and Plosser (1982) type of a random walk model. Even though the actual data appears to lie within the distribution confined by the standard deviation, the model fails to capture the general dynamics of the series. It appears that the model estimates the same value for each quarter. Thus, the model fails to forecast the quarters precisely.

The model that has the only precise estimates is ARIMA (1, 1, 2). Thus, the results of the findings of the analysis are correct. The U.S GDP series has a stochastic trend in its representation and is best described ARIMA (1, 1, 2).

[23] A method of forecast is not implemented for ARIMA (2, 1, 1) because when estimated, the coefficients becomes greater than one suggesting that the process becomes explosive.

4. Conclusions

The purpose of this analysis is to specify an ARIMA (p, d, q) model to examine the impact of short-run variations on the long-run trend for the U.S GDP series. The analysis is divided into two parts. In the first part, time series concepts are briefly introduced to study an ARIMA (p, d, q) model. In the second part, a literature review on U.S GDP series is provided together with the empirical findings.

The models that are proposed in the literature[24] are compared against the ARIMA (1, 1, 2) model by using a forecast method which involves creating an in-sample data by excluding the last few observations and forecasting the out-of-sample data. As a result, it is found that amongst all the models, ARIMA (1, 1, 2) yields the best forecast accuracy and hence the dynamics of the U.S GDP series is best described by an ARIMA (1, 1, 2) model. This specific model points out that in every period the growth rate is affected by the previous event and the last two shocks, and more importantly it points out that the U.S economy undergoes short-run fluctuations that have permanent effects on the long-run trend of the economy.

There can be three limitations of the findings of this analysis. One limitation may be due to the brevity of the sample making it difficult to fully identify the long-run dynamics of the series. Usually an infinite sample is needed to characterize the long-run dynamics of the series (Romer, 2001). Thus, an attempt to claim that the long-run trend is stochastic under a sample size of 228 may be too assertive. The second limitation is that there may be a long memory process in the U.S GDP as indicated by the ACF in the top panel of Figure 2. Thus, a partially integrated process may have been even more

[24] Except for ARIMA (2, 1, 1), AR (1) coefficient is greater than one.

appropriate to capture the dynamics of the underlying GDP series. The third limitation is that the studied models here are linear. As Potter (1995) suggested, a nonlinear model outperform the standard linear models for the U.S GNP.

Further research could be conducted by using longer time spans; the dynamics of the long memory could be further examined and also a non-linear model could be designed. Overall, these possible models can be compared by the forecast technique described and the one that has the best predictive can be studied.

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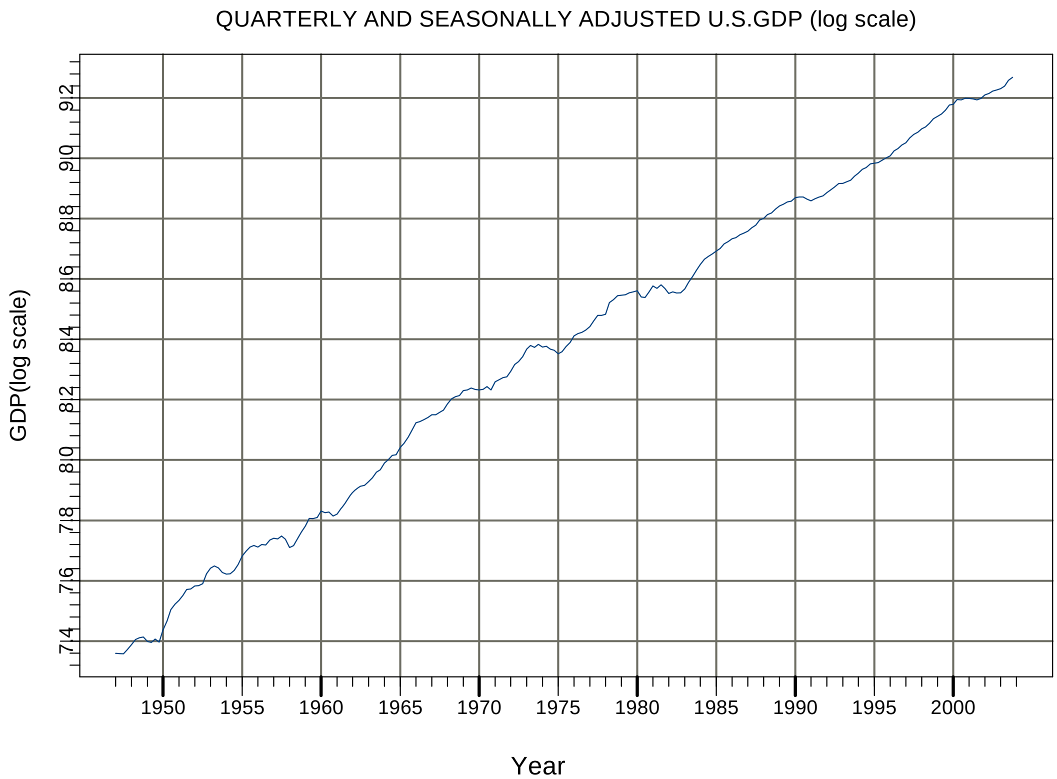

FIGURE 1: Quarterly and seasonally adjusted Gross Domestic Product in log scale. The sample period is from January 1, 1947 to October 1, 2003. x-axis represents the years of the sample and y-axis represents the sample in log scale. The presence of trend indicates non-stationarity. Source: U.S. Bureau of Economic Analysis, National Income and Product Accounts.

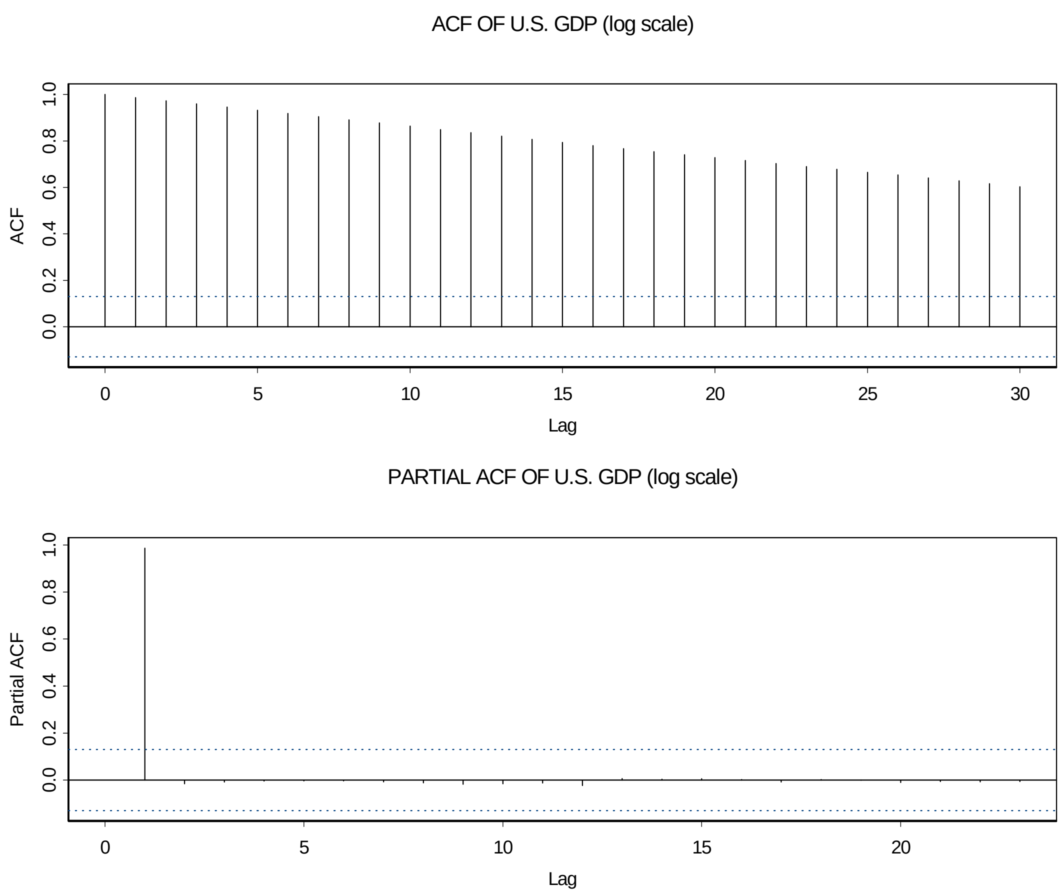

FIGURE 2: Top: Autocorrelation function of quarterly and seasonally adjusted GDP series. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. If the series is identically and independently distributed, the correlation coefficients should be within confidence interval bands. The confidence interval bands are represented by the dotted line. The figure indicates the significance of the autocorrelations in the series. Bottom: Partial autocorrelation function of quarterly and seasonally adjusted U.S. GDP series. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. If the lag is insignificant, the correlation coefficient lies within confidence interval bands. The confidence interval bands are represented by the dotted line. The figure indicates the 1st lag is statistically significantly.

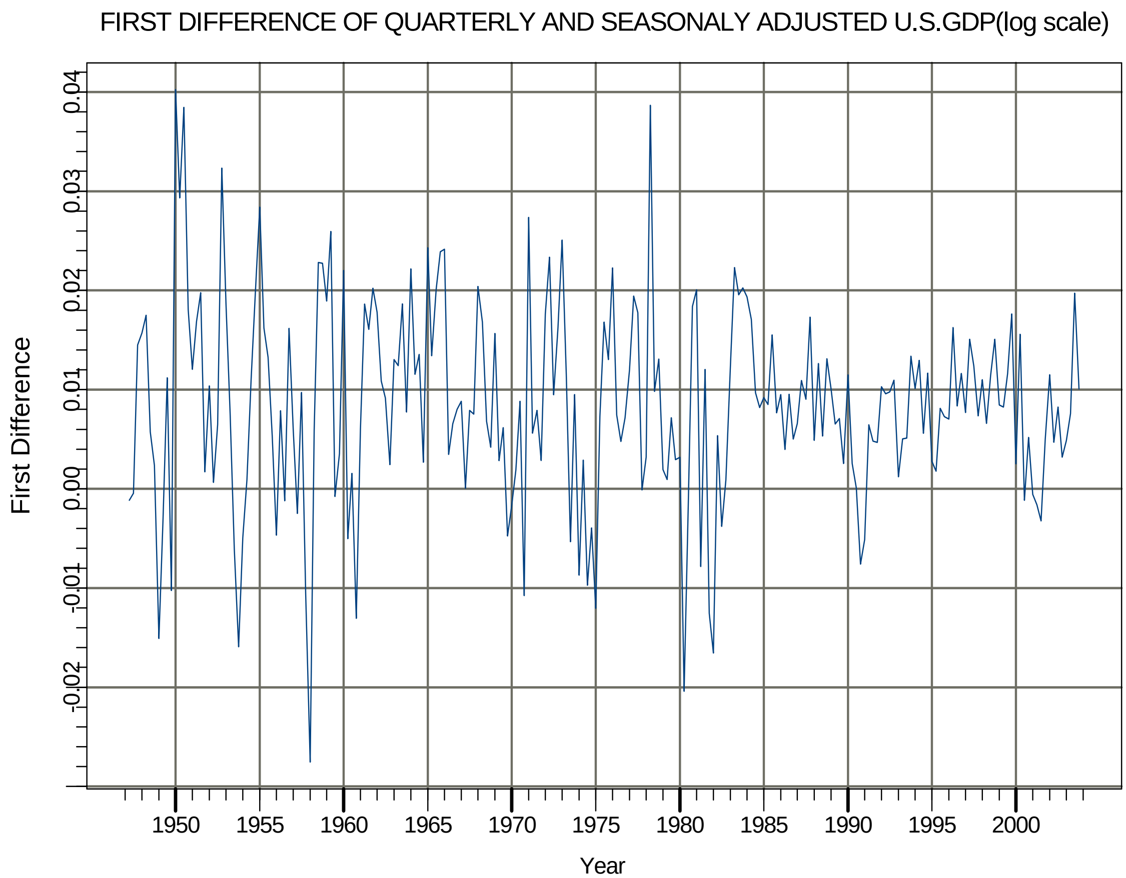

FIGURE 3: The first difference of quarterly and seasonally adjusted U.S. GDP in log scale. x-axis represents the years of the sample and y-axis represents the first difference of the series. The sample period is from January 1, 1947 to October 1, 2003. The mean reverting dynamics is an indication of stationarity.

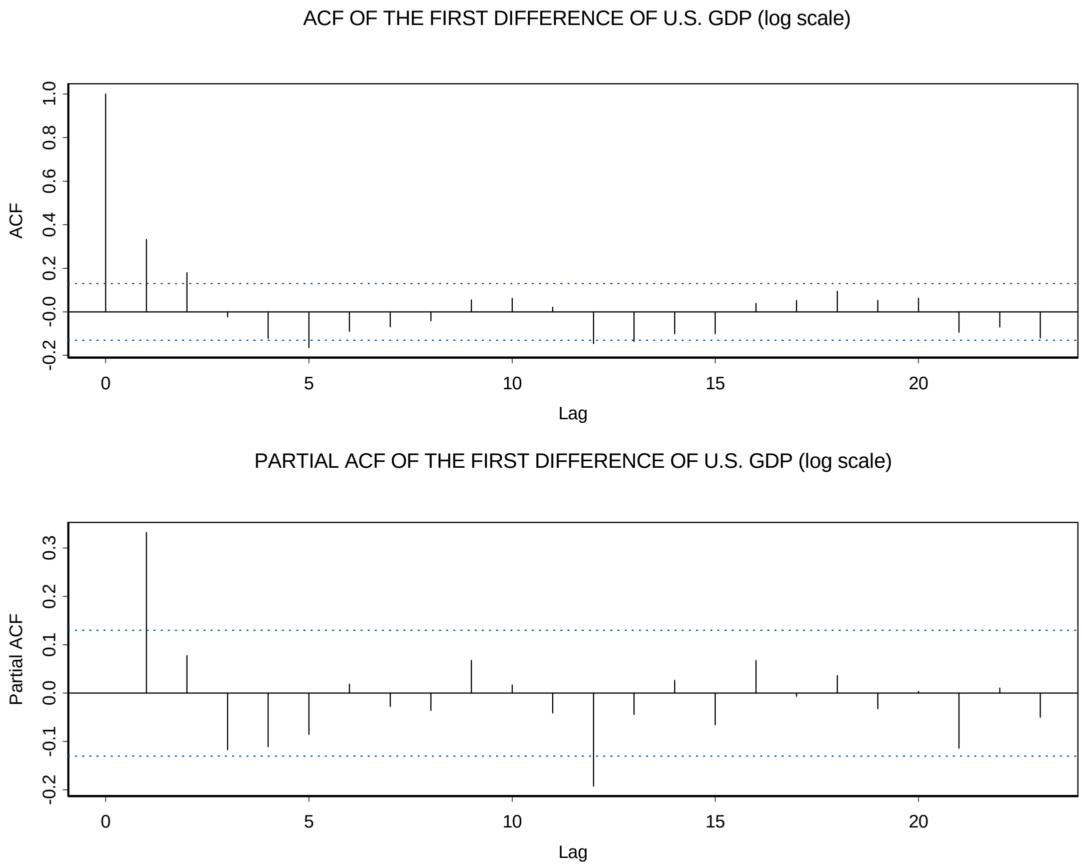

FIGURE 4: Top: Autocorrelation function of the first difference of quarterly and seasonally adjusted U.S. GDP series. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. The presence of damping and cosine waves may be an indicative of AR(2) process. Bottom: Partial autocorrelation function of quarterly and seasonally adjusted U.S. GDP series. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. If the lag is insignificant, the correlation coefficient lies within confidence interval bands. The confidence interval bands are represented by the dotted line. The figure indicates the 1^{st} and the 12^{th} lags are statistically significant. MA(2) process is suspected due to the duality between AR and MA processes.

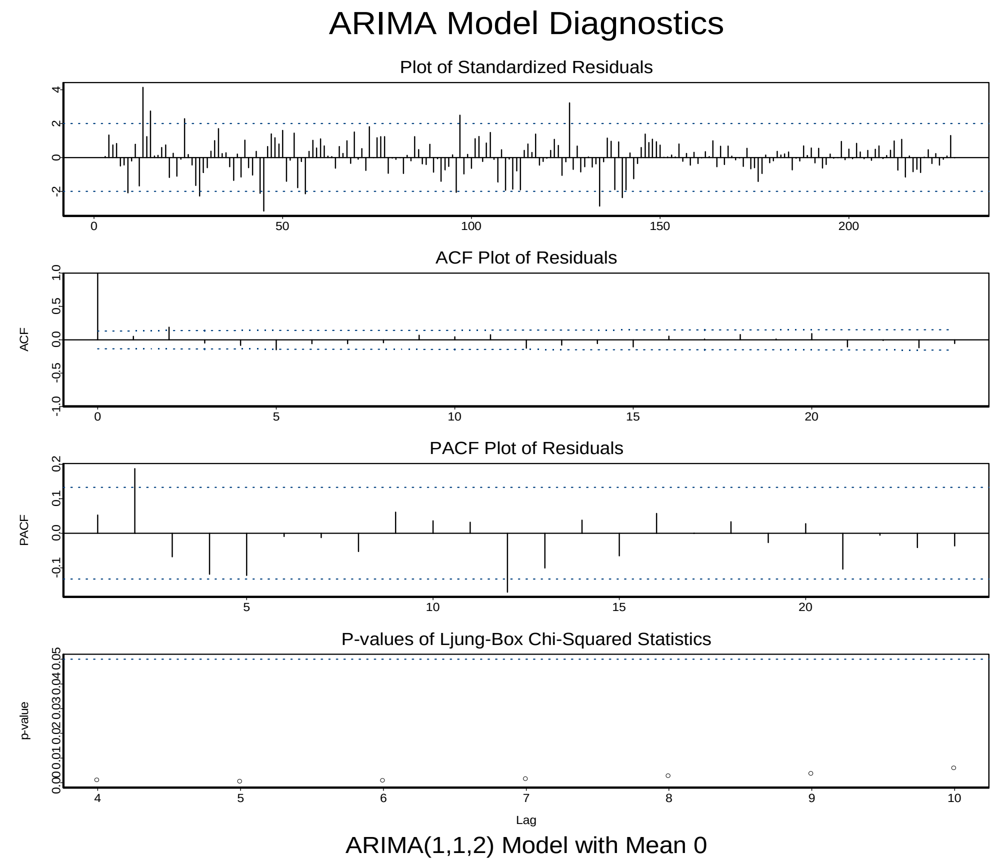

Figure 5: 1st Top: Plot of standardized residuals of the ARIMA (1, 1, 2) model. Each residual is divided by the standard deviation (σ). x-axis represents the number of standardized residuals and y-axis represents standard deviations. 2nd Top: Autocorrelation function of residuals of the ARIMA (1, 1, 2) model. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. The model is correctly specified if the residuals are identically and independently distributed. Since the correlation coefficient for the second lag is statistically significant, the figure indicates that the model is misspecified. 2nd Bottom: Partial autocorrelation function of the residuals of the ARIMA (1, 1, 2) model. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. The correlation coefficients on the y-axis are measured for the original residual series and with other lags on the x-axis. If the lag is insignificant, the correlation coefficient lies within confidence interval bands. The confidence interval bands are represented by the dotted line. The figure indicates that the 2nd and 12th lag is statistically significant. Bottom: Ljung-Box test for detecting autocorrelation in the residuals of the ARIMA (1, 1, 2) model. x-axis represents the lags and y-axis represents p-values from the test. If the model is correctly specified, then the p-values would be lower than the 0.05 critical value. The critical p-value is represented by the dotted line. Since the p-value's up to 10th lag is almost zero, the figure indicates that the model is correctly specified.

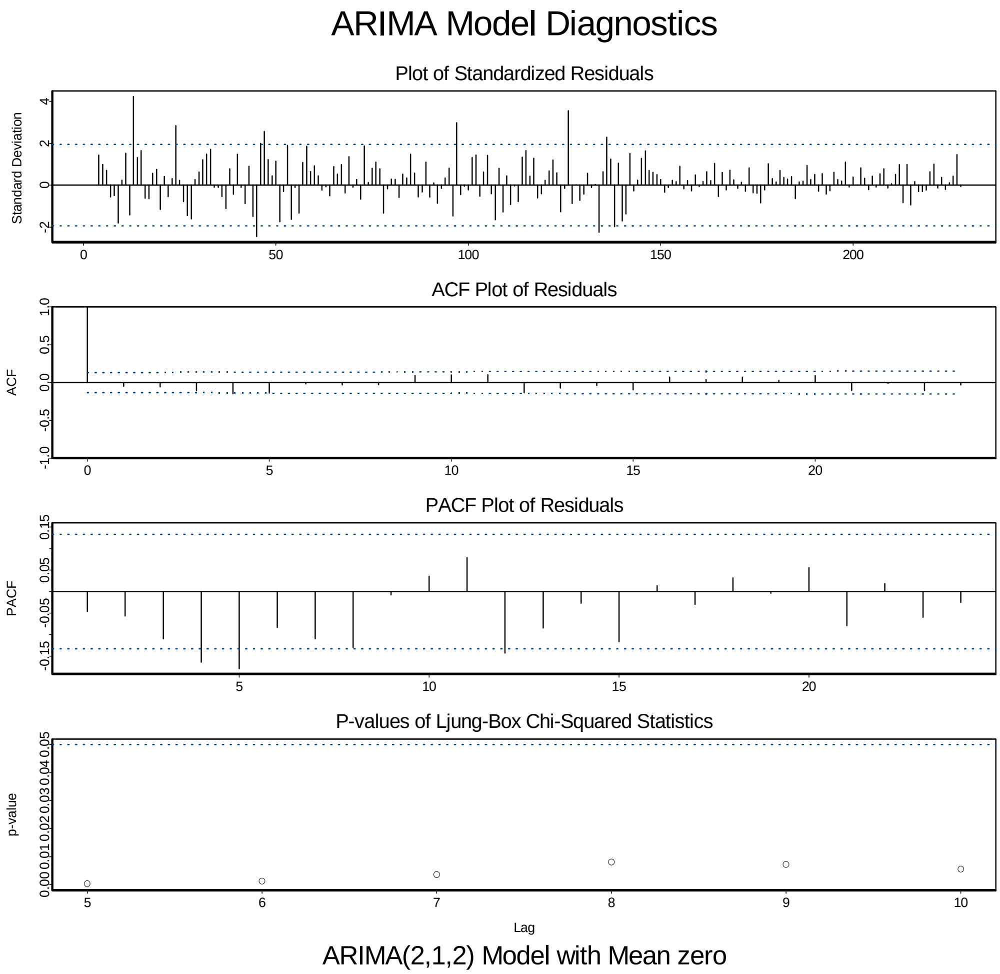

Figure 6: 1st Top: Plot of standardized residuals of the ARIMA (2, 1, 2) model. Each residual is divided by the standard deviation (σ). x-axis represents the number of standardized residuals and y-axis represents standard deviations. 2nd Top: Autocorrelation function of residuals of residuals of the ARIMA (2, 1, 2) model. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. The model is correctly specified if the residuals are identically and independently distributed. Since no lag is significant, the figure indicates that the model is correctly specified. 2nd Bottom: Partial autocorrelation function of the residuals of the ARIMA (2, 1, 2) model. x-axis corresponds to lags and y-axis corresponds to correlation coefficients of the series. The correlation coefficients on the y-axis are measured for the original residual series and with other lags on the x-axis. If the lag is insignificant, the correlation coefficient lies within confidence interval bands. The confidence interval bands are represented by the dotted line. The figure indicates that the 4th, 5th, and 12th lags are statistically significant. Bottom: Ljung-Box test for detecting autocorrelation in the residuals of the ARIMA (2, 1, 2) model designed. x-axis represents the lags and y-axis represents p-values from the test. If the model is correctly specified, then the p-values would be lower than the 0.05 critical value. The critical p-value is represented by the dotted line. Since none of the p-values is greater than the critical p-value, the figure indicates that the model is correctly specified.

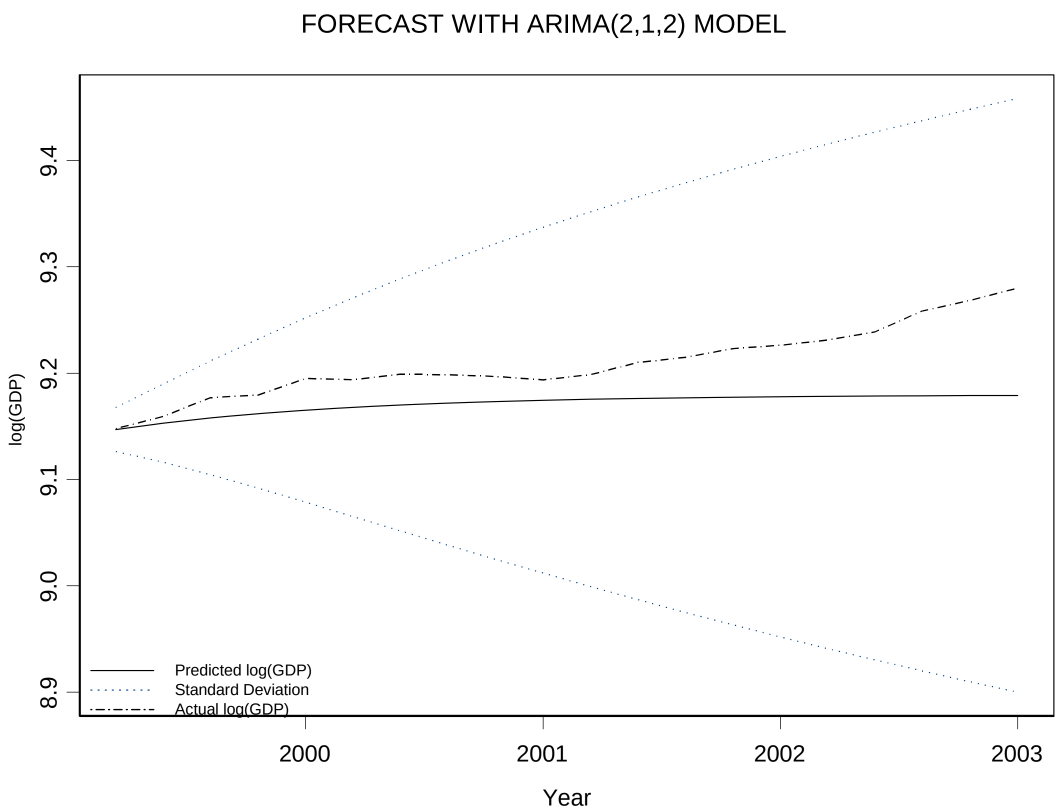

Figure 7: Forecast with ARIMA (2, 1, 2) for the out-of sample series from the first quarter of 1999 to last quarter of 2003. x-axis represents the truncated years and y-axis represents the log of quarterly and seasonally adjusted U.S series. The solid line indicates the forecasted series with ARIMA (2, 1, 2). The broken line lines represent the predictions. The dotted lines represent 95% confidence intervals. The figure indicates that ARIMA (2, 1, 2) do not forecasts the actual data accurately.

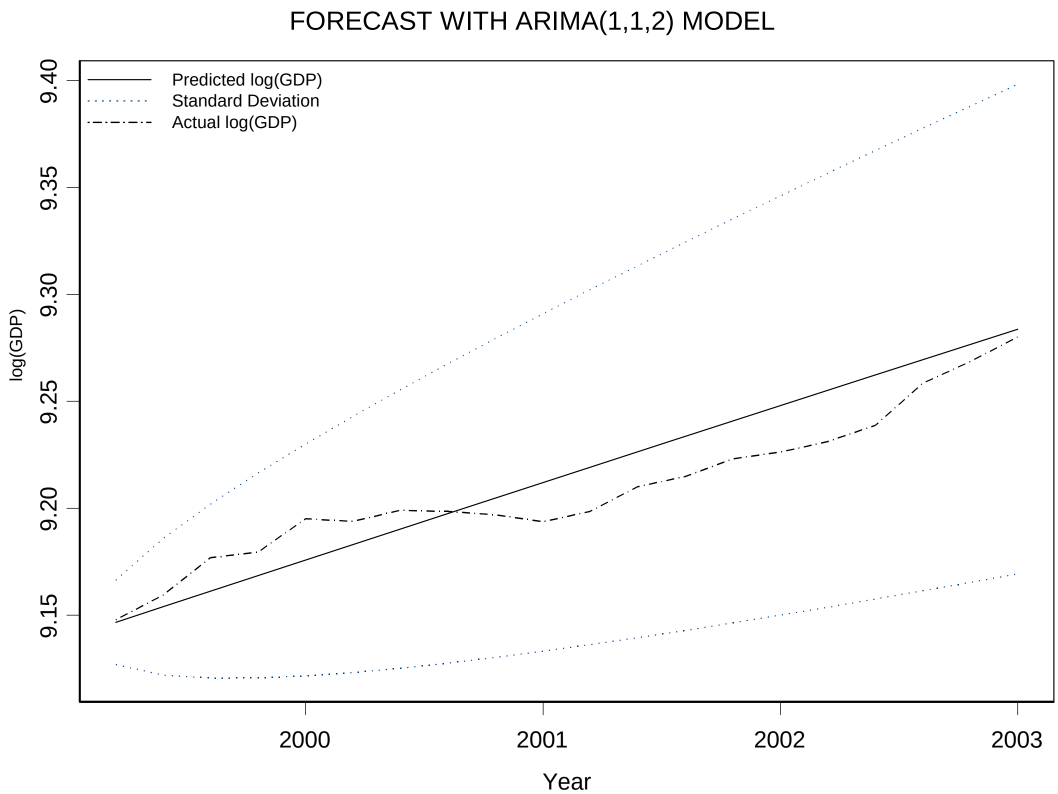

Figure 8: Forecast with ARIMA (2, 1, 1) for the out-of sample series from the first quarter of 1999 to last quarter of 2003. x-axis represents the truncated years and y-axis represents the log of quarterly and seasonally adjusted U.S series. The solid line indicates the forecasted series with ARIMA (2, 1, 1). The broken line lines represent the predictions. The dotted lines represent 95% confidence intervals. The figure indicates that ARIMA (2, 1, 1) forecasts the actual data accurately.

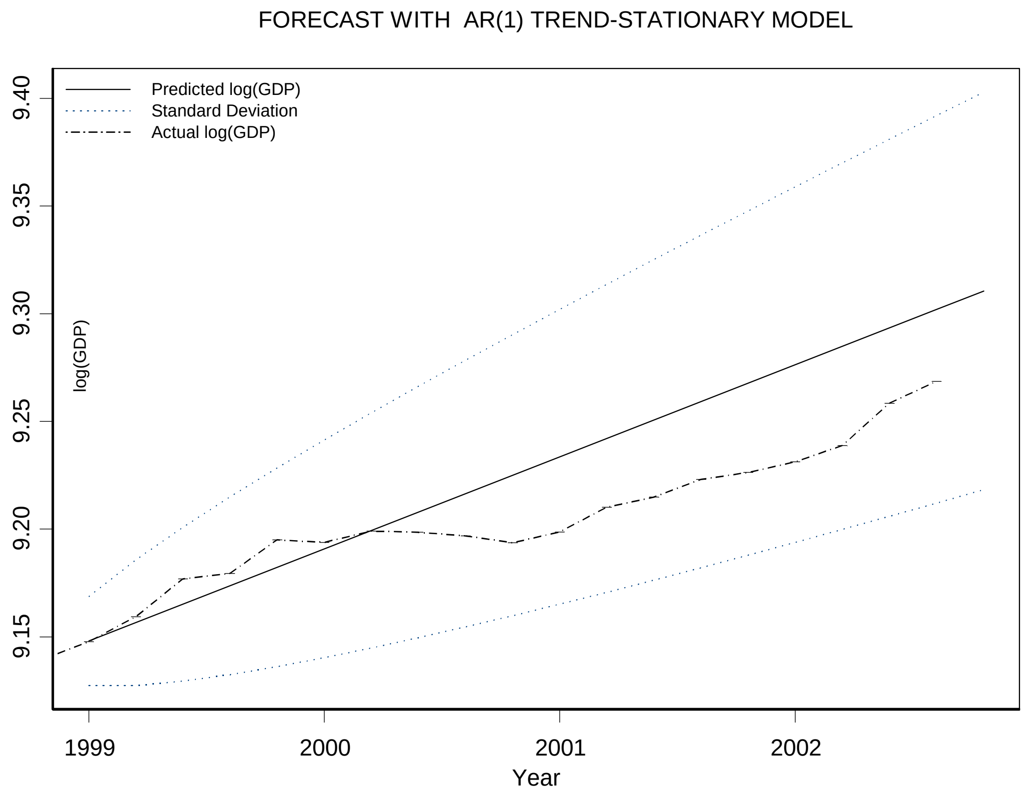

Figure 9: **Forecast with AR(1) trend-stationary model for the out-of sample series from the first quarter of 1999 to last quarter of 2003. x-axis represents the truncated years and y-axis represents the log of quarterly and seasonally adjusted U.S series. The solid line indicates the forecasted series with ARIMA (2, 1, 1). The broken line lines represent the predictions. The dotted lines represent 95% confidence intervals. The figure indicates that AR(1) trend-stationary model do not forecast the actual data accurately.**

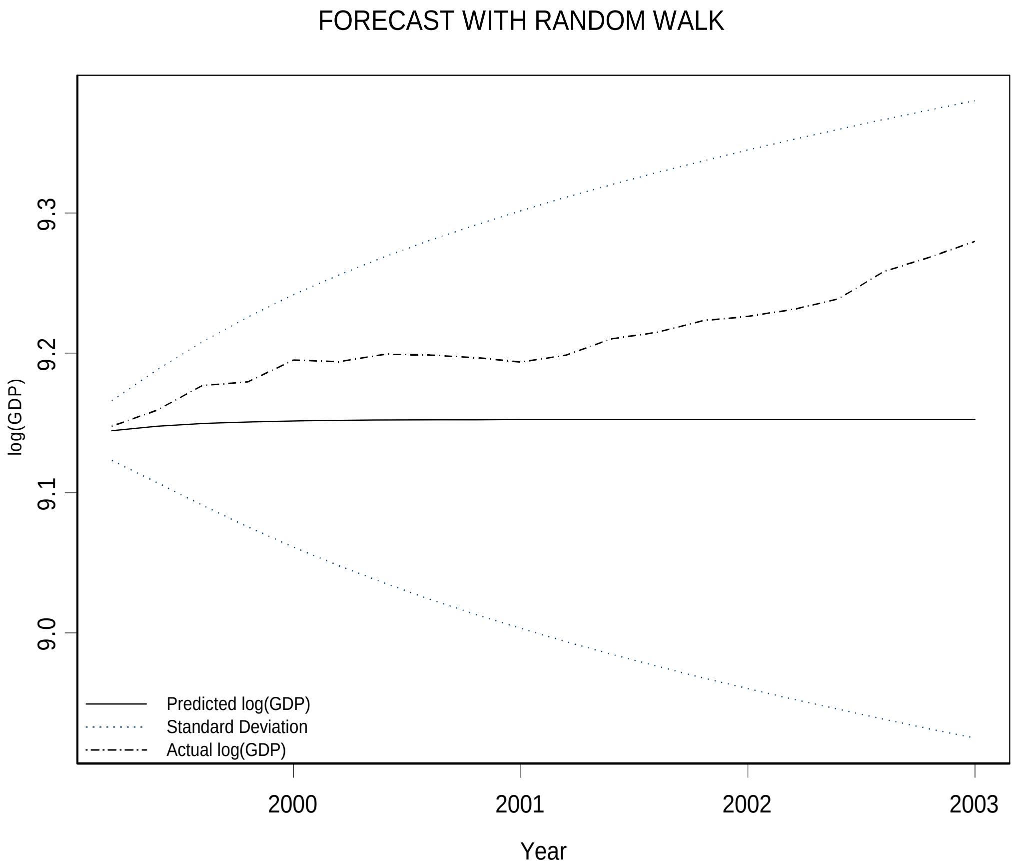

Figure 10: **Forecast with a random walk for the out-of sample series. from the first quarter of 1999 to last quarter of 2003. x-axis represents the truncated years and y-axis represents the log of quarterly and seasonally adjusted U.S series. The broken line lines represent the predictions. The dotted lines represent 95% confidence intervals. The figure indicates that random walk model fails to forecast the actual data precisely.**

The results of Augmented Dickey-Fuller Test

	t-value	P-value
Lag 1	-2.0178	0.0450
Lag 2	4.15	0.0
Lag 3	1.565	0.1192
Lag 4	-0.3767	0.7068
Lag 5	-0.6266	0.2317
Lag 6	-1.0494	0.2953
Lag 7	0.9706	0.3330
Lag 8	-0.6740	0.5011
Lag 9	-0.8835	0.3780
Lag 10	0.5926	0.5541
Lag 11	0.8976	0.3705
Lag12	0.8031	0.4229
Lag 13	-2.4864	0.0137
Lag 14	-0.6324	0.5279
Lag 15	0.3422	0.7326
Test Statistic	-2.018	0.5879

Table1: The results of Augmented Dickey-Fuller Test for the maximum lag length of 15. For this particular test if the absolute value of the t-value is grater lower than the absolute value of 1.6, which is assumed to be the critical value, then the lag is statistically insignificant.

The results of Augmented Dickey-Fuller Test

	t-value	P-value
Lag 1	-2.1843	0.0301
Lag 2	4.2507	0.0000
Lag 3	1.9020	0.0586
Lag 4	-0.6864	0.4933
Lag 5	-0.4922	0.6231
Lag 6	-1.0274	0.3055
Lag 7	0.7815	0.4354
Lag 8	-0.6677	0.5051
Lag 9	-0.9124	0.3627
Lag 10	0.6570	0.5119
Lag 11	0.8972	0.3707
Lag 12	0.7932	0.4286
Lag 13	-2.5453	0.0117
Lag 14	-0.6007	0.5487
Test Statistic	-2.184	0.4955

Table 2: The results of Augmented Dickey-Fuller Test for the maximum lag length of 14. For this particular test if the absolute value of the t-value is lower than the absolute value of 1.6, which is assumed to be the critical value, then the lag is statistically insignificant.

The results of Augmented Dickey-Fuller Test

	t-value	P-value
Lag 1	-2.3722	0.0186
Lag 2	4.8195	0.0000
Lag 3	1.8218	0.0700
Lag 4	-0.6583	0.5111
Lag 5	-0.5109	0.6100
Lag 6	-1.0652	0.2881
Lag 7	0.8373	0.4034
Lag 8	-0.6958	0.4874
Lag 9	-0.8191	0.4137
Lag 10	0.7467	0.4561
Lag 11	0.9540	0.3412
Lag 12	0.7012	0.4840
Lag 13	-2.8270	0.0052
Test Statistic	-2.372	0.3931

Table 3: The results of Augmented Dickey-Fuller Test for the maximum lag length of 15. For this particular test if the absolute value of the t-value is lower than the absolute value of 1.6, which is assumed to be the critical value, then the lag is statistically insignificant.