Optimal Deposit Contracts
with Transfers

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Chapter 1: Introduction

The advancement of human civilization is a long and complicated process. Through ages of evolution, human beings have invented many great things and concepts. Two of them are certainly the bank and the closely-related financial market. Over centuries of development, the banking has transformed into almost a daily necessity of most people in developed countries. However, we have not yet seen the end of this long and intricate evolution of the banking industry. Riding on the trend of ever increasing free trades and globalizations, the banks play a very pivotal role in facilitating the growth and stabilization of the financial industry. Thus, a good understanding of how banks should operate with its vast financial resources in order to provide the maximum financial stability to the society has become certainly a very urgent need.

Without taking well-thought actions, the banks could bring very severe impacts to the stability of financial industry and in turn lead to turmoil in the society. For instance, a major US banking crisis occurred in October 1907. During that time, half of the bank loans in New York City were invested in the stock and bonds with high risk. This let financial market be brought into an extreme speculating situation. During the panic period, the economy in the United States went down, and there were runs on many banks and trust companies. In the recent two decades, there were also bank runs in Mexico in 1995, in Thailand, Indonesia, Korea in 1997, and in Argentina in 2001. Even more recently, in 2007, British’s fifth largest credit bank

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1 The panic of 1907 caused a half fall in the New York Stock Exchange from its top.
had runs and the financial troubles of banks in Iceland essentially bankrupted the entire country. Apparently, a better effort is needed to understand how to run the banks.

In the academia, the economists have been studying the banks and their economic impacts over two centuries. However, only until recently were some seminal models of banking proposed by Bryant in 1980 and Diamond and Dybvig in 1983. Essentially, these models of banking aimed to explain and understand bank runs. After Bryant (1980) and Diamond and Dybvig (1983) published their research papers, most of the later theories were built on these fundamental ideas and model. In 2007, Franklin Allen and Douglas Gale published a book “Understanding Financial Crisis”. In the book, they described a model of banking based on the Diamond and Dybvig’s model. In their model, they show how banks maximize investors’ utility by carefully allocating financial resources in two different time periods when there is an uncertainty of runs on banks.

For this paper, I followed Dr. Zhang’s suggestions to modify Allen and Gale’s model. In the original Allen and Gale’s model, they do not explicitly consider a transfer between the two time periods. However, we know from real life experiences that banks can either liquidate long term investments to meet short term needs or transfer short term excesses to long term investments. Thus, I and Dr. Zhang are curious to find out whether an inclusion of such a transfer in the Allen model.

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2 In the original DD model and its derivatives, they acknowledge the possibility of transfer of assets between periods through liquidation. However, they never apply this concept in the model to derive the maximized utility functions. In contrast, our new model explicitly integrates this transfer into the model.
and Gale’s model may yield new insights into the way how banks should manage their financial resources. In other words, a more technical explanation is that we would like to see if we can better optimize the expected utility function of the banks’ customers if we include transfer in the model.

In chapter 2, I first introduce the role of banks, the liquidity problem, the causes of the liquidity problem in the real life, and the actual influences of the liquidity problem in our banks and the economy. In chapter 3, I describe the Diamond-Dybvig view of bank runs and the business cycle view of bank. In chapter 4, I describe Allen and Gale’s model in greater details as this model is the one I modify. In chapter 5, I explain how I modify Allen and Gale’s model and provide some motivations why our model could be better. In chapter 6, I derive some general properties of our model. Finally, a conclusion is given in chapter 7. All the proofs are put in Appendix A-E.
Chapter 2: Banks

Nowadays, banks have more and more impacts on facilitating the development and stabilization of the global economy. So, I would like to first explain the links between the banks and real economic activities. This chapter provides an introduction to the central problem, the bank runs, that all models presented in chapter 3, 4, and 5 try to understand.

2.1 The functions of banks

The most common sense functions of the banks involve providing accounts to help manage customers’ money, providing investment portfolios for individuals, and providing loans or mortgage services to selected companies or individuals. However, banks have much more functionalities. In the commercial world, investment banks facilitates large amount of transactions in the stock market and foreign currency market, various trading between transnational corporations, and numerous mergers and acquisitions in the corporate world.

To carry out some activities listed above, sometimes, banks have to transform short-term assets (liquid deposits) into long-term assets (illiquid liabilities). During this process, banks can be considered as an “intermediary”. Customers bring their savings into a bank; the bank can transfer this money to its credit clients in the form of loans. This flexibility of banks to re-allocate financial resources in order to fuel the economic growth, if used properly, will certainly benefit the society at large; however, sever consequences are ensured if banks do not act prudently at all times.
2.2 Liquidity problem

As mentioned in previous section, banks have the flexibility to transform their short term assets into long term assets. This flexibility also comes with a potential danger that the banks might not have enough short term assets to meet the amount of withdrawal demanded by the depositors of the banks. This mismatch between demand and supply of liquid assets is called the liquidity problem.

Hereby, I provide three possible scenarios that may lead to the liquidity problem. First of all, for some reasons, a bank may encounter an unexpected demand of withdrawal that exceeds the bank’s ability to re-allocate their resources to meet the demand. For instance, this scenario happened to the British’s fifth largest bank in 2007. Secondly, banks suffer from the inability to collect a significant amount of money from debtors. For instance, many Japanese banks had this problem after their bubble economy busted towards the end of the last century. Thirdly, a bank can also be viewed as a company. Just like any other companies, if banks make a huge mistake in their business or investment strategy, bankruptcy is always a possible consequence. Many banks in Iceland certainly had similar problems.

Disregard of whatever the reason that a mismatch between the demand and supply of withdrawal emerge, the customers have causes to worry that the bank may become insolvent in the future. This fear motivates the majority of the depositors to withdraw from the bank. This aggregation of withdrawals, due to the news of liquidity problem, certainly exacerbates the bank’s situation even further.
If the bank could not somehow contain the situation in time, a bank run would surely follow soon.

In the following models, efforts are made to understand how the banks should manage their financial resources to maximize customers’ expected utility function and also avoid the potential of a bank runs.
Chapter 3: Classical views of bank runs

In this chapter, I re-explain the liquidity problem and bank runs by using some economic theories, the Diamond-Dybvig view of bank runs and the business cycles. This chapter provides all necessary backgrounds to understand the following models in Chapters 4 and 5.

3.1 The Diamond-Dybvig view of bank runs

In 1983, Diamond and Dybvig published their influential research paper. In this paper, they developed a very famous model of banking in order to explain and understand bank runs. Their goal of study was to determine the optimal allocation of banks’ financial resources in the form of illiquid assets (such as loans) and liquid assets under the assumption of the possibility of a mismatch between the demand and supply of liquid assets due to the self-fulfilling prophecies or panics so that bank runs may occur.

In the Diamond-Dybvig model, the time was divided into three periods, denoted by \( t = 0, 1, 2 \). They assumed some depositors of some banks would like to consume early (consume in the period 1), the rest of the depositors want to consume late (consume in the period 2). After the customers of the banks deposit

---

3 “The self-fulfilling prophecies or panics” mean that a lot of banks’ customers do not believe banks will fulfill their commitment in the future.

4 A customer of a bank is called as “early consumer” if and only if this customer only consumes his/her asset goods at the period 1. More details are in the next chapter.

5 A customer of a bank is called as “late consumer” if and only if this customer only consumes goods at the period 2. More details are shown in the next chapter.
their wealth into the banks, the banks would want to invest these deposits. Since Diamond and Dybvig assumed that the return yielded by long-term investments (illiquid assets) could be higher than the return of short-term investments (liquid assets), the banks would be inclined to invest these assets in long-term projects. However, the banks had the obligations to their customers to allow them to withdraw all their savings at any time. Thus, the banks always needed to keep a portion of their financial resources in the form of short term assets.

In the Diamond-Dybvig model, it was also assumed that the depositors did not know whether they would withdraw early or late at the beginning of the period, \( t = 0 \). As a consequence, the banks would not know this information at the beginning of the period either. The banks had to estimate the fraction of the early consumers and late consumers. If this information was obtained, then the banks could decide how much assets they should keep as short term assets for the period 1. In the period 1, the withdrawals of the early consumers would not cause any problems. The problems might happen when some late consumers change their behaviors (They did not want to be late consumers anymore.), so they would like to withdraw their assets in the period 1 too. Therefore, there was a liquidity shock, banks suffer unexpected repayments. In that situation, banks had to use some of illiquid assets at a loss in order to meet all the unexpected payments. This also resulted in the loss of banks in the period 2. It was obvious that when some of late consumers withdraw their assets in the period 1, it is in the best interests of other late consumers to withdraw their assets in the period 1 too. These behaviors would lead to a bad
feedback cycle for the banks. Thus, a bank runs might occur. A bank crisis might take place if several banks runs happen simultaneously.

In the Diamond-Dybvig paper, they pointed out that a run on a bank depended on the late consumers’ beliefs. If all the late consumers believed the banks could fulfill their obligations in the period 2, then the late consumers would keep their plan and withdraw their deposits at time 2. However, if the late consumers did not believe banks would be able to fulfill the commitments in the period 2, they would withdraw their deposits in the period 1. A bank runs then could possibly occur. All of these were called as “self-fulfilling prophecies or panics” in the Diamond-Dybvig paper.

3.2 The business cycle view of bank runs

The fluctuations in the aggregate economic activities of nations are called the business cycle. These periodic fluctuations alternate between growth of economy and stagnation of economy. These fluctuations in the economy always correspond to the changes in business. With a rapid growth in real GDP, which is called an expansion of economy, business is good. During the period of an economic expansion, outputs, investment spending, personal income, corporate profits and industrial production all will grow by different amounts, and unemployment rate will fall. However, when real GDP falls, which is called a recession, business is in trouble. During the recession, investment spending, corporate profits, personal income and industrial production all will fall by different amounts, and unemployment rate will go up.
Now, let us assume our economy is in the period of recession. With a fall in real GDP, the entire economy and businesses are in trouble. Just as Allen and Gale said in their book (2007), the value of bank assets are reduced as well in this period, and the probability of not being able to fulfill banks’ obligations in the future rises at the same time. In this case, if the depositors realize there is an economic depression in the cycle, they will believe banks may have troubles to meet their commitments in the future so that they would like to withdraw their wealth from banks at this moment. Then, bank runs may take place if there is a large amount of withdrawals in the meantime. It is also obvious to realize that bank runs can be considered as a rationale response to a recession in the economy by using business cycle view to explain it.
Chapter 4: The model without a transfer

In this chapter, I describe Allen and Gale’s original model of banking from the book “Understanding Financial Crisis”. Figure (4.1) illustrates all possible events in each time period. Before proceeding to the details of the model, I would like to first define each variable.

**Figure (4.1): Timeline for a bank without transfer**

<table>
<thead>
<tr>
<th>Time</th>
<th>No bank runs</th>
<th>Bank runs</th>
<th>Late cons. : withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0</td>
<td>Early cons. : withdrawal</td>
<td>Early cons. : withdrawal</td>
<td>Late cons. : withdrawal</td>
</tr>
<tr>
<td>t = 1</td>
<td>Late cons. : withdrawal</td>
<td>Late cons. : withdrawal</td>
<td>Banks have to liquidate long term assets at a loss.</td>
</tr>
<tr>
<td>t = 2</td>
<td>Late cons. : withdrawal</td>
<td>Late cons. : withdrawal</td>
<td></td>
</tr>
</tbody>
</table>

1) Individual doesn't know their type (early or late)

2) Bank invest assets (short term or long term)

Note: cons. is an abbreviation of consumer.
**Date**

In the model, time is divided into three periods (also called dates), and labeled by t=0, 1, 2, respectively. It is assumed that there is a unit of the full-purpose goods that can be used to consume and invest in each period.

**Assets**

The banks have two types of assets, namely the liquid assets and illiquid assets, that their customers may consume and invest during each period. For simplifying the model, authors suppose that each type of assets has a constant-returns-to-scale investment technology. Details of each type of assets are given as follows:

- **The liquid assets**

  The liquid assets are also named the short assets. With a constant-returns-to-scale technology, if customers take one unit of short-term goods at date t, it will be converted into one unit of goods at date t+1, where t=0, 1. The short assets seem to represent a storage technology in the bank.

- **The illiquid assets**

  The illiquid assets are called the long assets as well. With a constant-returns-to-scale technology, if the customers take one unit of the long assets at date 0, it will be converted into R units of the goods at date 2, where R>1; but if one unit of the long assets is paid off prematurely at date 1 then it has to pay r units of goods for each unit invested before. The long assets seem to represent an investment technology. It could be considered as a reward for
inconvenience of holding the long assets until date 2, and a penalty for paying off prematurely. In contrast, the short assets gain a lower return after one period; the long assets will yield a higher return after two periods.

Figure (4.2): The endowment, short assets and long assets

<table>
<thead>
<tr>
<th></th>
<th>T = 0</th>
<th>T = 1</th>
<th>T = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short assets</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Endowment</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Long assets</td>
<td>\begin{cases} 0 &amp; \text{R (Reward) } R&gt;1 \ r &amp; \text{0 (Penalty) } 0 \leq r \leq 1 \end{cases}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Looking at the figure (4.2), if a consumer has one unit of the initial endowment at the time 0, he/she can either invest it in the short assets or in the long asset. With a constant-returns-to-scale technology, if customers take one unit of short-term goods at date 0, he/she can obtain only one unit of goods at date 1. If the customers take one unit of the long assets at date 0, he/she can get R units of the goods at date 2, where R > 1; but if one unit of the long assets is paid off prematurely at date 1 then it has to pay r units of goods for each unit invested before.

**Liquidity preference**

In the model, a consumer begins with only one unit of goods as his/her endowment at the date 0 and nothing at the date 1 and 2. Furthermore, we assume the consumers do not know whether they will be early or late consumer
until the date 1; even though the banks already make an estimation on the ratio of the two types of consumer back at the date 0.

- **Early consumer**

  An early consumer only wants to consume at the date 1. In the model, $\lambda$ is used to denote the probability of a customer being an early consumer.

- **Late consumer**

  A late consumer only wants to consume at date 2. And $(1-\lambda)$ is used to denote the probability of a person being a late consumer.

  For $\lambda$, it is between zero and one. If $\lambda$ is zero, all the consumers are the late consumers. They will only consume at date 2, and only invest their endowments in the long assets since there are higher returns from the long assets. If $\lambda$ is one, all consumers are the early consumers. They will only consume at the date 1, and only invest their endowments in the short assets.

**Expected utility function**

Let $U(C)$ denote a random utility function of a consumer. In the model, it is necessary to assume that $U(C)$ is a strictly increasing, concave, twice continuously differentiable utility function. And $C_1$ and $C_2$ are used to denote the consumptions per capital in the date 1 and 2. Then, the consumer's utility function can be written as

\[
U(C_1, C_2) = \begin{cases} 
U(C_1) & \text{with probability } \lambda \\
U(C_2) & \text{with probability } (1-\lambda) 
\end{cases}
\]
If \((C_1, C_2)\) denotes the allocations of resources for early consumers and late consumers respectively when the bank is solvent, then the expected utility function for the consumptions can be written as

$$\lambda U(C_1) + (1-\lambda) U(C_2).$$

(4.1)

**Efficient solution**

As we know, it is assumed that each customer of banks has one unit of goods in the period 0. The consumer is able to invest the initial endowment either in the short assets or in the long assets. In the model, \((x, y)\) denotes a consumer’s portfolio, where \(x\) is the amount of goods per capital invested in the long assets, and \(y\) is the amount of goods per capital invested in the short assets. In the period 0, there is a consumer’s budget constrain, namely \(x + y \leq 1\). When a bank maximizes an investor’s utility function, the entire investments per capital have to be equal to one unit of goods of an endowment at the date 0:

$$x + y = 1.$$  

(4.2)

Moreover, there is another constrain at period 1, which is that the payoff of the short assets has to be more than or equal to the total consumptions per capital. According to the liquidity preference section, a customer will know the particular type of customers he/she is at the date 1. In this model, \(\lambda\) denotes the proportion of early consumers, and \(C_1\) is the consumptions per capital of the individual customer. Thus, the total consumptions per capital at time1 equal to \(\lambda C_1\). Under the feasibility condition, \(\lambda C_1\) must be less than or equal to \(y\), \(\lambda C_1 \leq y\).

(4.3)
Similarly, the total consumptions per capital have to be less than or equal to all assets owned at the date 2. In this case, \((1 - \lambda)\) denotes the proportion of late consumers, and \(C_2\) is the consumptions per capital of the individual customer at time 2. The total consumptions per capital at time 2 are equal to \((1 - \lambda) C_2\). In the model, all assets owned at the date 2 are divided into the payoff from long assets and some excesses from short assets, namely \(Rx + (y - \lambda C_1)\). Then, \((1 - \lambda) C_2\) has to be less than or equal to \(Rx + (y - \lambda C_1)\).

\[ (1 - \lambda) C_2 \leq Rx + (y - \lambda C_1). \] (4.4)

Authors assume that in an optimal plan equations (4.3) and (4.4) will be reduced simply to \(\lambda C_1 = y\) and \((1 - \lambda) C_2 = Rx + (y - \lambda C_1)\). We can base on these to calculate the optimal portfolio \((x, y)\). After we obtain the optimal solutions of \(x\) and \(y\), the optimal consumptions \(C_1\) and \(C_2\) are determined as well.

\[ C_1 = \frac{y}{\lambda} \quad C_2 = \frac{Rx}{(1 - \lambda)} \]

On the basis of all the above analyses, the equation (4.1) can be defined as

\[ \lambda u \left( \frac{y}{\lambda} \right) + (1 - \lambda) u \left( \frac{Rx}{(1 - \lambda)} \right) \]

Subject to:

\[
\begin{align*}
    x &= 1 - y \\
    0 &\leq y \leq 1
\end{align*}
\]

When we compute an optimal solution of \(y\), we need to calculate the first order condition of equation (4.5) with respect to \(y\) via differentiating the above equation and setting its derivative be zero.

\[ u' \left( \frac{y}{\lambda} \right) - R \times u' \left( \frac{R(1-y)}{(1-\lambda)} \right) = 0 \quad \Rightarrow \quad u' (C_1) = R \times u' (C_2) \] (4.5.1)
**Equilibrium bank runs**

All previous analyses of banking arrangement are based on the liquid assets and illiquid assets, but it is not sufficient to fully understand the equilibrium in the banking system. In general, banks will provide a portfolio \((x, y)\) and deposit plan \((C_1, C_2)\) for their individual customer at the date 0. However, runs on banks are not predictable at the period 0. There may be runs on banks at the period 1 because of some unexpected withdrawals from their customers. In this situation, the equilibrium at time 1 can be defined, which is different from the equilibrium at date 0. The expected utility of a depositor that Allan and Gale adopted can be defined as

\[
\pi u(y + rx) + (1 - \pi)(\lambda u(C_1) + (1 - \lambda)u(C_2))
\]  

(4.6)

Subject to:

\[
\begin{align*}
x + y &= 1 \\
0 &\leq y \leq 1 \\
C_1 &= \frac{y}{\lambda}, \quad C_2 = \frac{Rx}{(1 - \lambda)}, \quad R > 1, \quad 0 < \lambda < 1.
\end{align*}
\]

where \(\pi\) is the probability of bank runs, \(0 \leq \pi \leq 1\).

Now the consumer can achieve the optimal portfolio \((x^*, y^*)\) by differentiating equation (4.6) and setting its derivative to be zero because the optimal portfolio must satisfy the first order condition. From the equation (4.7), if \(\pi\) is equal to 0, it implies there is no bank runs. The equation (4.5) is held when \(\pi\) is equal to 0.

FOC:

\[
\pi u'(y + rx)(1 - r) + (1 - \pi)(u'(C_1)) = (1 - \pi)u'(C_2)R
\]  

(4.7)
Chapter 5: The model with a transfer

(This chapter is based on Dr. Zhang jiangkang’ s suggestions.)

This model is a very simple modification of the Allen and Gale’s model presented in the previous chapter. We still aim to determine the optimal values of $C_1$, $C_2$, $x$, and $y$ by solving equations similar to (4.1) – (4.6). However, before I introduce the details of the model with a transfer, I would like to illustrate the usefulness of our model with a simple example. Let us consider a case in which the investor’s utility function, $u(C)$, is a logarithmic function, denoted by $u(C) = \log(C)$, which is an increasing, strictly concave, twice continuously differentiable utility function. Other variables are given as $\lambda = 0.5$, $R = 1.7$, $r = 0.1$, and $\pi = 0.3$ in this example.

So we begin by solving this particular case with the original model. Based on the equation (4.1) – (4.6), the expected utility function is written out explicitly as

$$U = 0.3 \ln((y + 0.1x)) + (1 - 0.3)(0.5 \times \ln(C_1) + (1 - 0.5) \times \ln(C_2))$$

Subject to:

$$x + y = 1$$

$$0 \leq y \leq 1$$

$$C_1 = \frac{y}{0.5}, \quad C_2 = \frac{1.7x}{1 - 0.5}$$

(5.1)

The planners can achieve the optimal portfolio $(x^*, y^*)$ by differentiating equation (5.1) and setting its derivative to be zero because the optimal portfolio must satisfied the first order condition.

$$\text{FOC: } \frac{0.27}{0.9y + 0.1} + \frac{0.35}{y} = \frac{0.35}{1 - y} \quad \Rightarrow \quad y^* = 0.63$$
Thus, the optimal portfolio \((1 - y^*, y^*)\) is (0.37, 0.63). As \(y^*\) is equal to 0.63, I will compute \(C_1^*\) and \(C_2^*\).

\[C_1^* = \frac{y}{\lambda} = \frac{0.63}{0.5} = 1.26\]

\[C_2^* = \frac{(1-y)R}{1-\lambda} = \frac{(1-0.63) \times 1.7}{1-0.5} = 1.258\]

Finally, we can plug \(y^*\) into the equation (5.1) to compute \(u(C_1^*, C_2^*)\):

\[
U = 0.3 \ln (0.63 + 0.1 (0.37)) + 0.35 (\ln (1.26) + \ln (1.258))
\]

\[= 0.0397\]

**Figure (5.1): The graph of the equation (5.1) in the range of (0, 1)**

By solving this optimal problem, we get the optimal portfolio \((x^*, y^*)\) is (0.37, 0.63), in turn, it implies \(C_1^*(y) = 1.26\) and \(C_2^*(y) = 1.258\). Finally, I plug \(C_1^*\) and \(C_2^*\) into the utility function, and we get \(u(C_1^*, C_2^*) = 0.0397\).
Now we will solve the same example by using our model. I obtain an optimal portfolio \((x^*, y^*)\) is \((0.35, 0.65)\). The value of utility function, \(u(C_1^*, C_2^*)\), is 0.0399, which is barely higher than the one computed from the Allen and Gale’s model. We have showed that our model gives a higher utility in this particular case.

The major difference is that we postulate a transfer of assets between the period 1 and 2 with certain amount specified by new variables \(\tau_1\) and \(\tau_2\), where \(\tau_1\) specifies the percentage of assets that will be transferred from the period 1 to the period 2 and \(\tau_2\) specifies the percentage of assets that will be transferred the other way round. Since it does not make sense for the banks to transfer assets back and forth between the two periods at the same time, we assume that \(\tau_1\) and \(\tau_2\) represent two mutually exclusive events. Thus, at most one variable is non-zero under all possible circumstances.

As mentioned, we just add a very intuitive and straightforward idea on top of the original. The introduction of \(\tau_1\) and \(\tau_2\) into the model leads to the modifications of few equations. First, we re-define the early and late consumer consumption functions in terms of \(y\) and \(x\).

\[
\begin{align*}
\lambda C_1 &= (1 - \tau_1)y + \tau_2 xR & \Rightarrow & & c_1 = \frac{(1 - \tau_1)y + \tau_2 xR}{\lambda} \\
(1 - \lambda) C_2 &= (1 - \tau_2) xR + \tau_1 y & \Rightarrow & & c_2 = \frac{(1 - \tau_2) xR + \tau_1 y}{1 - \lambda}
\end{align*}
\]

where \(0 \leq \tau_i \leq 1, i = 1, 2\). \hspace{1cm} (5.2)

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Please refer to Appendix A for more details.
In the equations above, early consumers consume either \((1 - \tau_1)y\) amount of wealth or consume \(y + \tau_2xr\) amount of wealth at the time 1. Similarly, we interpret that the late consumer either consumes \(xR + y\tau_1\) amount of wealth or \((1 - \tau_2)xR\) amount of wealth at time 2. We will then solve the optimization problem of exactly the same utility function (equation (4.6) in Chapter 4), but with \(C_1\) and \(C_2\) defined in equation (5.2) and with a new set of constraints as follows:

\[
\max \pi u(y + r x) + (1 - \pi)(\lambda u \left(\frac{(1-\tau_1)y + \tau_2xr}{\lambda}\right) + (1 - \lambda) u \left(\frac{(1-\tau_2)xR + \tau_1y}{1-\lambda}\right)) \quad (5.3)
\]

Subject to:
\[
\begin{align*}
0 &\leq y \leq 1 \\
x + y &= 1 \\
0 &\leq \tau_i \leq 1, \ i = 1, 2; \ 0 \leq \pi \leq 1; \ 0 \leq \lambda \leq 1; \ 0 \leq r \leq 1; \ R > 1.
\end{align*}
\]

Now I will provide more details to solve the above example by using our model. In our model, we use all the same parameters as above, and additionally we set \(\tau_2\) to be equal to zero.\(^7\) In this example, the utility function can be re-written as

\[
U = 0.3 \times \ln((y + 0.1 \times x)) + (1 - 0.3)(0.5 \times \ln(C_1)) + (1 - 0.5) \times \ln(C_2))
\]

Subject to:\(^8\)
\[
\begin{align*}
x + y &= 1 \\
0.63 &< y < 1 \\
C_1 &= \frac{(1 - \tau_1)y}{0.5}, \ C_2 = \frac{(1-y) \times 1.7 + \tau_1y}{1-0.5}
\end{align*}
\]

---

\(^7\) In our model, we need just one of \(\tau\)'s to be non-zero. For this particular example, we set \(\tau_2\) equal to zero. Please refer to Appendix A - C for more explanations.

\(^8\) The range of \(y\) is calculated as we solve an optimal problem. Please refer to the equation (A-4) – (A-6) in Appendix A for more details.
By the same approach, we have the optimal portfolio \((x^*, y^*) = (0.35, 0.65)\). As \(y^*\) is 0.65, \(\tau^*_1\) is equal to 4.2% and \(C^*_1(y) = C^*_2(y) = 1.245\). Then, I compute the value of utility function \(u(C^*_1, C^*_2) = 0.0399.^9\)

Even though, this is just one example; it certainly makes the point that the original model might not always give the most optimal portfolios that maximize the customer’s utility function. A higher utility is shown in this particular case. This example gives us a very strong motivation to carry out further analyses on the new model. We would like to improve on the original model and get a more comprehensive understanding of the complicated banking problem.

---

9 The whole calculations are shown in Appendix A.
Chapter 6 General properties of the model with a transfer

In this section, we would like to explore the general properties of our new model in two situations. In the first case, we consider the situation in which the banks transfer assets from the period 1 to period 2. In the next case, we consider the transfer from the other way round. Figure (6.1) and its associated table summarize all the possible events that could happen in our more general model.

Figure (6.1): Timeline for a bank with a transfer

<table>
<thead>
<tr>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1) Individual doesn’t know their type (early or late)
2) Bank estimate all information they need
3) Banks invest assets (short term or long term)

Bank runs
Early cons.: withdrawal
Late cons.: withdrawal
Banks have to liquidate long term assets at a loss.

No bank runs

Case 1: \(0 < \tau_1 < 1, \tau_2 = 0\)
Early cons.: withdrawal
Transfer: \(\tau_1 y\) \(t=1 \Rightarrow t=2\)

Case 2: \(0 < \tau_2 < 1, \tau_1 = 0\)
Early cons.: withdrawal
Transfer: \(\tau_2 x_r\) \(t=2 \Rightarrow t=1\)

Note: cons. is an abbreviation of consumer.
6.1 Case 1: $0 \leq \tau_1 \leq 1$ and $\tau_2 = 0$

In the original model, we learn that an optimal portfolio solution has to satisfy a relation like equation (4.5.1). In this section, I derive some similar yet different relations that an optimal portfolio has to satisfy in our first case.

Based on the conditions of this case, the equation (4.5) is defined as

$$\lambda u \left( \frac{(1-\tau_1)y}{\lambda} \right) + (1 - \lambda) u \left( \frac{(1-y)R + \tau_1 y}{1-\lambda} \right)$$

Subject to:

$$\begin{cases} 
0 < y < 1 \\
0 \leq \tau_1 \leq 1, \ \tau_2 = 0 \\
R > 1, \ 0 < \lambda < 1.
\end{cases}$$

By calculating the first order condition of equation (6.1) with respect to $\tau_1$, we obtain the following equations:

$$-\frac{y}{\lambda} \times \lambda u'(C_1^*) + \frac{y}{1-\lambda} \times (1 - \lambda)u'(C_2^*) = 0$$

$$\frac{y}{\lambda} \times \lambda u'(C_1^*) = \frac{y}{1-\lambda} \times (1 - \lambda)u'(C_2^*)$$

$$y \times u'(C_1^*) = y \times u'(C_2^*)$$

$$u'(C_1^*) = u'(C_2^*)$$

As promised, I obtain this new relation, equation (6.2), that an optimal portfolio has to satisfy which differs from the original model’s. As $U(C)$ is a strictly increasing, concave, twice continuously differentiable utility function, I know surely that $C_1^*$ is equal to $C_2^*$ as $u'(C_1^*) = u'(C_2^*)$. Hence, it is easy to compute the following equations to obtain the formula of $\tau_1$ in terms of $y$. 

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\[ C_1^* = C_2^*. \]
\[
\frac{(1-\tau_1)y}{\lambda} = \frac{(1-y)R + \tau_1 y}{1-\lambda}
\]
\[ (1 - \lambda)(1 - \tau_1)y = \lambda(1 - y)R + \lambda \tau_1 y \]
\[ (1 - \lambda)y - \lambda(1 - y)R = (1 - \lambda)\tau_1 y + \lambda \tau_1 y \]
\[ \tau_1^* = \frac{(1-\lambda)y - \lambda(1-y)R}{y} \quad (6.3) \]

where \( R > 1, 0 < \lambda < 1, 0 < y < 1, 0 < \tau_1^* < 1. \)

As \( 0 < \tau_1^* < 1, \) we derive the range\(^{10}\) of \(y, \frac{\lambda R}{1 + \lambda R - \lambda} < y < 1.\)

6.2 Case 2: \( 0 < \tau_2 < 1 \) and \( \tau_1 = 0 \)

Similarly, I tried to derive similar relations like equation (6.2) for the case 2 when there is a transfer from time 2 to time 1, namely \( 0 \leq \tau_2 \leq 1 \) and \( \tau_1 = 0. \) First of all, I start with re-defining equation (4.5) for this specific condition.

\[
\lambda u\left(\frac{y + \tau_2(1-y)r}{\lambda}\right) + (1 - \lambda) u\left(\frac{(1-\tau_2)(1-y)R}{1-\lambda}\right)
\]

Subject to: \[
\begin{align*}
0 < y < 1 \\
0 \leq \tau_2 \leq 1, \quad \tau_1 = 0 \\
R > 1, \quad 0 < \lambda < 1.
\end{align*}
\]

Obviously, we can obtain a relation between \(u'(C_1^*)\) and \(u'(C_2^*)\) by computing the first order condition of the equation (6.4) with respect to \(\tau_2.\)

\(^{10}\) Please see Appendix D for more details.
\[
\frac{(1-y)r}{\lambda} \times \lambda u'(C_1^*) - \frac{(1-y)R}{1-\lambda} \times (1 - \lambda)u'(C_2^*) = 0
\]

\[
(1 - y)r \times u'(C_1^*) = (1 - y)R \times u'(C_2^*)
\]

\[
r \times u'(C_1^*) = R \times u'(C_2^*)
\]

\[
\frac{u'(C_1^*)}{u'(C_2^*)} = \frac{R}{r} \tag{6.5}
\]

For the second case, again, we derive another specific relation, equation (6.5), that an optimal solution has to satisfy. However, due to the fact that it is harder to make general inferences on other properties of system without specifying a particular utility function in this case, I cannot give a general formula for \( \tau_2 \) and the range of \( y \) in the case 2.\(^{11}\) However, we should have no problem in solving for \( \tau_2 \) and the range of \( y \) for a particular problem once we are given a specific utility function.

\(^{11}\) There is a specific example to show how to calculate \( \tau_2 \) and the range of \( y \) in Appendix E.
Chapter 7: Conclusions

In this work, we started with Dr. Zhang’s argument that a lack of consideration for a transfer of assets between periods in the original model may not yield the most optimal portfolio as claimed by Franklin Allen and Douglas Gale. We first worked out the necessary modifications to the original model if we would like to incorporate a transfer into the model. Next, we carefully selected an example and evaluate the utility function with two different optimal solutions that we got from calculations done with the original and the new model respectively. Not surprisingly, we were able to show that at least for some specific examples, our new model with a transfer might indeed return a better “optimal” solution. I then proceeded to derive some general relations such as equation (6.2) and equation (6.5) for the two possible scenarios in our model. Since our general relations are very different from the original one, equation (4.5.1), we should expect that our model might yield very different results from the original one under many potential circumstances. Therefore, we have improved the original model.
Appendix A: The Numerical example for the model with a transfer-case 1

\[ 0 \leq \tau_1 \leq 1 \text{ and } \tau_2 = 0, \ r = 0.1, \ R = 1.7, \ \lambda = 0.5, \ \pi = 0.3, \ u(C) = \ln(C) \]

First of all, I need to re-define the equation (4.5)

\[ 0.5 \times \log \left( \frac{(1-\tau_1)y}{0.5} \right) + (1 - 0.5) u \left( \frac{(1-y) \times 1.7 + \tau_1 y}{1 - 0.5} \right) \]  

Subject to: \[ \begin{align*}
0 < y < 1 \\
0 \leq \tau_1 \leq 1, \ \tau_2 = 0
\end{align*} \]

Based on Dr. Zhang’s suggestions\(^{12}\), I will compute the first order condition of the equation (A-1) with respect to \( \tau_1 \).

FOC: \[ - \frac{y}{0.5} \times 0.5 \times u' \left( C_1^* \right) + \frac{y}{1 - 0.5} \times (1 - 0.5) u' \left( C_2^* \right) = 0. \]  

\[ \Rightarrow \quad u' \left( C_1^* \right) = u' \left( C_2^* \right) \quad \Rightarrow \quad C_1^* = C_2^* \]

\[ \Rightarrow \quad \frac{(1 - \tau_1)y}{0.5} = \frac{(1-y) \times 1.7 + \tau_1 y}{1 - 0.5} \]

\[ \Rightarrow \quad (1 - \tau_1)y = (1 - y) \times 1.7 + \tau_1 y \]

\[ \Rightarrow \quad y - \tau_1 y = 1.7 - 1.7y + \tau_1 y \]

\[ \Rightarrow \quad \tau_1^* = \frac{2.7y - 1.7}{2y} \]  

\(^{12}\) Dr. Zhang suggested that I should use two steps to solve the equation (5.3) to obtain the optimal solution. In the first step, he advised me to compute the optimal contract \((C_1^*(y), C_2^*(y))\) by maximizing the equation 

\[ (\lambda u \left( \frac{(1-\tau_1)y + \tau_2 x}{\lambda} \right) + (1 - \lambda) u \left( \frac{(1-\tau_2)x + \tau_1 y}{1 - \lambda} \right)) \]

which is subject to \( 0 \leq \tau_i \leq 1, \ (i=1, 2) \). In this step, I should be able to compute a new range of \( y \). In the second step, once I obtain the optimal contract \((C_1^*(y), C_2^*(y))\), I will plug \((C_1^*(y), C_2^*(y))\) into the equation (5.3) to find the optimal solution \((x^*, y^*)\).
As $\tau_1$ is between 0 and 1, we can derive the range of $y$ in the following steps.

$$0 < \tau_1^* < 1$$

$$\Rightarrow \quad 0 < \frac{2.7y-1.7}{2y} < 1 \quad (0<y<1)$$

$$\Rightarrow \quad 0 < 2.7y - 1.72 < y$$

(A-4)

- The left hand side inequality

$$0 < 2.7y - 1.7 \quad \Rightarrow \quad y > \frac{1.7}{2.7} = 0.63 \quad \text{(A-5)}$$

- The right hand side inequality

$$2.7y - 1.7 < 2y \quad \Rightarrow \quad y < \frac{1.7}{0.7} = 2.43 \quad \text{(A-6)}$$

**Figure (A1): The range of $y$ for case 1**

```
<table>
<thead>
<tr>
<th>0</th>
<th>0.63</th>
<th>2.43</th>
</tr>
</thead>
</table>
```

```
| 0.5 | 1    |
```

$0.63 < y < 1$

From the figure A1, it is easy to show the range of $y$ is $0.63 < y < 1$. Once we know the equation (A-3), I can compute $C_i^*(y), (i = 1, 2)$. $C_1^*$ and $C_2^*$ are equivalent in the case 1. For simplification, I will set $C_i^*(y), (i = 1, 2)$ equal to equation $\frac{(1-\tau_1)y}{0.5}$.

$$C_i^*(y) = C_2^*(y) = \frac{(1-\tau_1)y}{\lambda} = \frac{y-1.35y+0.85}{0.5} = y + (1 - y)1.7$$
Now, I start to work on solving the expected utility function of the representative depositor. The utility function can be re-defined as the following equations.

\[
0.3 \log (y + 0.1(1 - y)) U + 0.7\log(1.7 - 0.7y) \quad (A-7)
\]

Subject to: \[0.63 < y < 1\]

Accordingly, I obtain the optimal portfolio \((1 - y^*, y^*)\) by maximizing the equation \((A-7)\).

\[
\text{FOC:} \quad 0.3 (1 - 0.1)u'(y + 0.1(1 - y)) + 0.7(1 - 1.7)u'((y + (1 - y)1.7)) = 0.
\]

\[
\Rightarrow \quad \frac{0.27}{0.1(1-y)+y} = \frac{0.49}{1.7(1-y)+y}
\]

\[
\Rightarrow \quad 0.49[0.1(1-y) + y] = 0.27[1.7(1-y) + y]
\]

\[
\Rightarrow \quad y = 0.65
\]

The result of \(y^*\) is 0.65, which lies between 0.63 and 1. Thus, the optimal portfolio \((1 - y^*, y^*)\) is \((0.35, 0.65)\). As \(y^*\) is 0.65, the relative the \(\tau_1^*\) and \(C_1^*(y), C_2^*(y)\), and \(u (C_1^*(y), C_2^*(y))\) can be calculated.

\[
\Rightarrow \quad \tau_1^* = \frac{2.7y - 1.7}{2y} = \frac{2.7 \times 0.65 - 1.7}{2 \times 0.65} = 0.042
\]

\[
\Rightarrow \quad C_1^*(y) = C_2^*(y) = \frac{(1-\tau_1)y}{\lambda} = \frac{(1-0.042) \times 0.65}{0.5} = 1.245
\]

\[
\Rightarrow \quad U = 0.3 \log (y^* + 0.1 (1 - y^*)) + 0.7\log(1.7 - 0.7y^*)
\]

\[
= 0.3 \log (0.65 + 0.1(1 - 0.65)) + 0.7\log (1.7 - 0.7 \times 0.65)
\]

\[
= 0.0399
\]
Table (A1): Results for the case 1

<table>
<thead>
<tr>
<th>((x^<em>, y^</em>))</th>
<th>(0.35, 0.65)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_1^<em>(y) = C_2^</em>(y))</td>
<td>1.254</td>
</tr>
<tr>
<td>Maximum utility</td>
<td>0.0399</td>
</tr>
</tbody>
</table>

Figure (A2): The graph of the equation (A-7) in the range of \((0, 1)\)
Appendix B: The Numerical example for the model with a transfer-case 2

\[ 0 \leq \tau_2 \leq 1 \text{ and } \tau_1 = 0, \ r = 0.1, \ R = 1.7, \ \lambda = 0.5, \ \pi = 0.3, \ u(C) = \ln(C) \]

In the same way I did in case 1, I will re-write equation (4.5) in this example.

\[
0.5 \times \log \left( \frac{1 + \tau_2 (1 - y) \times 0.1}{0.5} \right) + (1 - 0.5) \times \left( \frac{1.7 \times (1 - \tau_2)(1 - y)}{1 - 0.5} \right) \quad (B-1)
\]

Subject to: \[ \begin{align*}
0 < y < 1 \\
0 \leq \tau_2 \leq 1, \ \tau_1 = 0
\end{align*} \]

The next step is to compute first order condition of the equation (B-1) respect to \( \tau_2 \).

\[
\text{FOC:} - \frac{0.1 \times (1-y)}{0.5} \times 0.5 \times u'(C_1^*) - \frac{1.7 \times (1-y)}{1-0.5} \times (1-0.5) u'(C_2^*) = 0 \quad (B-2)
\]

\[
\Rightarrow \ r \times u'(C_1^*) = R \times u'(C_2^*) \Rightarrow 0.1 \times u'(C_1^*) = 1.7 \times u'(C_2^*)
\]

\[
\Rightarrow 0.1 \times \left[ \frac{1}{y + \tau_2 (1 - y) \times 0.1} \right] = 1.7 \times \left[ \frac{1}{1 - 0.5} \right]
\]

\[
\Rightarrow \frac{1.7 \times (1 - \tau_2)(1 - y)}{[y + \tau_2 (1 - y) \times 0.1]} = \frac{\log(C_1^*)'}{\log(C_2^*)'} = \frac{C_z^2}{C_1^*} = \frac{R}{r} = 17
\]

From the above calculations, it is easy to verify that the ratio of \( \frac{u'(C_1^*)}{u'(C_2^*)} \) is equal to \( \frac{R}{r} \) in the case 2.

Since I got a specific utility function \( \ln(C) \) in this example, then we can calculate the formula of \( \tau_2 \).
\[
\frac{[1.7 \times (1 - \tau_2)(1 - y)]}{0.5} = \frac{u'(C_1)}{u'(C_2)} = \frac{R}{r}
\]

\[\Rightarrow\]
\[
\frac{1.7 \times (1 - \tau_2)(1 - y)}{y + \tau_2(1 - y) \times 0.1} = \frac{R}{r} = \frac{1.7}{0.1}
\]

\[\Rightarrow\]
\[
\tau_2 = \frac{0.1(y - 0)}{0.2(1 - y)}
\]

(B-3)

The equation (B-3) has to be greater than 0 and less than 1.

\[0 < \tau_2^* < 1\]

\[\Rightarrow\]
\[
0 < \frac{0.1(y - 0)}{0.2(1 - y)} < 1
\]

\[\Rightarrow\]
\[
0 < 0.1(1 - y) - y < 0.2(1 - y)
\]

(B-4)

- The left hand side inequality

\[0 < 0.1(1 - y) - y \quad \Rightarrow \quad y < \frac{1}{11} = 0.09\]

(B-5)

- The right hand side inequality

\[0.1(1 - y) - y < 0.2(1 - y) \quad \Rightarrow \quad y > \frac{-2}{0.9} = -2.22\]

(B-6)

**Figure (B1): The range of y for the case 2**

On the other hand, once I have the equation of \(\tau_2\), I can calculate \(C_1^*\) and \(C_2^*\).

\[C_1^* = \frac{y + \tau_2(1 - y)r}{\lambda} = \frac{y + [0.1(y - 0)] \times (1 - y) \times 0.1}{0.5} = y + 0.1(1 - y)\]

(B-7)
Based on the equation \((B-7)\) and equation \((B-8)\), it is shown that the ratio of \(C_2^*/C_1^*\) is \(R/r\), where \(R/r\) is 17 in this example.

Finally, the expected utility of the depositor in this example can be written as the following equation \((B-9)\).

\[
0.3 \log (y + 0.1 (1 - y)) + 0.35 \left( \log (y + 0.1 (1 - y)) + \log(1.7 (1 - y) + 17y) \right)
\]

Subject to: \(0 < y < 1/11\) \hspace{1cm} \(B-9\)

As the investors’ utility function is a strictly increasing, concave, twice continuously differentiable logarithmic function, it means that the value of \(U(y^*, C_1^*(y), C_2^*(y))\) is smaller than or equal to \(U\left(\frac{1}{11}, C_1^*\left(\frac{1}{11}\right), C_2^*\left(\frac{1}{11}\right)\right)\) as \(y^* < 1/11\). By calculations, I am able to know the value of \(U\left(\frac{1}{11}, C_1^*\left(\frac{1}{11}\right), C_2^*\left(\frac{1}{11}\right)\right)\) is smaller than 0.

\[
\begin{align*}
C_1^* &= y + 0.1(1 - y) \quad \Rightarrow \quad C_1^*\left(\frac{1}{11}\right) = \frac{1}{11} + 0.1 \left(1 - \frac{1}{11}\right) = 0.18 \\
C_2^* &= 1.7 (1 - y) + 17y \quad \Rightarrow \quad C_2^*\left(\frac{1}{11}\right) = 1.7 \left(1 - \frac{1}{11}\right) + 17 \times \left(\frac{1}{11}\right) = 3.09
\end{align*}
\]

\[
U\left(\frac{1}{11}, C_1^*\left(\frac{1}{11}\right), C_2^*\left(\frac{1}{11}\right)\right)
\]

\[
= 0.3 \log \left(\frac{1}{11} + 0.1 \left(1 - \frac{1}{11}\right)\right) + 0.35 \left( \log \left(\frac{1}{11} + 0.1 \left(1 - \frac{1}{11}\right)\right) + \log(1.7 \left(1 - \frac{1}{11}\right) + 17 \times \frac{1}{11}) \right)
\]

\[
= -0.72 < 0
\]
Thus, the value of $U(y^*, C_1^*(y), C_2^*(y))$ is smaller than 0 as well in this example, namely $U(y^*, C_1^*(y), C_2^*(y)) \leq U\left(\frac{1}{11}, C_1^*(\frac{1}{11}), C_2^*(\frac{1}{11})\right) \leq 0$.

Table (B1): Results for the case 2

| Maximum utility | $U(y^*, C_1^*(y), C_2^*(y)) \leq U\left(\frac{1}{11}, C_1^*(\frac{1}{11}), C_2^*(\frac{1}{11})\right) < 0$ |
Appendix C: Results

According to the previous calculations for three different situations, I achieve three different solutions. All results are shown in the following table (C1).

Table (C1): Results for three different cases

<table>
<thead>
<tr>
<th>The model with transfer between time 1 and time 2</th>
<th>Case 1(^{13}): (0 \leq \tau_1 \leq 1) and (\tau_2 = 0)</th>
<th>Max (U_1 = 0.0399)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 2(^{14}): (0 \leq \tau_2 \leq 1) and (\tau_1 = 0)</td>
<td>Max (U_2 &lt; 0)</td>
<td></td>
</tr>
<tr>
<td>The model without transfer between time 1 and time 2</td>
<td>The model without transfer(^{15}): (\tau_1 = \tau_2 = 0)</td>
<td>Max (U_3 = 0.0397)</td>
</tr>
</tbody>
</table>

Looking at the table (C1), it is obvious to know that the largest value of the utility function is generated by using the case 1 method when we solve an optimal problem in the example. Based on Dr. Zhang’s idea of criteria that an optimal solution maximizes the utility function, we may conclude the case 1 method is the best way to solve an optimal problem in this particular example. It proves that it is possible to maximize investors’ utility more efficiently if there is an appropriate

\(^{13}\) In the case 1, there is a transfer from time 1 to time 2. I obtain \(u'(C_1^*) = u'(C_2^*)\) or \(C_1^* = C_2^*\).

\(^{14}\) Case 2 means there is only a transfer from time 2 to time 1. In this case, \(\frac{u'(C_1)}{u'(C_2)} = \frac{R}{r}\).

\(^{15}\) The model without transfer is the Franklin Allen and Douglas Gale’s model in the book “Understanding Financial Crisis” (2007). In this model, \(C_1\) is equal to \(\frac{x}{1-x}\), and \(C_2\) is equal to \(\frac{R_x}{(1-x)^2}\), and \(u'(C_1^*) = R u'(C_2^*)\).
transfer between period 1 and period 2 for specific case. Hence, a transfer between
date 1 and date 2 should be considered in the model when I want to maximize
depositor’s utility function.
Appendix D: Derivation of the range of y of the model with \( \tau_1^* \)

As we get \( \tau_1 \), we can use it to derive the range of y because \( \tau_1 \) is greater than 0 and less than 1.

\[
0 < \tau_1^* < 1
\]

\[
\Rightarrow 0 < \frac{(1-\lambda)y - \lambda(1-y)R}{y} < 1
\]

\[
\Rightarrow 0 < (1 - \lambda)y - \lambda(1 - y)R < y
\]

(D-1)

- The left hand side inequality

\[
0 < (1 - \lambda)y - \lambda(1 - y)R \Rightarrow y > \frac{\lambda R}{1 + \lambda R - \lambda} > 0
\]

(D-2)

\[
0 < (1 - \lambda)y - \lambda(1 - y)R \Rightarrow \frac{(1-\lambda)y}{\lambda(1-y)} > R > 1
\]

(D-3)

- The right hand side inequality

\[
(1 - \lambda)y - \lambda(1 - y)R < y \Rightarrow y < \frac{R}{R-1}
\]

(D-4)

\[
(1 - \lambda)y - \lambda(1 - y)R < y \quad 0 < \frac{y}{1-y} < R
\]

(D-5)

From the inequality (D-3), the value of \( \frac{(1-\lambda)y}{\lambda(1-y)} \) has to be greater than 1 as well because R is greater than 1, namely \( \frac{(1-\lambda)y}{\lambda(1-y)} > R > 1 \).

\[
\frac{(1-\lambda)y}{\lambda(1-y)} > 1
\]

\[
\Rightarrow (1 - \lambda)y > \lambda(1 - y)
\]

\[
\Rightarrow y - \lambda y > \lambda - \lambda y
\]

\[
\Rightarrow y > \lambda
\]

(D-6)
The inequality (D-5) is true all the time since \( R \) is always greater than \(-\frac{y}{1-y}\), where \(-\frac{y}{1-y}\) is always negative.

In summary, the ratio of \( \frac{R}{R-1} \) is always greater than 1 when \( R \) is always greater than 1. As we know, the value of \( y \) has to be smaller than 1. Thus, the upper bound of \( y \) is 1. Additionally, I need to compare with the inequality (D-2) and the inequality (D-6) to obtain the lower bound of \( y \). I will subtract \( \lambda \) from \( \frac{\lambda R}{1+\lambda R-\lambda} \) in order to gain the difference between them. If this difference is always positive, then \( \frac{\lambda R}{1+\lambda R-\lambda} \) is greater than \( \lambda \). Otherwise, \( \frac{\lambda R}{1+\lambda R-\lambda} \) is smaller than \( \lambda \).

\[
\frac{\lambda R}{1+\lambda R-\lambda} - \lambda = \frac{\lambda R - \lambda (1 + \lambda R - \lambda)}{1 + \lambda R - \lambda} = \frac{\lambda (R-1-\lambda R+\lambda)}{1 + \lambda R - \lambda} = \frac{\lambda (1-\lambda)(R-1)}{1 + \lambda R - \lambda} \quad (D-7)
\]

By calculations, the difference is \( \frac{\lambda (1-\lambda)(R-1)}{1 + \lambda R - \lambda} \). As the range of \( \lambda \) and \( R \) are known, it is simple to know that the ratio of \( \frac{\lambda (1-\lambda)(R-1)}{1 + \lambda R - \lambda} \) is always greater than 0. This implies that \( \frac{\lambda R}{1+\lambda R-\lambda} \) is greater than \( \lambda \). As \( \frac{\lambda R}{1+\lambda R-\lambda} \) is always greater than 0, it is considered \( \frac{\lambda R}{1+\lambda R-\lambda} \) as the lower bound of \( y \). Therefore, the value of \( y \) should lie between \( \frac{\lambda R}{1+\lambda R-\lambda} \) and 1, denoted as \( \frac{\lambda R}{1+\lambda R-\lambda} < y < 1 \).
Figure (D1): The range of $y$ in case 1

$0 < \frac{\lambda R}{1+\lambda R-\lambda} < y < 1$
Appendix E: The example for the case 2

In the section 6.2, I only get a specific relation, equation (6.5) because it is harder to compute other properties without specifying a particular utility function in the case 2. Hence, I will suppose a particular investor’s utility function to show how to obtain other properties of case 2. For example, the investor’s utility function, \( u(C) \), is \( e^{1-\sigma} \), where \( \sigma \) is defined between 0 and 1.

For the specific utility function, the equation (4.5) becomes

\[
\max U = \lambda \frac{(y + \tau_2(1-y)r)1-\sigma}{1-\sigma} + (1 - \lambda) \frac{(1-\tau_2)1-\sigma}{1-\sigma} \tag{E-1}
\]

Subject to:
\[
\begin{align*}
0 &< y < 1 \\
0 &\leq \tau_2 \leq 1, \quad \tau_1 = 0 \\
R &> 1, \quad 0 < \lambda < 1, \quad 0 < \sigma < 1.
\end{align*}
\]

By the same approaches, I need to compute the first order condition of the equation (E-1) with respect to \( \tau_2 \).

\[
\text{FOC:} \quad \frac{(1-y)r}{\lambda} \times \lambda \times u'(c_1^*) - \frac{(1-y)R}{1-\lambda} \times (1 - \lambda) \times u'(c_2^*) = 0
\]

\[
\Rightarrow \quad \frac{(1-y)r}{\lambda} \times \lambda \times \left(\frac{y + \tau_2(1-y)r}{\lambda}\right)^{-\sigma} - \frac{(1-y)R}{1-\lambda} \times (1 - \lambda) \times \left(\frac{(1-\tau_2)(1-y)R}{1-\lambda}\right)^{-\sigma} = 0
\]

\[
\Rightarrow \quad (1 - y)r \times \left(\frac{y + \tau_2(1-y)r}{\lambda}\right)^{-\sigma} = (1 - y)R \times \left(\frac{(1-\tau_2)(1-y)R}{1-\lambda}\right)^{-\sigma}
\]

\[
\Rightarrow \quad \frac{u'(c_1^*)}{u'(c_2^*)} = \frac{\left(\frac{y + \tau_2(1-y)r}{\lambda}\right)^{-\sigma}}{\left(\frac{(1-\tau_2)(1-y)R}{1-\lambda}\right)^{-\sigma}} = \frac{R}{r}
\]

\[
\Rightarrow \quad \left(\frac{y + \tau_2(1-y)r}{(1-\tau_2)(1-y)R}\right)^{-\sigma} = \frac{y + \tau_2(1-y)r}{(1-\tau_2)(1-y)R} - \sigma = \frac{R}{r}
\]
\[
\frac{(1-\tau_2)(1-y)R}{y+\tau_2(1-y)r} = \left(\frac{R}{r}\right)^{1/\sigma} = k
\]

\[
\frac{c_2^*}{c_1^*} = \left(\frac{R}{r}\right)^{1/\sigma} = k
\]

\[
c_2^* = k \times c_1^*
\]

(E-2)

\[
\frac{(1-\tau_2)(1-y)R}{1-\lambda} = k \times \frac{y+\tau_2(1-y)r}{\lambda}
\]

\[
(1-\tau_2)(1-y)R \times \lambda = k \times (1-\lambda) \times (y+\tau_2(1-y)r)
\]

\[
(1-y)R \times \lambda - \tau_2 \times (1-y)R \times \lambda = k \times (1-\lambda) \times y + k \times (1-\lambda) \times \tau_2(1-y)r
\]

\[
\tau_2 \times (1-y) \times (\lambda R + (1-\lambda) \times k \times r) = (1-y)R \times \lambda - k \times (1-\lambda) \times y
\]

\[
\tau_2^* = \frac{(1-y)R \times \lambda - k \times (1-\lambda) \times y}{(1-y) \times (\lambda R + (1-\lambda) \times k \times r)}
\]

(E-3)

where \( R > 1, 0 < \lambda < 1, 0 < y < 1, 0 < \tau_2^* < 1, \left(\frac{R}{r}\right)^{1/\sigma} = k, 0 < \sigma < 1. \)

Once I obtain the formula of \( \tau_2^* \), I will calculate the range of \( y \). Since \( \tau_2 \) is defined between 0 and 1, then the equation (E-3) has to satisfy these constrains:

\[
0 < \tau_2^* < 1
\]

\[
0 < \frac{(1-y)R \times \lambda - k \times (1-\lambda) \times y}{(1-y) \times (\lambda R + (1-\lambda) \times k \times r)} < 1
\]

\[
0 < (1-y)R \times \lambda - k \times (1-\lambda) \times y < (1-y) \times (\lambda R + (1-\lambda) \times k \times r)
\]

(E-4)

where \( R > 1, 0 < \lambda < 1, 0 < y < 1, 0 < \tau_2^* < 1, \left(\frac{R}{r}\right)^{1/\sigma} = k, 0 < \sigma < 1. \)
The left hand side inequality

\[ 0 < (1 - y)R \times \lambda - k \times (1 - \lambda) \times y \]
\[ \Rightarrow k \times (1 - \lambda) \times y < (1 - y)R \times \lambda \]
\[ \Rightarrow (k \times (1 - \lambda) + \lambda R) \times y < \lambda R \]
\[ \Rightarrow y < \frac{\lambda R}{k \times (1 - \lambda) + \lambda R} < 1 \]  
(E-5)

where \( R > 1, 0 < \lambda < 1, 0 < y < 1, 0 < \tau^*_2 < 1, \left(\frac{R}{r}\right)^{1/\sigma} = k, 0 < \sigma < 1. \)

\[ 0 < (1 - y) \times R \times \lambda - k \times (1 - \lambda) \times y \]
\[ \Rightarrow k \times (1 - \lambda) \times y < (1 - y) \times R \times \lambda \]
\[ \Rightarrow \left(\frac{R}{r}\right)^{1/\sigma} \times (1 - \lambda) \times y < (1 - y) \times R \times \lambda \]
\[ \Rightarrow \left(\frac{R^{1-\sigma}}{r}\right)^{\frac{1}{\sigma}} < \frac{(1 - y) \times \lambda}{(1 - \lambda) \times y} \]  
(E-6)

where \( R > 1, 0 < \lambda < 1, 0 < y < 1, 0 < \tau^*_2 < 1, \left(\frac{R}{r}\right)^{1/\sigma} = k, 0 < \sigma < 1. \)

The right hand side inequality

\[ (1 - y)R \times \lambda - k \times (1 - \lambda) \times y < (1 - y) \times (\lambda R + (1 - \lambda) \times k \times r) \]
\[ \Rightarrow \lambda R - \lambda R \times y - k \times (1 - \lambda) \times y < (\lambda R + (1 - \lambda) \times k \times r) - y \times (\lambda R + (1 - \lambda) \times k \times r) \]
\[ \Rightarrow (\lambda R + (1 - \lambda) \times k \times r - \lambda R - k \times (1 - \lambda)) \times y < \lambda R + (1 - \lambda) \times k \times r - \lambda R \]
\[ \Rightarrow -(1 - \lambda) \times k \times (1 - r) \times y < (1 - \lambda) \times k \times r \]
\[ \Rightarrow y > 0 > \frac{-r}{1-r} \]  
(E-7)

where \( 0 < r < 1, 0 < y < 1. \)
\[(1 - y) \times R\lambda - k \times (1 - \lambda) \times y < (1 - y) \times (\lambda R + (1 - \lambda) \times k \times r)\]

\[\Rightarrow -k \times (1 - \lambda) \times y < (1 - y) \times (1 - \lambda) \times k \times r\]

\[\Rightarrow \frac{-y}{1-y} < 0 < r < 1 \quad (E-8)\]

By calculating the left hand side inequality, I get an inequality (E-6). It is easy to learn \(\left(\frac{R^{1-\sigma}}{r^{1/\sigma}}\right) > 1\) when R is always greater than 1, and r is always between 0 and 1. Then we can compute the following inequality to get a range of y.

\[
\frac{(1-y) \times \lambda}{(1-\lambda) \times y} > 1
\]

\[\Rightarrow (1 - y) \times \lambda > (1 - \lambda) \times y\]

\[\Rightarrow y < \lambda \quad (E-9)\]

Based on the inequality (E-6) – (E-9), we can find the range of y in this example. As r is always greater than 0, the ratio of \(\frac{-r}{r-1}\) is always negative. As the original range of y is between 0 and 1, the lower bound of y is 0. In addition, I will calculate the difference between \(\lambda\) and \(\frac{\lambda R}{k \times (1 - \lambda) + \lambda R}\) to decide the upper bound of y.

\[
\frac{\lambda R}{k \times (1 - \lambda) + \lambda R} - \lambda = \frac{\lambda R - \lambda \times [k \times (1 - \lambda) + \lambda R]}{k \times (1 - \lambda) + \lambda R}
\]

\[
= \frac{\lambda \times [k \times (1 - \lambda) - \lambda \times R]}{k \times (1 - \lambda) + \lambda R}
\]

\[
= \frac{\lambda \times (1 - \lambda) \times (R-k)}{k \times (1 - \lambda) + \lambda R}
\]

\[
= \frac{\lambda \times (1 - \lambda) \times (R - \left(\frac{R^{1/\sigma}}{r^{1/\sigma}}\right)}{k \times (1 - \lambda) + \lambda R} \quad (E-10)
\]
where \( R > 1, 0 < \lambda < 1, 0 < y < 1, \ 0 < \tau_2^* < 1, (\frac{R}{r})^{1/\sigma} = k, 0 < \sigma < 1. \)

From the equation (E-10), \((k \times (1 - \lambda) + \lambda R)\) is always positive. \((R - (\frac{R}{r})^{1/\sigma})\) is less than 0 as \( 0 < \sigma < 1, R > 1, 0 < r < 1. \) Thus, the value of \( \frac{\lambda \times (1 - \lambda) \times (R - (\frac{R}{r})^{1/\sigma})}{k \times (1 - \lambda) + \lambda R} \) is negative in this case. It means that \( \frac{\lambda \times (1 - \lambda) \times (R - (\frac{R}{r})^{1/\sigma})}{k \times (1 - \lambda) + \lambda R} \) is smaller than \( \lambda, \) and the upper bound of \( y \) is \( \frac{\lambda R}{k \times (1 - \lambda) + \lambda R} \). Therefore, the value of \( y \) should lie between 0 and \( \frac{\lambda R}{k \times (1 - \lambda) + \lambda R} \), denoted as \( 0 < y < \frac{\lambda R}{k \times (1 - \lambda) + \lambda R} \).

**Figure (E1): The range of \( y \) for a particular example of the case 2**

As I obtain the formula of \( \tau_2^* \), I can compute the equations of \( C_1^*(y) \) and \( C_2^*(y) \) in terms of \( y \) as the following equations.

\[
C_i^*(y) = \frac{y + \tau_2^* (1 - y) \times r}{\lambda} = \frac{y + [(1 - y) \times (1 - \lambda) \times k \times \lambda] \times (1 - y) \times r}{(1 - y) \times (\lambda R + (1 - \lambda) \times k \times r)} \times \frac{\lambda R + (1 - \lambda) \times k \times r}{\lambda}
\]
\[
\begin{align*}
&= \frac{y \times \lambda R + (1 - \lambda) \times k \times x \times y + (1 - y) R \times \lambda \times x \times r - k \times (1 - \lambda) \times y \times x \times r}{\lambda \times [\lambda R + (1 - \lambda) \times k \times x \times r]} \\
&= \frac{y \times \lambda R + (1 - y) R \times \lambda \times x \times r}{\lambda \times [\lambda R + (1 - \lambda) \times k \times x \times r]} \\
&= \frac{y \times R + (1 - y) \times R \times x \times r}{[\lambda R + (1 - \lambda) \times k \times x \times r]} \\
&= \frac{R}{[\lambda R + (1 - \lambda) \times k \times x \times r]} \times [y + r \times (1 - y)] \quad \text{(E-11)}
\end{align*}
\]

where \( R > 1, 0 < \lambda < 1, 0 < r < 1, 0 < y < \frac{\lambda R}{k \times (1 - \lambda) + \lambda R}, \left(\frac{R}{r}\right)^{1/\sigma} = k. \)

\[C_2^*(y) = \frac{(1 - \tau_2)(1 - y) \times R}{1 - \lambda} \]

\[= \frac{(1 - \lambda) \times k \times R \times (y + r \times (1 - y))}{(1 - \lambda) \times [\lambda R + (1 - \lambda) \times k \times x \times r]} \times [y + r \times (1 - y)] \quad \text{(E-12)}
\]

where \( R > 1, 0 < \lambda < 1, 0 < r < 1, 0 < y < \frac{\lambda R}{k \times (1 - \lambda) + \lambda R}, \left(\frac{R}{r}\right)^{1/\sigma} = k. \)

From the equations (E-11) and (E-12), it is easy to show that \( C_2 \) is as much as \( k \) times of \( C_1 \) in the case 2, where \( k \) is equal to \( \left(\frac{R}{r}\right)^{1/\sigma} \). Additionally, it is computed that the ratio of \( \frac{u'(C_1)}{u'(C_2)} \) is equal to \( R/r \) in the beginning. Then, I should be able to use the equation (E-11) and (E-12) to verify it

\[
\frac{u'(C_1)}{u'(C_2)} = \frac{(C_1)^{-\sigma}}{(C_2)^{-\sigma}} = \frac{R}{r}.
\]
References


