Lending Relationships and Optimal Monetary Policy*

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Abstract

We develop a monetary model of internal and external finance with bank entry and endogenous formation of lending relationships through search and bargaining to study various shocks to banking. Following an unanticipated destruction of relationships, optimal monetary policy under commitment lowers the interest rate in the aftermath of the shock and uses forward guidance to promote bank entry and rebuild relationships. Absent commitment, optimal policy is subject to a deflationary bias that delays recovery. If there is a temporary freeze of relationship creation, then the interest rate is set at the zero lower bound for some period of time.

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1 Introduction

Most businesses, especially small ones, rely on secure access to credit through stable relationships with banks to finance investments. According to the 2003 National Survey of Small Business Finances, 85% of small businesses have some form of credit line or revolving credit arrangement. In times of financial crisis, some of these relationships are severed due to bank failures or tighter lending standards (see Appendix B for additional empirical support). During the 2007–2008 financial crisis, the number and value of new small business loans declined by approximately 70% and 50%, respectively (see left panel of Figure 1). Similarly during the Great Depression, Cohen, Hachem, and Richardson (2016) find the destruction of lending relationships explains one-eighth of the economic contraction.

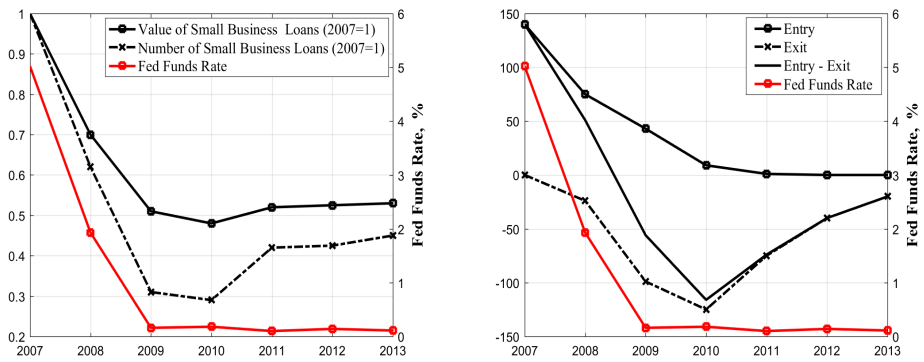

Figure 1: Left: Fed funds rate, number and value of small business loans; Right: bank entry and exit (source: see Appendix B)

Monetary policy responded to the 2007–2008 financial crisis by expanding the monetary base and lowering short-term nominal rates. According to different views of the transmission mechanism, such expansionary monetary policy lowers the holding cost of cash (money channel) and the cost of external finance through credit (lending channel).[1] However, low interest rates can also reduce banks' incentives to participate in the credit market, as indicated by declines in the number of loans offered or net bank entry (see right panel of Figure 1). This adverse effect creates a tradeoff for the policymaker between lowering the holding cost of liquid assets and promoting access to external

[1] Restoring credit was a key motivating factor behind the policy response following the Great Recession. At the July 2010 Federal Reserve Meeting Series "Addressing the Financing Needs of Small Businesses," Chairman Ben Bernanke stated that "making credit accessible to sound small businesses is crucial to our economic recovery and so should be front and center among our current policy challenges [...] the formation and growth of small businesses depends critically on access to credit, unfortunately, those businesses report that credit conditions remain very difficult."

finance through banks.[2]

To capture this trade off, we develop a general equilibrium model of lending relationships and corporate finance that studies optimal monetary policy in the aftermath of a crisis. In the model economy, entrepreneurs receive idiosyncratic investment opportunities, as in Kiyotaki and Moore (2005), which can be financed with bank credit or retained earnings in liquid assets. Liquid assets are modeled as fiat money, the rate of return of which is controlled by the central bank. The role of money is to provide insurance against idiosyncratic investment opportunities. Similarly, we emphasize the insurance role of relationship lenders to secure funding of future investment opportunities.[3] Due to search and information frictions, relationships take time to form, and the terms of the loans are negotiated bilaterally (as in, e.g., Wasmer and Weil 2004).

In order to generate a role for banks (even at the Friedman rule when liquidity is abundant), we assume bank financing is superior to financing with money, e.g., banks' liabilities have broader acceptability than fiat money, or they are less subject to counterfeiting or theft.[4] In addition, we assume banks' bargaining power is low relative to their share in the creation of lending relationships, thereby generating inefficiently low bank entry at the Friedman rule. We rule out direct subsidies to banks to focus on second-best analysis with non-trivial implications for monetary policy.[5]

In our model, the transmission mechanism of monetary policy works through both firms' choice of real balances (internal finance) and the creation of lending relationships (external finance). As the *nominal* rate increases, it becomes more costly to hold liquid assets, thereby making firms more eager to enter a lending relationship, which raises the *real* lending rate negotiated with banks. In addition, a higher real lending rate increases banks' incentives to enter the credit market thereby promoting bank entry and the creation of lending relationships.

[2] There is a related literature in monetary economics on the monetary policy trade-off between enhancing the rate of return of money to promote self-insurance and providing risk sharing through lump-sum transfers financed with money creation in models with non-degenerate distributions of money holdings. See, e.g., Wallace (2013). Similar to our model, the trade-off arises because policy affects differently the intensive and extensive margins of trade. Different from this literature, our model has both money and credit and the extensive margin corresponds to the endogenous formation of lending relationships with banks.

[3] We downplay other roles of banks and lending relationships, such as monitoring in the presence of agency problems or screening under informational asymmetries, which we think are of lesser importance for the optimal design of monetary policy. See Section 1.1 for a literature review.

[4] In most models that abstract from distributional effects, the Friedman rule, which sets the nominal interest rate permanently to zero, is optimal and makes credit inessential. This property holds for the models of money and credit in Rocheteau and Nosal (2017, Ch. 8) or Gu, Mattesini, and Wright (2016). Sanches and Williamson (2010) make the coexistence of money and credit essential by introducing theft, which is broadly consistent with our formalization.

[5] In reality, some central banks can pay interests on reserves or dividends on their stocks but such payments compensate for the opportunity cost of holding idle reserves or stocks of the central bank, hence do not constitute direct subsidies to bank creation.

We parameterize the model to match moments in the U.S. economy from the 2003 National Survey of Small Business Finances (SSBF) and use the calibrated model to describe the economy's response to a negative shock described as an exogenous and unanticipated destruction of lending relationships starting from a steady state. With constant money growth, the banking shock generates a flight to liquidity (in our model, real balances) as entrepreneurs lose access to credit. The initial deflation is followed by positive inflation that falls gradually over time as the banking sector recovers and firms' demand for cash shrinks. Inflation makes it more costly to hold liquid assets, thereby generating higher real lending rates, higher bank profits, and a higher rate of credit creation. In contrast, if the monetary authority targets a constant nominal interest rate, the credit creation rate remains constant throughout the transition to steady state. As these two policy rules are arbitrary, we then turn to the optimal monetary response to a banking shock under different assumptions on the policymaker's power to commit.

If the policymaker can commit over long time horizons, optimal policy entails setting low nominal interest rates close to the zero lower bound (ZLB) at the outset of the crisis to promote internal finance by newly unbanked firms. To maintain banks' incentives to participate in the credit market despite low interest rates, the policymaker uses "forward guidance" by promising high inflation and high nominal interest rates in the future.

However, forward guidance is not time consistent. Indeed, once lending relationships have been rebuilt, the policymaker faces the temptation to set low interest rates to maximize investment by unbanked firms. We therefore relax the commitment assumption and let the policymaker set the interest rate period by period taking as given future policies. We focus on Markov perfect equilibria and find that in order to maintain banks' incentives to create lending relationships, the policymaker does not lower the initial interest rate as much as it does under commitment. The interest rate falls over time but by a small amount and is typically lower than under commitment. Hence, optimal policy absent commitment exhibits a low interest rate bias, or deflationary bias, i.e. the policymaker cannot credibly promise high interest rates in the future.[6] The recovery is considerably slower than under commitment, e.g. by about two years in the calibrated model.

In our benchmark model, it is always optimal (with or without commitment) to keep the interest rate bounded away from zero when the Friedman rule is suboptimal. Since the zero lower bound (ZLB) has been a significant constraint in the aftermath of the 2007–2008 financial crisis, we provide

[6] This deflationary bias is in sharp contrast with the inflationary bias emphasized in the time consistency literature since Kydland and Prescott (1977) and Barro and Gordon (1983).

several explanations for why setting the interest rate at the ZLB may be optimal in times of crisis even though a deviation from the ZLB is optimal in normal times. First, we introduce imperfect competition in the capital goods market by assuming the price is set at a positive markup over marginal cost. In that case, the welfare loss from being away from the Friedman rule for unbanked entrepreneurs is first order. Following a shock that destroys lending relationships, it becomes optimal to set the nominal interest rate at zero for some period of time.

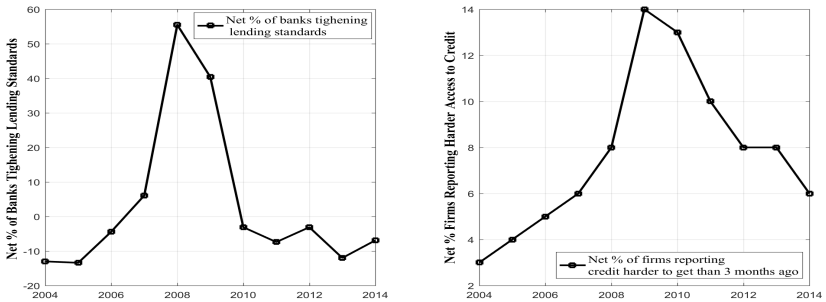

Figure 2: Lending standards and availability of credit

Alternatively, it can be optimal to set the interest rate at the ZLB if there is a temporary freeze in the creation of relationships. For instance, following the Great Recession banks tightened lending standards and firms had a harder time accessing credit, as illustrated in Figure 2.[7] If the freeze is severe, the trade-off between the rate of return of money and the creation of lending relationships vanishes: the only objective of the policymaker is to maximize the rate of return of money, which is achieved at the ZLB.

Finally, we extend the model to incorporate both transaction and relationship lenders to study how optimal policy differs in the presence of an alternative source of external finance. Transaction lenders offer one-time loans and the terms of the loan contract are negotiated at the time the investment opportunity materializes. In that case, interest payments to relationship lenders are larger than those to transaction lenders when the nominal rate is low. Moreover, access to transaction lenders reduces the optimal interest rate.

[7] The left panel of Figure 2 shows the net percentage of senior loan officers reporting tighter standards on commercial and industrial loans from the Senior Loan Officer Opinion Survey on Bank Lending; the right panel shows the net percentage of firms reporting more difficult access to loans from the Survey of Small and Independent Business Owners.

1.1 Literature

Our model, which builds on Lagos and Wright (2005) and its competitive version by Rocheteau and Wright (2005), is part of the New Monetarist literature that explains the coexistence of money and credit from first principles. Recent surveys include Lagos, Rocheteau, and Wright (2017) and Rocheteau and Nosal (2017). Our approach to make fiat money an imperfect substitute to banks' liabilities is different but related to Sanches and Williamson (2010). The role of banks is related to Gu, Mattesini, Monnet, and Wright (2014) and references therein. Given our focus on monetary policy and corporate finance, the closest paper is Rocheteau, Wright, and Zhang (2016) that studies transaction lenders who provide one-time loans under pledgeability constraints. Models of money and credit with long-term relationships include Corbae and Ritter (2004) with indivisible money and Rocheteau and Nosal (2017, Ch. 8) with divisible money. Our description of the credit market with search frictions is analogous to Wasmer and Weil (2004) and the textbook treatment by Petrosky-Nadeau and Wasmer (2017). Our credit market also resembles the OTC markets with middlemen of Duffie, Garleanu, and Pedersen (2005) and, more precisely, its version with divisible assets by Lagos and Rocheteau (2009). These papers do not model long term relationships; for a recent exception, see Hendershott, Li, and Divdan (2017) for a model of repeat trading relationships in an OTC corporate bond market. Our description of a crisis as an unanticipated shock to the distribution of entrepreneurs' states is analogous to the description of a crisis in OTC markets in Weill (2007) and Lagos, Rocheteau, and Weill (2011).

The corporate finance literature on relationship lending was pioneered by Sharpe (1990) and is surveyed by Boot (2000) and Elyasiani and Goldberg (2004). There are four main approaches to the role of lending relationships: insurance in the presence of idiosyncratic investment opportunities (Berger and Udell 1992, Berlin and Mester 1999), monitoring in the presence of agency problems (Diamond 1984; Holmstrom and Tirole 1997; Boualam 2017), screening with hidden types (Agarwal and Hauswald 2010), and dynamic learning under adverse selection (Rajan 1992, Hachem, 2011, Bolton, Freixas, Gambacorta, and Mistrulli 2016). We adopt the insurance approach as it is central to the monetary policy tradeoff we are focusing on. In addition, we also emphasize the role of relationship lenders to eliminate the holdup problem that would arise if loan contracts were negotiated ex post, once investment opportunities have materialized.

The focus of this paper is on monetary policy and its effects on corporate finance. Bolton and Freixas (2006) analyze corporate finance and monetary transmission but take the real interest rate

on government bonds as an exogenous policy tool. By raising the real rate on bonds, monetary policy affects the external financing mix between corporate bonds and bank loans. In contrast, a key aspect of our theory is to deliver an explicit transmission mechanism from the nominal policy rate to the real lending rate. Monetary policy in our model sets the money growth rate and, by the Fisher equation, the nominal rate on bonds, which affects real lending rates through the decentralized negotiation of loan contracts. In contrast with Bolton and Freixas (2006), monetary policy affects the mix between internal finance through retained earnings in cash and external finance through bank loans. We do not explicitly model corporate bonds here since few businesses in reality are able to access the corporate bonds market, as acknowledged in Bolton, Freixas, Gambacorta, and Mistrulli (2016).[8] One could also interpret corporate bonds as a kind of transaction lending to fund investment in our model. In Hachem (2011), monetary policy is modeled as an exogenous cost of funds whereas we formalize explicitly monetary policy in terms of money growth and nominal interest rates. Complementary to what we do, Dreschsler, Savov, and Schnabl (2017) formalize the transmission of monetary policy as arising from market power in the deposits market where an increase in the nominal interest rate widens spreads banks charge on deposits.

Importantly, the existing literature on lending relationships does not study optimal monetary policy. This question is especially challenging in general equilibrium models with money and credit since the Friedman rule is typically optimal and it eliminates credit altogether. In contrast, we design an environment where the Friedman rule is suboptimal due to externalities associated with bank entry. Related models where the constrained-efficient allocation requires both the Friedman rule and Hosios condition include Berentsen, Rocheteau, and Shi (2007) and Rocheteau and Wright (2005, 2009). Those models do not have credit, banks, and lending relationships, and do not characterize optimal monetary policy.

Our recursive formulation of the Ramsey problem is related to Chang (1998). Aruoba and Chugh (2010) study optimal monetary and fiscal polices in the Lagos-Wright model when the policymaker has commitment. Our approach to the policy problem without commitment is similar to Klein, Krusell, and Rios-Rull (2008), i.e. we focus on Markov perfect equilibria. Martin (2011, 2013) studies fiscal and monetary policy absent commitment in a New Monetarist model where the

[8] For a model where firms can finance with bank loans or corporate bonds, see de Fiore and Uhlig (2011). Relative to our model, they add bond finance to an information model of banking where corporate lending occurs without costly ex ante information acquisition while bank lending does. Thus, bond finance resembles transaction lending in our model. However, they do not model long term banking relationships and do not consider optimal policy.

government finances the provision of a public good with money, nominal bonds, and distortionary taxes.[9]

2 Environment

Time is denoted by $t \in \mathbb{N}_0$. Each period is divided in three stages. In the first stage, there is a competitive market for capital goods. The second stage is a credit market with search frictions where long-term lending relationships are formed. The last stage is a frictionless centralized market where agents trade money and consumption goods and settle debts. See Figure 3. The capital good k is storable across stages but not across periods. The consumption good c is taken as the numéraire.

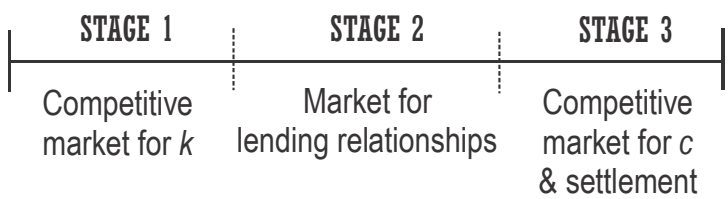

Figure 3: Timing of a representative period

There are three types of agents: *entrepreneurs* who need capital, *suppliers* who can produce capital, and *banks* who can finance the acquisition of capital by entrepreneurs as explained below. The population of entrepreneurs is normalized to one. Given CRS for the production of capital goods (see below), the population size of suppliers is immaterial. The population of active banks is endogenous and will be determined through free entry. All agents have linear preferences, $c - h$, where c is consumption of numéraire and h is labor. They discount across periods according to $\beta = 1/(1+\rho)$, $\rho > 0$.

At the start of each period, an entrepreneur gets an investment opportunity with probability λ, in which case he can transform k into $f(k)$ units of c in stage 3, where $f(0) = 0$, $f'(0) = \infty$, $f'(\infty) = 0$ and $f'(k) > 0 > f''(k)\ \forall k > 0$. Capital is produced by suppliers in the first stage with a linear technology, $k = h$. We define $k^* = \arg\max_k \{f(k) - k\}$. Entrepreneurs can also produce c using their labor in the last stage with a linear technology, $c = h$. Banks cannot produce neither c nor k.

Entrepreneurs lack commitment and their trading histories are private. As a result, suppliers do

[9] See also Díaz-Giménez, Giovannetti, Marimon, and Teles (2008) on monetary policy with nominal bonds and Dominguez (2007) on fiscal policy with public debt. Cooley and Quadrini (2004) study optimal monetary policy in a cash-in-advance model with search-matching frictions in the labor market.

not accept IOUs issued by entrepreneurs since they understand entrepreneurs could renege without fear of retribution.[10] In contrast, banks can issue one-period liabilities and can commit to repay them in the last stage. Moreover, banks can accept entrepreneurs' IOUs because they can enforce repayment by inflicting arbitrarily large punishments in case of default.

A bank can only extend a loan to an entrepreneur it is matched with. In the spirit of Pissarides' (2000) one-firm-one-job assumption, a bank manages at most one lending relationship. One can think of actual banks as a large collection of such relationships.[11] In a lending relationship, all actions are observable and the bank can enforce the terms of the contract. At the beginning of the second stage, banks without a lending relationship decide whether to participate in the credit market at a disutility cost, $\zeta > 0$. There is then a bilateral matching process between unbanked entrepreneurs and unmatched banks. The number of new lending relationships formed in the second stage of period t is $\alpha_t = \alpha(\theta_t)$, which depends on credit market tightness, θ_t, defined as the ratio of unmatched banks to unbanked entrepreneurs. We assume $\alpha(\theta)$ is increasing and concave, $\alpha(0) = 0$, $\alpha'(0) = 1$, $\alpha(\infty) = 1$, and $\alpha'(\infty) = 0$. Because matches are formed at random, the probability an entrepreneur matches with a bank is α_t, and the probability a bank matches with an entrepreneur is $\alpha_t^b = \alpha(\theta_t)/\theta_t$. We denote the elasticity of the matching function as $\epsilon(\theta) \equiv \alpha'(\theta)\theta/\alpha(\theta)$. A match that exists for more than one period is terminated at the start of the last stage with probability $\delta \in (0, 1)$. Newly formed matches in the second stage are not subject to the risk of termination.

In addition to banks' short-term liabilities, there is another liquid asset, fiat money. Fiat money is storable across periods and its supply, M_t, grows at the gross growth rate $\gamma \equiv M_{t+1}/M_t \geq \beta$. Changes in the money supply are implemented through lump-sum transfers or taxes to entrepreneurs at the start of the last stage. The value of money in the last stage of period t in units of numéraire is ϑ_t. We define $\pi_{t+1} \equiv \vartheta_t/\vartheta_{t+1} - 1$ as the inflation rate.

To prevent banks from exiting the market at the Friedman rule, we divide investment opportunities into two groups. There is a fraction ν of investment opportunities that can be financed with either cash or banks' liabilities. As a result, such opportunities can be undertaken by both banked and unbanked entrepreneurs. The remaining fraction, $1 - \nu$, can only be financed by entrepreneurs

[10] Alternatively, we could relax this assumption and allow for direct finance or trade credit in a fraction of matches where the entrepreneur can obtain a loan directly from the supplier. In addition, trade credit can be subject to a pledgeability constraint where only a fraction of the entrepreneur's output is pledgeable due to a moral hazard problem. See Rocheteau, Wright, and Zhang (2016) for a model in a related set up.

[11] For a related formalization, see Atkeson, Eisfeldt, and Weill (2015) where banks trade derivative swap contracts in an OTC market.

Table 1: Financing methods

Investment $\longrightarrow$ Entrepreneur $\downarrow$	General $(\lambda\nu)$	Bank-financed $(\lambda(1-\nu))$
Matched (ℓ)	Bank loan or cash	Bank loan
Unmatched $(1-\ell)$	Cash	None

in a lending relationship. For instance, cash is not accepted while banks' liabilities are. This would be the case if cash is subject to a counterfeiting problem that only makes it partially acceptable (e.g., Lester, Postlewaite, and Wright 2012; Rocheteau, Wright, and Xiao 2017), or subject to theft and embezzlement with some probability (e.g., Sanches and Williamson 2010), so that an entrepreneur has no cash on hand when he receives an investment opportunity.[12] Alternatively, banks bring a measure $(1-\nu)\lambda$ of new investment opportunities to their clients, e.g., because by monitoring loans, they acquire information allowing them to identify new profit opportunities.[13] We summarize our assumption in Table 1 where columns indicate the types of investment opportunities and rows indicate entrepreneurs' states.

Finally, we assume that banks can misrepresent their entry decisions costlessly, thereby preventing the use of direct subsidies to promote entry. By ruling out direct subsidies to banks we can focus on a second-best analysis with non-trivial implications for policymaking.

3 Pure banking equilibria

In order to facilitate the introduction of the lending rate contract, we begin by studying nonmonetary equilibria where bank liabilities are the only means of payment. In the first stage of each period, suppliers choose the amount of k to produce at a linear cost taking its price in terms of numéraire, q_t, as given. Formally, they solve $\max_{k\geq 0}\{-k+q_t k\}$. If $q_t > 1$, the problem has no solution, and if $q_t < 1$, $k_t = 0$. Hence, if the capital market is active, $q_t = 1$.

[12] Here ν is exogenous though we can make partial acceptability endogenous along the lines of Lester, Postlewaite, and Wright (2012) and Li, Rocheteau, and Weill (2012). Similarly, Williamson (2012) assumes a fixed fraction of currency transactions are deemed illegal. Another interpretation is that in a life cycle model unbanked entrepreneurs have to pass on investment opportunities if they did not have the opportunity to produce and accumulate cash (e.g., Rocheteau and Nosal, Ch. 6.6).

[13] The fact that banks learn valuable information about firms when they form lending relationships is a common theme of the literature (e.g., Hachem 2011, Bolton, Freixas, Gambacorta, and Mistrulli 2016). This information can be used to enhance the profitability of firms – in our context, through additional investment opportunities. Some argued to us that these two roles of banks, i.e., lending and learning new investment opportunities, could be independent. We take the view that they are intertwined.

In the second stage, lending relationships are formed where newly matched entrepreneurs and banks negotiate the terms of the loan contract. The contract is a sequence, $\{(k_{t+\tau}, \phi_{t+\tau})\}_{\tau=0}^{\infty}$, that specifies for all dates: (i) a loan of size $k_{t+\tau}$ the entrepreneur obtains from the bank if he requests it and (ii) a repayment $\phi_{t+\tau}$ to the bank in units of numéraire. Each component, $(k_{t+\tau}, \phi_{t+\tau})$, is interpreted as a loan conditional on an investment opportunity.[14] The terms of the loan contract are determined through bargaining between the two parties.

To determine the bargaining outcome, we first compute the entrepreneur's surplus from a lending relationship. The lifetime expected utility of an unbanked entrepreneur at the beginning of period t is

$$U_t^e = \alpha_t \beta Z_{t+1}^e + (1 - \alpha_t)\beta U_{t+1}^e, \tag{1}$$

where Z_t^e is the lifetime utility of an entrepreneur in a lending relationship at the beginning of period t, which solves

$$Z_t^e = \lambda \left[f(k_t) - k_t - \phi_t \right] + \delta \beta U_{t+1}^e + (1 - \delta)\beta Z_{t+1}^e. \tag{2}$$

According to (1), an unbanked entrepreneur has no means of financing investment opportunities. Hence, his profits in the first stage of period t are zero. In the second stage, he enters a lending relationship with probability α_t and remains unmatched with probability $1 - \alpha_t$. From (2), the entrepreneur receives an investment opportunity with probability λ, in which case he can draw from his credit line to finance k_t in exchange for an interest payment ϕ_t. The lending relationship is terminated at the start of the last stage with probability δ, in which case his continuation value is βU_{t+1}^e. The relationship is maintained with probability $1 - \delta$, in which case his continuation value is βZ_{t+1}^e.

The entrepreneur's surplus from entering a lending relationship in period $t - 1$ is βS_t^e where $S_t^e \equiv Z_t^e - U_t^e$, and solves

$$S_t^e = \lambda \left[f(k_t) - k_t - \phi_t \right] - \left(U_t^e - \beta U_{t+1}^e \right) + (1 - \delta)\beta S_{t+1}^e. \tag{3}$$

The second term on the right side of (3) is a banked entrepreneur's outside option, $U_t^e - \beta U_{t+1}^e$, which is the flow value of being unbanked.

[14] There are many lending contracts that are payoff equivalent. Loan contracts in practice include a commitment fee where the bank charges the firm for setting up a credit line, including charges and penalties if the firm exceeds the line limit or violates contract terms. In our model, the absence of agency problems means all that matters for determining the loan contract is the discounted sum of payments to the bank. Further, loan contracts in principal could be even more general where (k_t, ϕ_t) is a function of the history of lending relationships.

Similarly, U^b is defined as the bank's lifetime profits of being unmatched and Z^b as the bank's lifetime value of being in a lending relationship. They solve

$$U_t^b = -\zeta + \alpha_t^b \beta Z_{t+1}^b + \left(1 - \alpha_t^b\right) \beta \max\left\{U_{t+1}^b, 0\right\}, \tag{4}$$

$$Z_t^b = \lambda \phi_t + \delta \beta \max\left\{U_{t+1}^b, 0\right\} + (1-\delta)\beta Z_{t+1}^b. \tag{5}$$

From (4), an unmatched bank incurs a cost ζ at the start of the second stage to participate in the credit market; there, the bank is matched with an entrepreneur with probability α_t^b and remains unmatched with probability $1-\alpha_t^b$. From (5), the bank's expected profits are composed of expected interest payments, $\lambda\phi_t$. At the beginning of the last stage, the lending relationship is terminated with probability δ and is maintained with probability $1-\delta$. Free entry of banks in the credit market means $U_t^b \leq 0$ (with equality if there is entry).

The surplus of a bank from being in a lending relationship in period $t-1$ is βS_t^b where $S_t^b \equiv Z_t^b - \max\left\{U_t^b, 0\right\}$. From (5),

$$S_t^b = \lambda\phi_t + \beta(1-\delta)S_{t+1}^b. \tag{6}$$

From (6), the total surplus from a lending relationship, $S_t \equiv S_t^b + S_t^e$, solves

$$S_t = \lambda[f(k_t) - k_t] - (U_t^e - \beta U_{t+1}^e) + \beta(1-\delta)S_{t+1}. \tag{7}$$

The terms of the contract negotiated at time $t-1$ specifies a sequence $\{k_{t+\tau}, \phi_{t+\tau}\}_{\tau=0}^{\infty}$, that maximizes the generalized Nash product:

$$\max_{\{k_{t+\tau},\phi_{t+\tau}\}_{\tau=0}^{\infty}} [S_{t+\tau}^b]^{\eta}[S_{t+\tau}^e]^{1-\eta}, \tag{8}$$

where $\eta \in (0,1)$ is the bank's bargaining power. To pin down the contract uniquely, we impose the Nash solution $S_{t+\tau}^b = \eta S_{t+\tau}$ at all dates in the lending relationship.

Lemma 1 *The lending contract solution to (8) specifies $k_t = k^*$ for all t and*

$$\phi_t = \eta\left[f(k^*) - k^*\right] - (1-\eta)\theta_t\zeta/\lambda \text{ for all } t. \tag{9}$$

The real lending rate, defined as $r_t \equiv \phi_t/k_t$, is

$$r_t = \frac{\eta\left[f(k^*) - k^*\right] - (1-\eta)\,\theta_t\zeta/\lambda}{k^*} \quad \text{for all } t. \tag{10}$$

From (9)-(10), the interest payment entails a share of the profits from an investment opportunity, $f(k^*) - k^*$, net of the fraction of the bank entry cost per investment opportunity. Given θ_t, $\partial r_t/\partial\lambda > 0$ and $\partial r_t/\partial\eta > 0$.

Substituting $U_t^b = 0$ and ϕ_t from (9) into (5), and assuming positive entry, credit market tightness solves

$$\frac{\theta_{t-1}}{\alpha(\theta_{t-1})} = \frac{\beta\lambda\eta\left[f(k^*) - k^*\right]}{\zeta} - \beta\left(1-\eta\right)\theta_t + \beta(1-\delta)\frac{\theta_t}{\alpha(\theta_t)}. \tag{11}$$

The measure of lending relationships at the start of a period evolves according to

$$\ell_{t+1} = (1-\delta)\ell_t + \alpha_t(1-\ell_t). \tag{12}$$

The number of lending relationships at the beginning of $t+1$ equals the measure of lending relationships at the beginning of t that have not been severed, $(1-\delta)\ell_t$, plus newly created relationships, $\alpha_t(1-\ell_t)$. An active nonmonetary equilibrium is a bounded sequence, $\{\theta_t, \ell_t, r_t\}_{t=0}^{\infty}$, that solves (10), (11), and (12) for a given ℓ_0. A steady state equilibrium corresponds to a constant sequence, (θ, ℓ, r).

Proposition 1 ***(Pure banking equilibria.)***

A unique steady-state nonmonetary equilibrium exists and features $\theta > 0$ if and only if

$$(\rho+\delta)\,\zeta < \lambda\eta\left[f(k^*) - k^*\right], \tag{13}$$

where comparative statics are given by the table below.

	η	λ	ζ	δ
θ	$+$	$+$	$-$	$-$
ℓ	$+$	$+$	$-$	$-$
r	$+$	$-$	$+$	$+$

Given $\ell_0 > 0$, an equilibrium where $\theta_t = \theta$ exists for all $t \in \mathbb{N}$. Equilibrium is unique if $\beta(1-\delta) - \beta(1-\eta)\alpha(\theta)/[1-\varepsilon(\theta)] \in (-1, 1)$. In addition, $\ell_t = \ell + (\ell_0 - \ell)[1-\delta-\alpha(\theta)]^t$, where $\ell = \alpha(\theta)/[\delta+\alpha(\theta)]$.

According to (13), the credit market is active if banks' entry costs are low, lending relationships are long lasting, and the profits from investment opportunities are high. Across steady states, ℓ and r covary negatively following changes in λ, ζ, and δ, and they covary positively following a change in η. An increase in λ raises the value of lending relationships, which generates more bank entry. Higher competition in the credit market drives the lending rate down. The opposite

comparative statics hold for an increase in either ζ or δ since costlier search and a higher likelihood of a severed credit match lowers the value of being in a lending relationship. Outside of steady state, the condition for uniqueness holds when $\varepsilon(\theta) = \eta$, which is the Hosios condition. It also holds if the time period is small so that $\alpha(\theta)$ and δ are small.

4 Liquidity and lending relationships

It is time to turn to monetary equilibria with both internal and external finance. If fiat money is introduced and valued, then unbanked entrepreneurs can accumulate liquid assets with retained earnings and finance k_t^m if an investment opportunity arises. Moreover, the loan contract in a lending relationship can now specify a down payment in real balances, d_t.

The lifetime expected value of an unbanked entrepreneur holding m real balances (units of money in terms of numéraire) in the last stage of period t is

$$W_t^e(m) = m + T_t + \max_{m_{t+1} \geq 0} \{-(1 + \pi_{t+1})m_{t+1} + \beta U_{t+1}^e(m_{t+1})\}, \tag{14}$$

where T_t is the lump-sum transfer in terms of numéraire (or taxes if $T_t < 0$), and $U_t^e(m)$ is the value of an unbanked entrepreneur at the beginning of period t with m real balances. Since the rate of return on money is $1/(1+\pi_{t-1})$, the entrepreneur accumulates $(1+\pi_{t+1})m_{t+1}$ real balances in period t to hold m_{t+1} at the start of period $t+1$. The lifetime utility of a banked entrepreneur with m real balances in the last stage of period t, $X_t^e(m)$, solves a similar Bellman equation and is therefore omitted. Both $W_t^e(m)$ and $X_t^e(m)$ are linear in real balances.

Entrepreneur's choice of real balances The lifetime expected utility of an unbanked entrepreneur at the beginning of the second stage (credit market) solves:

$$V_t^e(m) = \alpha_t X_t^e(m) + (1 - \alpha_t)W_t^e(m) = m + \alpha_t X_t^e(0) + (1 - \alpha_t)W_t^e(0). \tag{15}$$

With probability α_t, an unmatched entrepreneur enters a lending relationship and, with probability $1 - \alpha_t$, he proceeds to the last stage unmatched. From (15), $V_t^e(m)$ is linear in m.

The problem of an unbanked entrepreneur in the capital market upon receiving an investment opportunity is

$$k_t^m \in \arg\max_{k_t \geq 0} \{f(k_t) + V_t^e(m_t - k_t)\} \quad \text{s.t.} \quad k_t \leq m. \tag{16}$$

The entrepreneur's purchase of k_t^m is bounded above by his real balances because his IOUs are not accepted by sellers and he does not have access to bank credit. Using the linearity of V_t^e,

$f(k_t^m) + V_t^e(m_t - k_t^m) = f(k_t^m) - k_t^m + V_t^e(m_t)$. The solution to (16) is $k_t^m = k^*$ if $m \geq k^*$ and $k_t^m = m$ otherwise. As a result, the lifetime utility of an unbanked entrepreneur at the beginning of period t is

$$U_t^e(m_t) = \lambda\nu\left[f(k_t^m) - k_t^m\right] + V_t^e(m_t). \tag{17}$$

With probability $\lambda\nu$, the entrepreneur receives an investment opportunity where he purchases k_t^m.

Substituting $U_t^e(m_t)$ from (17) into (14) and using the linearity of V_t^e, an unmatched entrepreneur's choice of real balances solves

$$\Delta_t \equiv \max_{m_t \geq 0} \left\{-i_t m_t + \lambda\nu\left[f(k_t^m) - k_t^m\right]\right\}, \tag{18}$$

where $i_t \equiv (1 + \pi_t - \beta)/\beta$ can be interpreted as the nominal interest rate on an illiquid bond (if such bonds were introduced in the model).[15] The FOC associated with (18) is

$$i_t = \lambda\nu[f'(k_t^m) - 1]. \tag{19}$$

Internally financed investment k^m decreases with i and increases in the probability of an investment opportunity for a banked entrepreneur, $\lambda\nu$.

Lending contract. We now turn to the terms of the contract in a lending relationship. The contract negotiated at time $t-1$ specifies $\{k_{t+\tau}, \phi_{t+\tau}, d_{t+\tau}\}_{\tau=0}^{\infty}$ where $d_{t+\tau}$ is the down payment in real balances and $k_{t+\tau} - d_{t+\tau}$ is interpreted as the loan size. We assume the entrepreneur can commit to the terms of the contract, which requires $m_{t+\tau} \geq d_{t+\tau}$. The entrepreneur's surplus from being in a lending relationship in the third stage of $t-1$ is defined as the difference between the discounted value of being in a lending relationship, $X_{t-1}^e(m)$, and the value of not being in a lending relationship, $W_{t-1}^e(m)$, and we denote $S_t^e = \left[X_{t-1}^e(0) - W_{t-1}^e(0)\right]/\beta$. As before, the terms of the contract are determined through generalized Nash bargaining.

Lemma 2 *The lending contract specifies $k_t = k^*$, $d_t = 0$ $\forall t$, and*

$$\phi_t = \eta\left[f(k^*) - k^*\right] - \frac{(1-\eta)\theta_t\zeta + \eta\Delta_t}{\lambda} \ \forall t. \tag{20}$$

The real lending rate, $r_t \equiv \phi_t/k_t$, is

$$r_t = \frac{\eta\lambda\left[f(k^*) - k^*\right] - (1-\eta)\theta_t\zeta - \eta\Delta_t}{\lambda k^*} \ \forall t. \tag{21}$$

[15] One can also extend our framework to allow for additional policy instruments thereby generating a richer set of interest rates. For instance, Rocheteau, Wright, and Zhang (2016) introduce partially liquid government bonds in a similar model and show there is a one-for-one relation between the interest rate on illiquid bonds and the rate on partially liquid government bonds. As is standard in the literature, we interpret i_t in our model as the rate on illiquid bonds (or, inflation or money growth) to focus on the tradeoff between self-insurance and access to bank credit.

Any Pareto-efficient contract maximizes the joint surplus, which implies $k_t = k^*$ and $d_t = 0$ for all t. Entrepreneurs in a relationship have no motives for holding cash since banks provide a secure source of funds and cash is costly to hold. Moreover, there is no strategic reason for holding cash since interest rates are negotiated ex ante thereby eliminating holdup inefficiencies. Consequently, banked entrepreneurs do not hold real balances while unbanked entrepreneur do.[16] From (20), the interest payment is an intermediation premium that consists of the banks' share of profits from investment opportunities minus the entrepreneur's flow value of being unbanked, which depends on their outside option Δ_t. Given θ_t, an increase in the nominal interest rate i_t raises both ϕ_t and r_t. Intuitively, higher i_t lowers the entrepreneur's net profits from internal finance, which allows the bank to charge a higher r_t.

Equilibrium. To determine credit market tightness, we substitute ϕ_t from (20) into (5). From free entry of banks, $Z^b_{t+1} = \zeta\theta_t/\beta\alpha_t$ and hence $\{\theta_t\}_{t=0}^{\infty}$ solves

$$\frac{\theta_t}{\alpha(\theta_t)} = \beta\eta\left\{\frac{\lambda\left[f(k^*) - k^*\right] - \Delta_{t+1}}{\zeta}\right\} - \beta(1-\eta)\theta_{t+1} + \beta(1-\delta)\frac{\theta_{t+1}}{\alpha(\theta_{t+1})}. \tag{22}$$

An increase in Δ_t reduces ϕ_t and hence bank profits and their incentives to enter the credit market. This provides a channel for monetary policy to affect credit market outcomes.

Market clearing implies the demand for real balances from the $1 - \ell_t$ unbanked entrepreneurs must equal the aggregate supply of real balances:

$$(1-\ell_t)k^m_t = \vartheta_t M_t. \tag{23}$$

From (23), the rate of return of money can be expressed as

$$\frac{\vartheta_{t+1}}{\vartheta_t} = \frac{M_t}{M_{t+1}}\frac{k^m_{t+1}}{k^m_t}\frac{1-\ell_{t+1}}{1-\ell_t}.$$

The rate of return of money decreases with the money growth rate, $\gamma = M_{t+1}/M_t$, and increases with the growth rate of the unbanked sector, $(1-\ell_{t+1})/(1-\ell_t)$. A monetary equilibrium is a bounded sequence, $\{\theta_t, \ell_t, k^m_t, r_t, \vartheta_t\}_{t=0}^{\infty}$, that solves (12), (19), (21), (22), and (23) for a given $\ell_0 > 0$.

[16] Key to this result is the fact that the loan contract is negotiated before investment opportunities occur and agents can commit to them. In Rocheteau, Wright, and Zhang (2016) and our extension with transaction lenders in Section 8, banks and entrepreneurs cannot form lending relationships and therefore cannot commit to the terms of long-term lending contracts. In this case, entrepreneurs hold real balances to improve their outside option and to negotiate better terms for their loans.

Proposition 2 ***(Transmission of monetary policy.)*** *A unique steady-state monetary equilibrium exists and features an active credit market if and only if*

$$(\rho+\delta)\zeta < \lambda\eta[f(k^*)-k^*]-\eta\Delta(i). \tag{24}$$

If $\eta > 0$, an increase in i raises θ and r, but lowers k^m and ϑM. In the neighborhood of $i = 0$,

$$\frac{\partial\theta}{\partial i} = \frac{\eta k^*}{\zeta\left\{(\rho+\delta)\left[1-\epsilon(\theta)\right]/\alpha(\theta)+1-\eta\right\}} \geq 0 \tag{25}$$

$$\frac{\partial r}{\partial i} = \frac{(\rho+\delta)\,\eta}{\lambda\left\{\rho+\delta+(1-\eta)\,\alpha(\theta)/\left[1-\epsilon(\theta)\right]\right\}} \geq 0. \tag{26}$$

The model delivers a pass through from the nominal policy rate to the real lending rate. An increase in i has two effects on r. First, higher i reduces Δ which tends to increase r. In words, an increase in i raises the opportunity cost of holding liquid assets. As a result, unbanked entrepreneurs reduce their money holdings. The outside option of entrepreneurs worsens, which allows banks to charge a higher r. Second, θ increases, thereby raising competition among banks, which tends to lower r. In the proof of Proposition 2, we show the first effect dominates.

Suppose that the matching function takes the form $\alpha(\theta) = \bar{\alpha}\theta/(1+\theta)$. From (57), we can solve for market tightness in closed form. Assuming an interior solution, and after some calculation, the pass through rate is

$$\frac{\partial r}{\partial i} = \frac{\eta}{\lambda\left[\bar{\alpha}(1-\eta)/\left(\rho+\delta\right)+1\right]}\frac{k^m}{k^*}. \tag{27}$$

The pass through rate is the product of the effective bargaining power of banks and the share of the efficient investment size that unbanked entrepreneur self-finance. It is higher for low interest rates since k^m is a decreasing function of i. Finally, the pass through rate increases with δ. Hence, the more stable the lending relationships the lower the pass through.

5 Calibration

We now describe the dynamic response of the economy to a banking shock modeled as an exogenous destruction of lending relationships starting from a steady state. To illustrate these dynamics, we calibrate the model with data on small businesses in the U.S. economy.

The period length is a month and $\rho = 1.04^{1/12} - 1$. The production function is $f(k) = k^a$ with $a = 0.75$.[17] Hence, $k^* = a^{\frac{1}{1-a}} = 0.316$. We adopt the matching function $\alpha(\theta) = \overline{\alpha}\theta/(1+\theta)$,

[17] The choice of a is important when calibrating the model to obtain plausible values for the acceptability of monetary assets, ν. For instance, for $a = 1/3$ we obtained $\nu = 0.23$.

where $\overline{\alpha} \in [0, 1]$. The parameters to calibrate are $(\overline{\alpha}, \delta, \lambda, \nu, \eta, \zeta)$. We obtain these parameters by matching targets from the 2003 National Survey of Small Business Finances (SSBF).

Table 2: Parameter values

Value	Target
$\rho = 1.04^{1/12} - 1$	Annual discount rate = 4%
$\overline{\alpha} = 0.0934$	Share of firms with access to credit = 84.9%, SSBF (2003)
$\delta = 0.009$	Duration of lending relationship = 111.5 months, SSBF (2003)
$\lambda = 0.037$	Credit utilization rate = 44.9%, SSBF (2003)
$\nu = 0.694$	Cash to credit ratio = 55.5%, SSBF (2003)
$\eta = 0.1016$	Annual pass through rate, 1994 to 2003 = 19.21%, FRED
$\zeta = 1.6641 \times 10^{-4}$	Annual lending rate, 1994 to 2003 = 9.78%, FRED

We define a credit relationship broadly to include a credit line, business credit card, or owner credit card used for business purposes.[18] A fraction 84.9% of small businesses report such a credit relationship. In our model, $\ell = \alpha/(\alpha + \delta) = 0.849$. The average duration of credit relationships is 9.3 years. This gives $\delta = 0.009$ and $\alpha = \delta\ell/(1 - \ell) = 0.0504$, i.e., it takes about 1.65 years to form a new lending relationship.

The average annual credit utilization rate, calculated as the amount drawn on current credit lines over the total available credit line, is 44.9%. To match this moment, we assume the total credit line corresponds to the optimal investment size, k^*, since it corresponds to the amount that the bank and the entrepreneur negotiate in an optimal contract. Hence, the credit utilization rate is $\ell\lambda k^*/\ell k^* = \lambda$. This implies 44.9% of firms receive an investment opportunity within a year, which gives $\lambda = 0.449/12 = 0.034$.

We take the 3 month T-bill secondary market rate as the interest rate targeted by monetary policy, which averages 4.9% per year from 1994 to 2003. Hence, $i = 0.049/12 = 0.0041$. The ratio of average cash holdings for unbanked small businesses to the average credit line for businesses in a lending relationship, k^m/k^* in our model, is 55.5%.[19] Hence, $\nu = i/\lambda[(k^m/k^*)^{a-1} - 1] = 0.69$. So

[18] The SSBF defines a credit line as an arrangement with a financial institution that allows a firm to borrow funds during a specified period up to a specific credit limit. In a different sample of larger firms, Sufi (2009) finds 74.5% of U.S. public, non-financial firms have access to lines of credit provided by banks. Since firms in the SSBF are heterogenous in size, we scale the related credit variables with firm size proxied by the net book value.

[19] An advantage of the SSBF is that we directly obtain average cash holdings conditional on being an unbanked firm and average credit line conditional on being a banked firm. The SSBF defines cash as the total amount of cash on hand (currency and coin used in operations, petty cash, etc.), balances in business checking accounts, and total balances of all savings accounts, money market accounts, time deposits, and certificates of deposit. In addition, some cash, like petty cash held in cashiers and minimal required balances in bank accounts, is needed to run a business no matter if an investment opportunity is available or not. Thus in our model, cash holdings for unbanked firms'

unbanked firms miss out approximately 30% of investment opportunities by not being in a lending relationship.

The bank's bargaining power, η, is chosen to target the pass through rate. We match the lending rate in the model to the prime loan rate, which is the best rate offered by banks to those with minimal concern of default. In our model, bank loans are repaid within a period.[20] So we interpret r as an intermediation premium equal to the difference between the nominal prime loan rate and nominal interest rate on government bonds. Regressing the 3-month T-bill secondary market rate on the prime loan rate we obtain $\partial r/\partial i = 0.1921$. See Figure 4 which plots the empirical relationship between the 3-month T-bill rate and the spread of the bank prime loan rate. From (27), $\eta = 0.1016$.

Banks' entry cost, ζ, is calibrated to match the average spread between the nominal prime loan rate and nominal interest rate, which is 4.87% per year from 1994 to 2003, $r = 0.0487/12 = 0.0041$. Hence, $\zeta = 1.6641 \times 10^{-4}$. To illustrate the model fit, we plot in Figure 4 the theoretical relationship between r and i for the calibrated values for η and ζ.

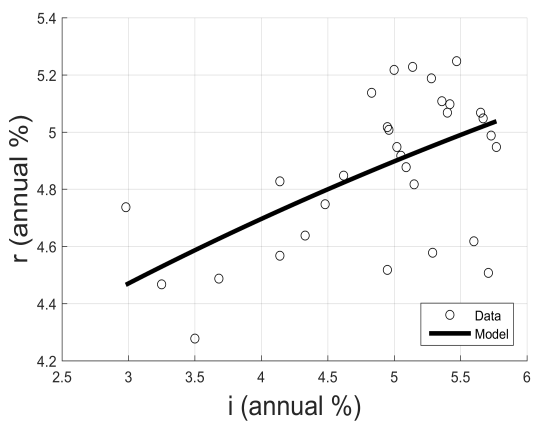

Figure 4: Pass through estimation

We now use our calibrated model to study the response of the economy to an unanticipated shock that destroys lending relationships. The size of the shock matches a measure of corporate credit contraction during the last financial crisis. See e.g. the contraction in the number of small business loans in Figure 1. Similarly, Ivashina and Scharfstein (2010) report the number of bank loans decreased on average by 59% during the peak of the U.S. banking crisis of 2008. In the

investment is the cash holding per firm net of operational cash.

[20] If loans were repaid with one period lag, then the interest on the loan would be approximately $\rho + r$ and hence does not affect the rate of time preference. It is a pure intermediation premium (see Rocheteau, Wright, and Zhang 2016 for more details on this interpretation).

context of our model, this gives $\ell_0 = 0.85 \times (1 - 0.59) = 0.35$, i.e., the measure of entrepreneurs with a line of credit falls from 85% to 35%. In Figure 5, we consider two policy rules: a constant money growth rate equal to 0.9% per year (solid blue line); a constant nominal interest rate equal to its calibrated value, 4.9% per year (dashed black line).

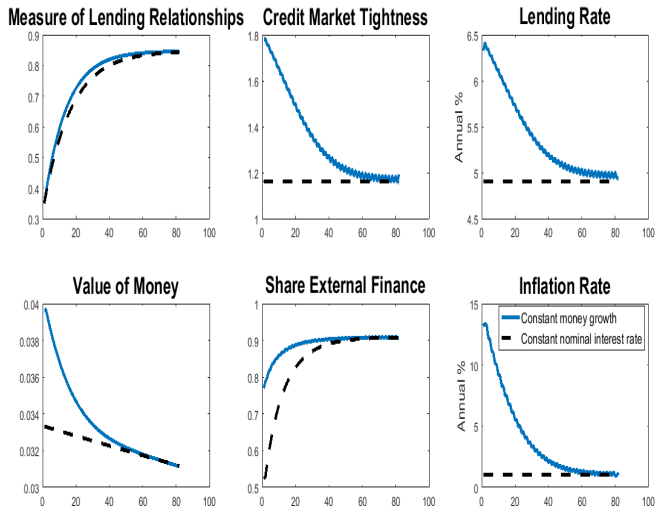

Figure 5: Effects of banking shock under two monetary policy rules

When the money growth rate is constant, the value of money ϑ_t jumps above its value in a stationary equilibrium since the aggregate demand for real balances by unbanked entrepreneurs increases. The initial deflation is followed by inflation at a higher rate than the money growth rate. Over time, the inflation rate returns to the money growth rate as the demand for money falls. The lending rate, r_t, jumps above its steady state value since inflation reduces the entrepreneur's profits from internally financed investment opportunities. Higher interest margins in turn promote bank entry. As lending relations recover, both r and θ fall gradually over time. When the nominal interest rate is constant, ϑ_t falls at a constant rate over time while market tightness and the real lending rate are constant. The share of external finance falls initially as lending relationships are destroyed then increases as credit recovers. The constant money growth rate implements a faster recovery than the constant interest rate.

6 Optimal monetary policy in the aftermath of a crisis

So far we considered two arbitrary monetary policies: a constant money growth rate and a constant nominal interest rate. We now characterize the optimal monetary policy described as a sequence of

nominal interest rates chosen to maximize social welfare. We consider two polar cases regarding the policymaker's ability to commit. In the first case, the policymaker commits to an infinite sequence of interest rates. In the second case, we take away the policymaker's power to commit to future interest rates. Instead, the policymaker reoptimizes its policy every period by setting a new interest rate taking as given the policies of future policymakers.

6.1 Policy with commitment

Before the credit market opens in stage 2, the policymaker announces a sequence of interest rates, $\{i_t\}_{t=1}^{\infty}$, and commits to it. We measure society's welfare, $\mathcal{W}$, starting in the second stage of $t = 0$ after the announcement is made,

$$\mathcal{W} = \sum_{t=0}^{\infty} \beta^t \left\{ -\zeta\theta_t(1-\ell_t) + \beta(1-\ell_{t+1})\lambda\nu \left[f(k_{t+1}^m) - k_{t+1}^m \right] + \beta\ell_{t+1}\lambda \left[f(k^*) - k^* \right] \right\}. \tag{28}$$

It is the sum of profits arising from investment opportunities net of bank entry costs. The policymaker chooses $\{i_t\}$ to maximize $\mathcal{W}$, taking into account that the equilibrium allocation is a function of $\{i_t\}$, i.e.,

$$\widetilde{\mathcal{W}}(\ell_0) = \max_{\{i_t\}_{t=1}^{\infty}} \mathcal{W} \text{ s.t. } \{\theta_t, \ell_t, k_{t+1}^m\}_{t=0}^{\infty} \text{ being an equilibrium given } \ell_0. \tag{29}$$

Since there is a one-to-one relation between i_t and k_t^m, i.e., $i_t = \lambda\nu[f'(k_t^m) - 1]$, the policymaker effectively chooses a path for k_t^m. When making this choice, the policymaker faces a tradeoff between providing insurance to unbanked entrepreneurs at time t by setting low interest rates, i.e., high k^m, and promoting entry of banks by setting high interest rates, i.e., low k^m.

We first establish the conditions under which the Friedman rule, which consists of setting $i_t = 0$ for all t, is suboptimal. We denote $\underline{\theta}$ the credit market tightness at the Friedman rule.

Proposition 3 ***(Suboptimality of the Friedman rule.)*** *Suppose*

$$\zeta < \frac{\alpha'(0)\lambda\eta(1-\nu)\left[f(k^*) - k^*\right]}{\rho + \delta}. \tag{30}$$

If $\eta < \varepsilon(\underline{\theta})$, a deviation from the Friedman rule is optimal, i.e., the optimal monetary policy does not require $i_t = 0$ for all t.

The condition (30) guarantees the credit market at the Friedman rule is active, $\underline{\theta} > 0$. A necessary condition for (30) to hold is that banked entrepreneurs receive more investment opportunities

than unbanked entrepreneurs. If $\nu = 1$, banks have no incentive to enter since entrepreneurs can self insure perfectly provided that $i_t = 0$ for all t. If $\nu < 1$, money only provides partial insurance so that banks have a role to play even when the Friedman rule is implemented. Provided that banks' bargaining power, η, is less than their contribution to the matching process in the credit market as measured by $\varepsilon(\underline{\theta})$, i.e., the Hosios condition is violated, it is socially beneficial to have positive nominal interest rates. Indeed, if $\eta < \varepsilon(\underline{\theta})$, banks' entry is inefficiently low as they fail to internalize the surplus they provide to entrepreneurs. In this case positive nominal rates raise bank profits and promote entry.[21] From now on, we suppose the conditions in Proposition 3 hold.

We now describe a recursive formulation of the Ramsey problem in (29). To simplify exposition, we adopt the functional forms $f(k) = Ak^a$, $\alpha(\theta) = \bar{\alpha}\theta/(1+\theta)$, and the parametric condition $\delta + \bar{\alpha}(1-\eta) < 1$. The policymaker takes as a constraint the relationship between current and future market tightness given by (22):

$$\theta_t = \frac{\bar{\alpha}\beta\eta\lambda\,(1-a)\,A}{\zeta}\left[(aA)^{\frac{a}{1-a}} - \nu(k_{t+1}^m)^a\right] + \beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta_{t+1} + \beta(1-\delta) - 1. \tag{31}$$

By setting θ_t in period t, the policymaker is making a promise of future profits to banks which must be honored in period $t+1$ by choosing k_{t+1}^m and θ_{t+1} consistent with the equilibrium condition. Since $k_t^m = f'^{-1}(1 + i_t/\lambda\nu)$ and $i_t \in \mathbb{R}_+$, the relevant state space is $k_t^m \in \mathbb{K} = [0, k^*]$ where $k^* = (aA)^{\frac{1}{1-a}}$. Values of θ_t consistent with an equilibrium are $\Omega = \left[\underline{\theta}, \bar{\theta}\right]$ where

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta\lambda\,(1-a)\,(1-\nu)A\,(aA)^{\frac{a}{1-a}}/\zeta + \beta(1-\delta) - 1}{1-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]} \tag{32}$$

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda\,(1-a)\,A\,(aA)^{\frac{a}{1-a}}/\zeta + \beta(1-\delta) - 1}{1-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}, \tag{33}$$

The quantity $\underline{\theta}$ is the steady-state value of θ in a monetary equilibrium at the Friedman rule. We assume (30) holds so that $\underline{\theta} > 0$. The quantity $\bar{\theta}$ is the steady-state value of θ in a non-monetary equilibrium.

Proposition 4 ***(Recursive formulation of the optimal policy.)*** *The policymaker's value function solves*

$$\widetilde{\mathcal{W}}(\ell_0) = \max_{\theta_0 \in \left[\underline{\theta}, \bar{\theta}\right]} \mathcal{W}(\ell_0, \theta_0) \tag{34}$$

[21] Note we do not allow the policymaker to make direct transfers to banks that participate in the credit market in order to correct for inefficiently low entry. Such transfers may not be feasible if the policymaker cannot distinguish between active and inactive banks in the credit market (i.e., one could create a bank but not search actively in the credit market). Also, we focus on optimal monetary policy taking fiscal policy and other (banking) regulations as given.

where $\mathcal{W}$ is the unique solution in $\mathcal{B}([0,1]\times\Omega)$ to

$$\begin{aligned}\mathcal{W}(\ell_t,\theta_t) &= \max_{\theta_{t+1}\in\Gamma(\theta_t),\ell_{t+1},k^m_{t+1}} \left\{-\zeta\theta_t(1-\ell_t)+\beta(1-\ell_{t+1})\lambda\nu\left[f(k^m_{t+1})-k^m_{t+1}\right]\right. \\ &\quad \left.+\beta\ell_{t+1}\lambda\left[f(k^*)-k^*\right]+\beta\mathcal{W}(\ell_{t+1},\theta_{t+1})\right\},\end{aligned} \tag{35}$$

where

$$k^m_{t+1} = \nu^{\frac{-1}{a}}\left\{(aA)^{\frac{a}{1-a}} - \frac{\zeta\left[\theta_t-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta_{t+1}-\beta(1-\delta)+1\right]}{\bar{\alpha}\beta\eta\lambda\left(1-a\right)A}\right\}^{\frac{1}{a}} \tag{36}$$

$$\ell_{t+1} = (1-\delta)\ell_t+\frac{\bar{\alpha}\theta_t}{1+\theta_t}(1-\ell_t), \tag{37}$$

and the feasibility correspondence $\Gamma(\theta_t)$ is defined as

$$\begin{aligned}\Gamma\left(\theta_t\right) = \Bigg[&\frac{\theta_t+1-\beta(1-\delta)-\bar{\alpha}\beta\eta\lambda\left(1-a\right)A\left(aA\right)^{\frac{a}{1-a}}/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}, \\ &\frac{\theta_t+1-\beta(1-\delta)-\bar{\alpha}\beta\eta\lambda\left(1-a\right)A(1-\nu)\left(aA\right)^{\frac{a}{1-a}}/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}\Bigg]\cap\Omega.\end{aligned} \tag{38}$$

The recursive formulation of the optimal policy problem in (29) takes θ_t as a state variable and describes its law of motion by the promise-keeping constraint (31). Given the state, the Bellman equation (35) is obtained from the Principle of Optimality. The equilibrium allocation is then pinned down by choosing θ_0 to maximize (34). In the following proposition we prove that the equilibrium under the optimal policy is monetary, $\theta_t < \bar{\theta}$, since there are always unbanked entrepreneurs who need liquid assets to finance investments, and bank entry is larger than its level at the Friedman rule, i.e., $\theta_t > \underline{\theta}$. The optimal policy and associated equilibrium is computed by value function iteration (see the Appendix for a description of the algorithm).

Proposition 5 ***(Interior optimal policy.)*** *Assuming (30) holds and $\eta < \epsilon(\underline{\theta})$, the solution to (34) is such that $\theta_t \in \left(\underline{\theta},\bar{\theta}\right)$ for all t.*

Figure 6 illustrates the optimal policy response following a banking shock that destroys lending relationships based on the parameterizations in Table 2.

RESULT #1. With commitment, optimal policy consists of lowering i_t close to zero in the aftermath of a crisis while promising higher interest rates in the future (forward guidance).

The top left panel plots the optimal path for the nominal interest rate. At the onset of the crisis, i_t is set at a low value close to the Friedman rule, about 0.25%, in order to reduce the holding cost

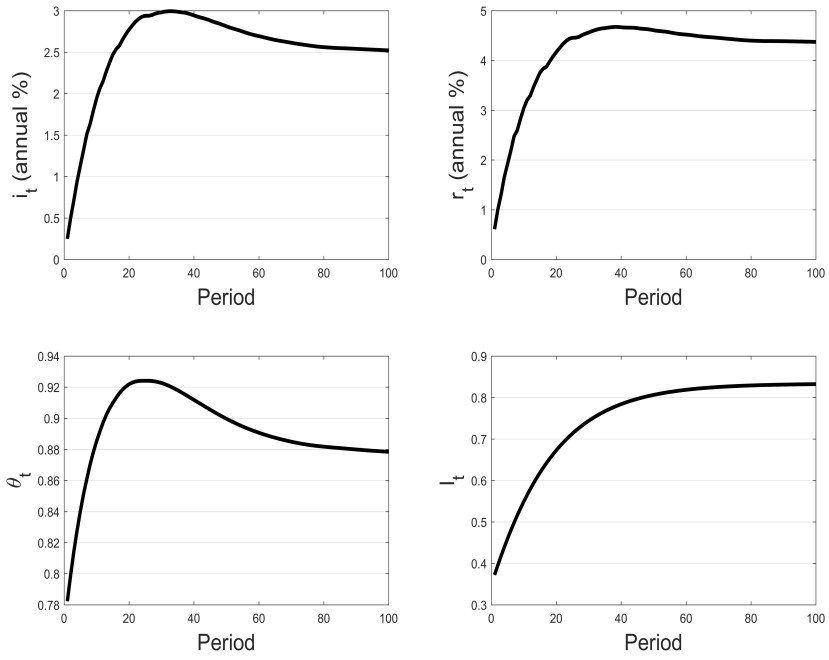

Figure 6: Optimal Ramsey policy with commitment

of liquidity for the 65% of unbanked entrepreneurs. In order to keep banks in the market despite such low interest rates, the policymaker promises a large increase in interest rates over the two years following the credit crunch. This "forward guidance" allows the policymaker to both provide liquidity at a low cost to entrepreneurs who lost access to a credit line and promote the creation of lending relationships by promising future profits to banks. The nominal interest rate peaks at about 3% and then falls to a steady-state level of 2.5%.

RESULT #2. With commitment, the real lending rate decreases at the onset of the crisis and then increases above its steady state value as the credit market tightens before returning to steady state.

The top right panel shows the real lending rate, r_t, which is set at a low value at the start of the banking crisis, less than 1%, but peaks at 4.6% after about two years before returning gradually to its steady state value of about 4.2%. In the bottom left panel, credit market tightness follows a hump-shaped path similar to the one for i_t, but θ_t reaches its maximum before i_t. The rate of creation of lending relationships, $\alpha(\theta_t)$, overtakes its steady state value about a year after the start of the crisis. The overshooting occurs because the policymaker had to commit to high interest rates

in order to maintain the creation of lending relationships early on.

6.2 Markov policy without commitment

The policy that consists in committing to an infinite sequence of interest rates is not time consistent. The policymaker promises high nominal interest rates to induce banks to keep supplying loans but would like to renege later in order to reduce entrepreneurs' cost of holding real balances. We now relax the assumption of commitment and assume that the policymaker sets i_{t+1} in period t but cannot commit to $\{i_{t+j}\}_{j\geq 2}$. The timing of actions within period t is analogous to Klein, Krusell, and Rios-Rull (2008), i.e., the policymaker moves first by choosing i_{t+1}, and the private sector moves next by choosing θ_{t+1} and k^m_{t+1}.

Given the one-to-one relationship between i_{t+1} and $k^m_{t+1} = f'^{-1}(1 + i_{t+1}/\lambda\nu)$, a strategy of the policymaker is a mapping, $k^m_{t+1} = \mathcal{K}\left(h^t\right)$, that assigns an investment level for unbanked entrepreneurs to each public history up to period t, h^t. In each period each atomistic bank chooses whether to enter or not the credit market given $\mathcal{K}$. Because banks are small they do not take into account how their entry decisions affects other banks' decisions and the policymaker's choices. We aggregate these individual decisions to obtain market tightness, $\theta_t = \Theta(h^t)$, for all histories ending at the beginning of the second stage of period t. Both the strategy of the policymaker and the strategies of banks must be sequentially rational to form a subgame perfect equilibrium. Because the set of subgame perfect equilibria of infinitely-repeated games is vast, we restrict our attention to Markov Perfect Equilibria composed of strategies that are only functions of the aggregate state, i.e., $k^m_{t+1} = \mathcal{K}(\ell_t)$ and $\theta_t = \Theta\left(\ell_t, k^m_{t+1}\right)$. In other words, in a Markov perfect equilibrium the policymaker chooses the same interest rate, and hence the same k^m_{t+1}, after any h^t leading to the same measure of lending relationships ℓ_t.

Given $\mathcal{K}(\ell)$ the market-tightness function, $\Theta(\ell, k)$, solves the following functional equation:

$$\frac{\Theta(\ell,k)}{\alpha\left[\Theta(\ell,k)\right]} = \beta\eta\left\{\frac{\lambda(1-a)A}{\zeta}\left[(aA)^{\frac{a}{1-a}} - \nu k^a\right]\right\} - \beta(1-\eta)\Theta\left(\ell',k'\right) + \beta(1-\delta)\frac{\Theta(\ell',k')}{\alpha\left[\Theta(\ell',k')\right]}, \quad (39)$$

where $\ell' = (1-\delta)\ell + \alpha(\theta)(1-\ell)$ and $k' = \mathcal{K}(\ell)$. From (39), θ_t depends on the interest rate set prior to the opening of the credit market, which determines the investment unbanked entrepreneurs can finance internally, and it depends on the current state, ℓ_t, which determines the future state through the law of motion. When forming expectations about market tightness in $t+1$, banks anticipate that the policymaker in period $t+1$ will adhere to his policy rule, $k^m_{t+2} = \mathcal{K}(\ell_{t+1})$, and hence $\theta_{t+1} = \Theta\left(\ell_{t+1}, k^m_{t+2}\right)$.

Given Θ from (39), the problem of the policymaker can be written recursively as

$$\begin{aligned}\mathcal{W}(\ell_t) &= \max_{k_{t+1}^m \in [0,k^*], \ell_{t+1} \in [0,1]} \left\{ -\zeta(1-\ell_t)\Theta\left(\ell_t, k_{t+1}^m\right) + \beta\lambda\nu\left(1-\ell_{t+1}\right)\left[f\left(k_{t+1}^m\right) - k_{t+1}^m\right] \right. \\ &\quad \left. + \beta\ell_{t+1}\lambda\left[f(k^*) - k^*\right] + \beta\mathcal{W}\left(\ell_{t+1}\right)\right\}, \end{aligned} \tag{40}$$

subject to

$$\ell_{t+1} = (1-\delta)\ell_t + \alpha\left[\Theta\left(\ell_t, k_{t+1}^m\right)\right](1-\ell_t). \tag{41}$$

We compute Markov perfect equilibrium using a two-dimensional iteration detailed in the Appendix. In a nutshell, in the nth iteration of the algorithm, we first compute $\Theta(\ell, k)$ as a fixed point to the mapping implied by (39) taking as given some policy rule $\mathcal{K}_n(\ell)$. Second, given $\Theta(\ell, k)$, we update $\mathcal{K}_{n+1}(\ell)$ from (40) by standard value function iteration. We repeat these iterations until $\mathcal{K}_n(\ell)$ converges up to a sufficient accuracy. We take as our initial guess for the Markov strategy, $\mathcal{K}_0(\ell)$, the policy function under the full commitment.

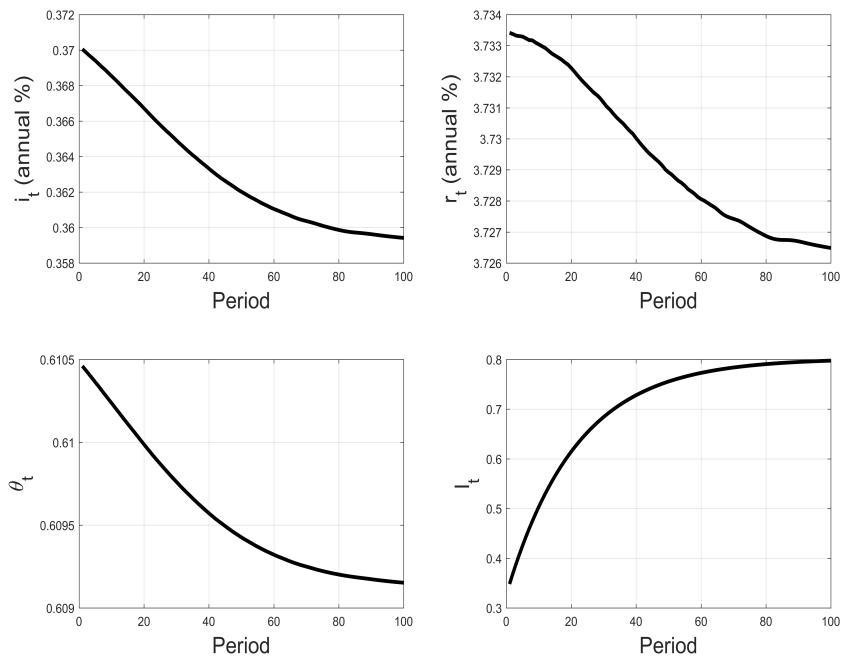

Figure 7: Optimal Markov policy without commitment

Figure 7 plots the optimal policy without commitment under the benchmark calibration. The top left panel shows that the time path for $\{i_t\}$ differs substantially from the one obtained with commitment.

RESULT #3. Without commitment, the optimal policy consists of raising i_t at the onset of a crisis.

The policymaker who cannot credibly promise higher future interest rates sets the interest rate at the outset of the crisis at a higher value than the steady state value. To promote bank entry, the interest rate is also higher than the one under commitment, 0.37% instead of 0.25%.

RESULT #4. Interest rates are lower than under commitment except for the initial periods.

As lending relationships recover and return to steady state, the policymaker reduces i_t slightly over time. Indeed i_t cannot be set too high since such policy is not time consistent. Quantitatively, changes in i_t are smaller in the absence of commitment, from 0.37% to 0.36%, and the level is closer to the Friedman rule. The policymaker's lack of commitment therefore generates a deflationary bias that is different from the inflationary bias that arises in Kydland and Prescott (1977) and Barro and Gordon (1983).

RESULT #5. Lending relationships are rebuilt more slowly without commitment.

Due to persistently low interest rates after the crisis, the stock of lending relationships take a longer time to rebuild than under commitment. The recovery in terms of lending relationships is weaker without commitment and credit market tightness is lower at all dates. After four years, the fraction of entrepreneurs in a lending relationship is 70% whereas under commitment such a fraction is reached in about two years. There is a welfare loss associated with the policymaker's lack of commitment, which we compute as the percentage loss in welfare between the Ramsey and Markov economies in terms of output, assuming the same initial states. In our example, the welfare loss amounts to 0.45% of total output.

7 Optimal policy and the ZLB

In Section 6, the optimal policy in the aftermath of a crisis is above the ZLB since the marginal social gain from being at $i_t = 0$ for unbanked entrepreneurs is second order whereas the social cost in terms of bank entry is first order. In the following, we extend our analysis to provide conditions under which the optimal steady-state interest rate is positive, but it is socially optimal to set it temporarily at the ZLB following a crisis.

7.1 Markup on capital goods

We first justify setting i_t at the ZLB by introducing imperfect competition in the capital goods market. Suppose the suppliers of k sell their output at a markup $\mu > 1$.[22] The surplus of the entrepreneur from an investment opportunity is $f(k) - \mu k$ which is maximized for $k = k^{**}$ solution to $f'(k) = \mu$. The ex ante surplus of unbanked entrepreneurs net of the cost of holding real balances is now

$$\Delta_t \equiv \max_{k_t^m \geq 0} \left\{ -i_t \mu k_t^m + \lambda \nu \left[f(k_t^m) - \mu k_t^m \right] \right\}, \tag{42}$$

where $m_t = \mu k_t^m$ are the real balances required to buy k_t^m units of goods, and the FOC is

$$i_t = \lambda \nu \left[\frac{f'(k_t^m)}{\mu} - 1 \right]. \tag{43}$$

So the markup reduces the demand for real balances. Moreover, at the ZLB, $i_t = 0$, $k_t^m = k^{**} < k^*$. A deviation from the ZLB lowers the profits of unbanked entrepreneurs by

$$\left. \frac{\partial \left[f(k_t^m) - k_t^m \right]}{\partial k_t^m} \right|_{k_t^m = k^{**}} = \mu - 1 > 0.$$

In the presence of markups, $\mu > 1$, this effect is first order. The rest of the analysis is unchanged.

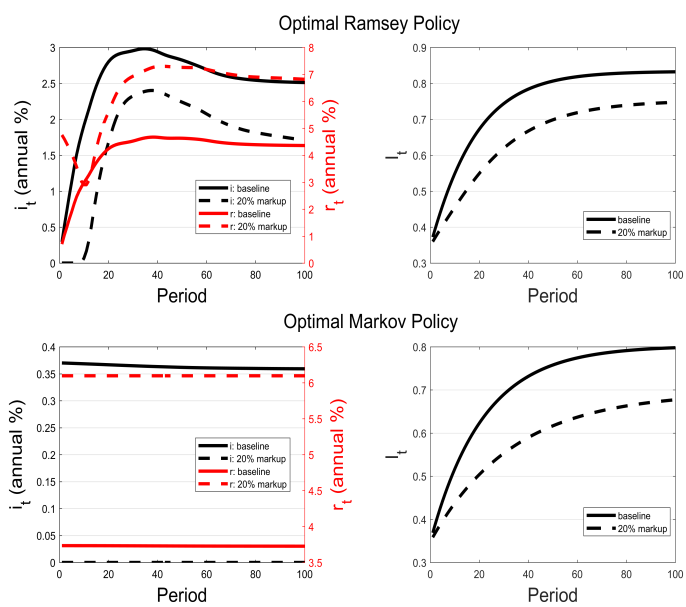

Figure 8: Optimal Ramsey and Markov policies with markup on capital goods

[22] There are many ways to introduce such a markup. We could assume that entrepreneurs and suppliers trade in bilateral matches and prices are determined through generalized Nash bargaining. Alternatively, we can assume monopolistic competition in the capital goods market as in Silva (2016).

In Figure 8, we keep the same parametrization as before but set $\mu = 1.2$. The top panel plots the optimal Ramsey policy and the bottom panel plots the optimal Markov policy. With a 20% markup, the optimal policy under commitment (dotted black line) is at the ZLB for about 10 months. Relative to the baseline model (solid black line), interest rates are lower with markups since deviations from the Friedman rule are more costly. Hence, the economy will spend a longer time at the ZLB the larger the markup. Positive markups reduce profits and hence credit market tightness and lending relationships at all dates. In addition, the real lending rate follows a non-monotone path. It decreases when the economy is at the ZLB since market tightness increases and entrepreneurs' outside options improve.

The bottom panel of Figure 8 plots the Markov policy response without commitment and shows the ZLB is now optimal throughout the crisis and recovery. Relative to the baseline model, credit market tightness and lending relationships are lower at all dates due to consistently low gains from trade from the zero interest rate.

7.2 Temporary freeze of relationship creations

Suppose now that the crisis takes the form of an unanticipated shock that temporarily shuts down the creation of lending relationships. This freeze could represent tightening of lending standards and the difficulty for banks to screen applicants in times of large uncertainty (see e.g. Figure 2). Then, we have the following result for optimal policy.

Proposition 6 ***(Temporary freeze of lending relationship creations.)*** *Suppose $\{\bar{\alpha}_t\}_{t=0}^{+\infty}$ is such that $\bar{\alpha}_t = 0$ for all $t = 0, ..., T$. The optimal monetary policy with and without commitment consists of setting $i_{t+1} = 0$ for all $t = 0, ..., T$.*

If the creation of lending relationships shuts down, the optimal policy with and without commitment is the Friedman rule. Indeed, the creation rate of lending relationships is zero irrespective of monetary policy. In that case, there is no trade off between enhancing the rate of return of liquid assets and providing incentives for banks to participate: the policymaker's only objective is to maximize the rate of return of currency. Moreover, even though the nominal rate is set at its lower bound, there is no sense in which the policymaker would like to implement negative rates if it could. At $i_t = 0$, the needs for liquidity are satiated.

We illustrate Proposition 6 with a simple example where $\overline{\alpha}$ is set to zero for 80 periods, and then returns to its calibrated value of $\overline{\alpha}$. Figure 9 plots the outcomes of the Ramsey problem with

commitment in the left panel and the Markov policy in the right panel.

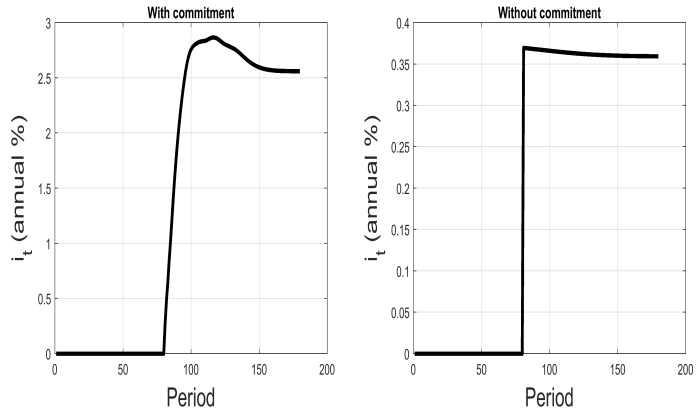

Figure 9: Optimal policy with temporary freeze of creations of lending relationships

Alternatively, we can consider a temporary increase in the entry cost of banks, ζ. For instance, it has been argued that the financing costs of banks increased sharply during the 2007–2008 financial crisis.[23] Denote $\bar{\zeta} \equiv \beta\lambda\eta\left[f(k^*) - k^*\right]$ the value for ζ above which banks have no incentive to enter irrespective of Δ_t. We obtain the following result for optimal policy.

Proposition 7 ***(Prohibitive entry costs.)*** *Suppose $\zeta_t > \bar{\zeta}$ for all $t = 0, ..., T$. The optimal monetary policy with and without commitment consists of setting $i_{t+1} = 0$ for all $t = 0, ..., T$.*

The logic is similar to that of Proposition 6. If banks have no incentive to enter irrespective of policy, it is optimal to set $i_t = 0$ to maximize the liquidity of unbanked entrepreneurs.

8 Transaction and relationship lending

So far, the only source of external finance is through lending relationships provided by banks. Firms in practice have access to alternative means of credit, namely both transaction and relationship loans (for evidence, see e.g. Sette and Gobbi 2014, Bolton, Freixas, Gambacorta, and Mistrulli 2016). To study how loan contracts and optimal monetary policy are affected in the presence of alternatives source of credit, we extend the model to introduce transaction lending interpreted as one-time lending contracts between a lender and entrepreneur. In contrast to long-term credit lines negotiated before investment opportunities occur, one-time loans are negotiated when the investment opportunity materializes in stage 1. Formally, an unmatched entrepreneur who received

[23] Illes, Lombardi, and Mizen (2011) provide cross country evidence on increased bank funding costs during the 2007 financial crisis as measured by the weighted-average cost of banks' liabilities.

an investment opportunity in stage 1 can find a transaction lender with probability $\alpha^s \in [0,1]$ ($\alpha^s = 0$ in the previous sections). In contrast, entrepreneurs meet relationship lenders in stage 2. We take α^s as exogenous, but could endogenize it through free entry of transaction lenders. We assume transaction lenders, just like relationship lenders, can fully enforce repayment of loans. Matches with transaction lender are short lived and destroyed at the end of the period with probability one. Here, the specific interpretation for ν matters for the role of transaction lenders. We maintain the assumption that the probability of receiving investment opportunities is larger with relationship lenders, λ, than without, in which case it is equal to $\lambda\nu$.

The terms of the loan contract between a transaction lender and an entrepreneur, (k^s, d^s, ϕ^s), are determined by generalized Nash bargaining where the bargaining power of the lender is $\eta^s \in (0,1)$, i.e.,

$$(k^s, d^s, \phi^s) \in \arg\max_{k,\phi,d} \left\{ f(k) - k - \phi - [f(k^m) - k^m] \right\}^{1-\eta^s} \phi^{\eta^s}, \tag{44}$$

where the repayment, $k - d + \phi$, is no greater than the entrepreneur's output, $f(k)$ (since output is fully pledgeable), and the down payment is no greater than the entrepreneur's real balances, $d \leq m$. The surplus of the transaction lender is simply the interest payment ϕ while the surplus of the entrepreneur is the difference between the profits if the entrepreneur takes a loan, $f(k) - k - \phi$, and the profits if investment is financed with money only, $f(k^m) - k^m$. The solution to the loan contract is

$$k^s = k^* \tag{45}$$

$$\phi^s = \eta^s \left\{ f(k^*) - k^* - [f(k^m) - k^m] \right\}. \tag{46}$$

Since output is fully pledgeable, transaction lenders can offer a loan to finance the first best. Interest payments to transaction lenders depends on the entrepreneur's real balances since his outside option improves with his ability to self finance investments. Hence, as real balances increase, the fee to transaction lender decreases. Moreover, as $k^m \to k^*$, $\phi^s \to 0$. The lifetime utility of an unbanked entrepreneur solves

$$U_t^e(m_t) = \lambda\nu \left[1 - \alpha^s(1-\eta^s)\right] \left[f(k_t^m) - k_t^m\right] + \lambda\nu\alpha^s(1-\eta^s) \left[f(k^*) - k^*\right] + V_t^e(m_t). \tag{47}$$

If the entrepreneur receives an investment opportunity, with probability $\lambda\nu$, then he enjoys profits equal to at least $f(k_t^m) - k_t^m$. If in addition he meets a transaction lender, he raises his profits by a fraction $1 - \eta^s$ of the additional surplus that the loan generates, $[f(k^*) - k^*] - [f(k_t^m) - k_t^m]$. The

expected surplus of an unmatched entrepreneur net of the cost of holding real balances is

$$\Delta_t \equiv \max_{m_t \geq 0} \left\{ -i_t m_t + \lambda\nu \left[1 - \alpha^s(1-\eta^s)\right] \left[f(k_t^m) - k_t^m\right] + \lambda\nu\alpha^s(1-\eta^s)\left[f(k^*) - k^*\right] \right\}. \tag{48}$$

This net surplus increases with α^s but decreases with η^s. The FOC is

$$f'(k_t^m) = 1 + \frac{i_t}{\lambda\nu\left[1 - \alpha^s(1-\eta^s)\right]}. \tag{49}$$

The entrepreneur's choice of real balances decreases with α^s but increases with η^s. Given the generalized expression for Δ_t, the rest of the model is unchanged. The intermediation fee in a lending relationship and the real lending rate solve (20) and (21). Market tightness solves (22).

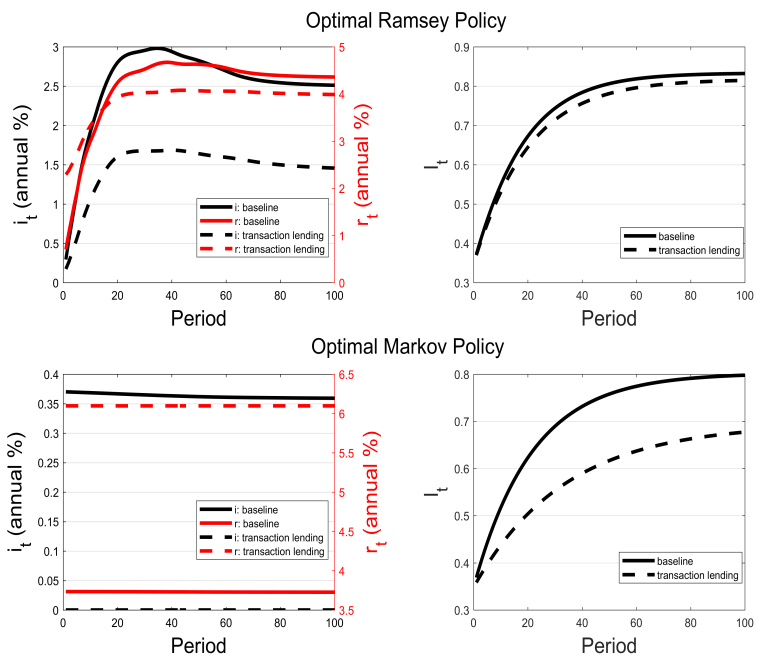

Figure 10: Optimal Ramsey and Markov policies with transaction lenders

Suppose $\eta^s = \eta$, i.e. all lenders have the same bargaining power. The intermediation fee paid to relationship lenders is larger than the fee paid to transaction lenders, $\phi \geq \phi^s$, if and only if

$$\eta\left[f(k^*) - k^*\right] - \frac{(1-\eta)\theta_t\zeta + \eta\Delta_t}{\lambda} \geq \eta\left\{f(k^*) - k^* - \left[f(k_t^m) - k_t^m\right]\right\}.$$

This condition holds in any equilibrium with bank entry provided i is close to zero. Indeed, in that case, entrepreneurs can self insure well so that the role of transaction lenders is limited whereas relationship lenders have an extra role by increasing the number of investment opportunities. In contrast, if i is very large, then $k_t^m \to 0$ and $\Delta_t \longrightarrow \lambda\nu\alpha^s(1-\eta)\left[f(k^*) - k^*\right]$ and $\phi < \phi^s$.

Figure 10 shows the optimal policy when $\eta^s = \eta = 0.10$ and the probability of a transaction loan is $\alpha^s = 1 - 0.2(1 - \alpha) = 0.81$, i.e. transaction loans have approximately a 20% lower rejection rate than relationship loans, consistent with some evidences from Bolton, Freixas, Gambacorta, and Mistrulli (2016). Under commitment (top panels), the optimal policy response with transaction lending (dotted lines) is qualitatively similar to the one in the benchmark model (solid lines) but interest rates are lower at all dates since transaction lending reduces the need for relationship lending. Without commitment (bottom panels), interest rates are at the ZLB throughout the crisis and recovery since the presence of transaction lending strengthens the deflationary bias. The lending rate on relationship loans (dotted red line) is higher than in the benchmark model (solid red line), while lending relationships are lower due to persistently lower interest rates.

9 Conclusion

This paper developed a corporate finance model of lending relationships and monetary policy. The formation of lending relationships between entrepreneurs and banks involves a time consuming matching process and the terms of the loan contract are negotiated bilaterally. Since entrepreneurs can remain unmatched for some time, there is a role for internal finance by retaining earnings in the form of cash. Our model delivers a transmission mechanism for monetary policy according to which the nominal interest rate set by the policymaker affects the real lending rate, banks' interest margins, and the supply of credit.

We used our model to determine the optimal monetary policy following a banking crisis described as an exogenous destruction of a fraction of the existing lending relationships. We made two assumptions regarding the power of the policymaker to commit to future interest rates. If the policymaker can commit over an infinite time horizon the optimal policy involves some "forward guidance": the interest is set close to its lower bound at the outset of the crisis and it increases over time as the economy recovers. It is the promise of future high interest and inflation rates that gives banks incentives to keep creating lending relationships in a low interest rate environment. However, such promises are not time consistent. If the policymaker cannot commit more than one period ahead, then the interest rate increases when the crisis hits and it falls slightly over time as the economy recovers. The inability to commit generates a more prolonged recession.

Our model of lending relationships and corporate finance can be extended in several ways. For instance, one could relax the assumption that banks can fully enforce repayment to study imperfect

pledgability of firms' returns and its relation with monetary policy (e.g., as in Rocheteau, Wright, and Zhang 2016). In addition, while we assume full commitment by banks, one could consider banks' limited commitment and analyze the dynamic contracting problem in the credit market (e.g., as in Bethune, Hu, and Rocheteau 2017). One could also introduce agency problems between firms and banks to capture additional benefits of lending relationships (e.g., Hachem 2011, Boualam 2017). It would be fruitful to formalize the life cycle of entrepreneurs to explain firms' cash accumulation patterns and their interaction with long term credit lines. Finally, our model of relationship lending could also be applied in other institutional contexts, like the interbank market (Brauning and Fecht 2016).

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Appendix A1: Proofs of Propositions and Lemmas

Proof of Lemma 1

The Nash problem (8) implies $(1-\eta)S^b_{t+\tau} = \eta S^e_{t+\tau}$, or equivalently, $S^b_{t+\tau} = \eta S_{t+\tau}$ for all τ. Hence, as is standard in the labor search literature (e.g. Pissarides 2000), we uniquely determine the full sequence, $\{k_{t+\tau}, \phi_{t+\tau}\}_{\tau=0}^{\infty}$, by imposing the Nash solution throughout the relationship.

Differentiating (7) with respect to k_t gives $f'(k_t) = 1$, and hence $k_t = k^*$ for all t. Since ϕ_t transfer utility perfectly between the entrepreneur and bank, all Pareto efficient contracts maximize S_t. Substituting S_t from (7) into $S^b_t = \eta S_t$ and solving for ϕ_t, we obtain

$$\phi_t = \eta\left[f(k^*) - k^*\right] - (1-\eta)\theta_t\zeta/\lambda \ \forall t,$$

where we used that $U^e_t - \beta U^e_{t+1} = \alpha_t \beta S^e_{t+1} = \frac{1-\eta}{\eta}\theta_t\zeta$. The real interest rate on a loan is then computed as $r_t = \phi_t/k_t$, which gives (10).

Proof of Proposition 1

A steady state equilibrium is a triple, (θ, ℓ, r), that solves

$$(\rho+\delta)\frac{\theta}{\alpha(\theta)} + (1-\eta)\theta = \frac{\lambda\eta\left[f(k^*)-k^*\right]}{\zeta}, \tag{50}$$

$$\ell = \frac{\alpha(\theta)}{\delta+\alpha(\theta)}, \tag{51}$$

$$r = \frac{\eta\left[f(k^*)-k^*\right] - (1-\eta)\theta\zeta/\lambda}{k^*}. \tag{52}$$

Credit market tightness is obtained uniquely from (50). Given θ, ℓ is given by (51) and r is given by (52). From (50), equilibrium features $\theta > 0$ if and only if

$$(\rho+\delta)\zeta < \lambda\eta[f(k^*)-k^*].$$

Given ℓ_0, an equilibrium with $\theta_t = \theta$ is unique if $\partial\theta_{t-1}/\partial\theta_t \in (-1,1)$, evaluated at the steady state. From (11), $\partial\theta_{t-1}/\partial\theta_t = \beta(1-\delta) - \beta(1-\eta)\alpha(\theta)/[1-\varepsilon(\theta)]$ where $\varepsilon(\theta) \equiv \theta\alpha'(\theta)/\alpha(\theta)$.

Proof of Lemma 2

To solve for the bargaining outcome, we follow the same steps as in Section 3. We first compute the surplus of being in a lending relationship for entrepreneurs and banks. At the beginning of a period, the lifetime expected utility of a banked entrepreneur with $m \geq d_t$ real balances solves

$$Z^e_t(m) = \lambda\left[f(k_t) - k_t - \phi_t - d_t\right] + \delta W^e_t(m) + (1-\delta)X^e_t(m), \tag{53}$$

where $W_t^e(m)$ is the lifetime expected value of an unbanked entrepreneur holding m real balances in the last stage of period t:

$$W_t^e(m) = m + T_t + \max_{m_{t+1} \geq 0} \{-(1+\pi_{t+1})m_{t+1} + \beta U_{t+1}^e(m_{t+1})\}.$$

Similarly, the lifetime expected value of a banked entrepreneur with m real balances is

$$X_t^e(m) = m + T_t + \max_{m_{t+1} \geq 0} \{-(1+\pi_{t+1})m_{t+1} + \beta Z_{t+1}^e(m_{t+1})\}.$$

From (53), the entrepreneur receives an investment opportunity with probability λ. In accordance with the terms of the lending contract, the investment size is k_t, which is financed with d_t real balances and a loan with interest payment ϕ_t. The lending relationship is destroyed with probability δ, in which case the value of the entrepreneur in the last stage is $W_t^e(m)$. Otherwise, the continuation value of the entrepreneur is $X_t^e(m)$. For all $m \geq d_t$, $Z_t^e(m) = m + Z_t^e(0)$. From the expression for X_t^e, it follows that $m = d_t$ if $1+\pi_t > \beta$. In words, if money is costly to hold, a banked entrepreneur does not carry more money than what is needed to honor the terms of the lending contract. In the following, we denote $Z_t^e = Z_t^e(d_t)$.

Following the same steps as in Section 3, the surplus of a banked entrepreneur solves

$$S_t^e = -i_t d_t + \lambda \left[f(k_t) - k_t - \phi_t - d_t\right] - \left[\Delta_t + V_t^e - W_t^e(0)\right] + (1-\delta)\beta S_{t+1}^e. \tag{54}$$

The first two terms on the right side of (54) represent the expected profit of the entrepreneur from an investment opportunity net of the cost of holding the real balances required for the down payment. The surplus of the bank solves $S_t^b = \lambda\phi_t + \beta(1-\delta)S_{t+1}^b$. Hence, the total surplus of a lending relationship, $S_t = S_t^e + S_t^b$, solves

$$S_t = -i_t d_t + \lambda \left[f(k_t) - k_t\right] + W_t^e(0) - (\Delta_t + V_t^e) + (1-\delta)\beta S_{t+1}. \tag{55}$$

The terms of the contract negotiated at time $t-1$ specifies a sequence $\{k_{t+\tau}, d_{t+\tau}, \phi_{t+\tau}\}_{\tau=0}^{\infty}$ that solves the generalized Nash product:

$$\max_{k_{t+\tau}, d_{t+\tau}, \phi_{t+\tau}} [S_{t+\tau}^b]^{\eta} [S_{t+\tau}^e]^{1-\eta}, \tag{56}$$

which implies $(1-\eta)S_{t+\tau}^b = \eta S_{t+\tau}^e$, or $S_{t+\tau}^b = \eta S_{t+\tau}$ for all τ. Hence as before, we impose the Nash solution at all dates throughout the relationship.

Differentiating (55) with respect to k_t gives $f'(k_t) = 1$, and hence $k_t = k^*$ for all t. Similarly, differentiating (55) with respect to d_t gives $d_t = 0$ for all t. Intuitively, any Pareto-efficient contract maximizes the joint surplus, which implies $k_t = k^*$ and $d_t = 0$ for all t. To uniquely determine the sequence of intermediation fees, we substitute the expressions for $S_t^b = \lambda\phi_t + \beta(1-\delta)S_{t+1}^b$ and S_t from (55). Solving for ϕ_t gives (70).

Proof of Proposition 2

In steady state, credit market tightness is the unique solution to

$$(\rho+\delta)\frac{\theta}{\alpha(\theta)}+(1-\eta)\theta=\frac{\lambda\eta\left[f(k^*)-k^*\right]-\eta\Delta}{\zeta}. \tag{57}$$

Given θ, closed-form solutions for $(\ell, k^m, r, \vartheta)$ are:

$$\ell = \frac{\alpha(\theta)}{\delta+\alpha(\theta)} \tag{58}$$

$$k^m = f'^{-1}\left(1+\frac{i}{\lambda\nu}\right) \tag{59}$$

$$r = \frac{\eta\lambda[f(k^*)-k^*]-(1-\eta)\theta\zeta-\eta\Delta}{\lambda k^*} \tag{60}$$

$$\vartheta M = \frac{\delta}{\delta+\alpha(\theta)}f'^{-1}\left(1+\frac{i}{\lambda\nu}\right). \tag{61}$$

The existence and uniqueness of the steady-state monetary equilibrium follow directly from (57)-(61). From (57) and $\partial\Delta/\partial i=-k^m<0$, $\partial\theta/\partial i>0$. To show $\partial r/\partial i>0$, we can rewrite (57) as

$$(\rho+\delta)\frac{\theta\zeta}{\alpha(\theta)}=\lambda r k^*.$$

Hence, r is increasing with θ. From (59),

$$\frac{\partial k^m}{\partial i}=\frac{1}{\lambda\nu f''(k^m)}<0.$$

From (61),

$$\frac{\partial(\vartheta M)}{\partial i}=\frac{-\delta\alpha'(\theta)k^m}{[\delta+\alpha(\theta)]^2}\frac{\partial\theta}{\partial i}+\frac{\delta}{\delta+\alpha(\theta)}\frac{\partial k^m}{\partial i}<0.$$

To obtain (25), we differentiate (57) in the neighborhood of $i=0$:

$$\frac{\partial\theta}{\partial i}=-\frac{\eta}{\zeta\left\{(\rho+\delta)\left[(1-\epsilon(\theta))/\alpha(\theta)+1-\eta\right]\right\}}\frac{\partial\Delta}{\partial i}.$$

Using $\partial\Delta/\partial i=-k^*$ gives (25). Finally, to obtain (26), we differentiate (60) in the neighborhood of $i=0$:

$$\frac{\partial r}{\partial i}=-\frac{(1-\eta)\zeta}{\lambda k^*}\frac{\partial\theta}{\partial i}-\frac{\eta}{\lambda k^*}\frac{\partial\Delta}{\partial i}.$$

Using (25) and $\partial\Delta/\partial i=-k^*$ then gives (26).

Proof of Proposition 3

We consider an economy that starts with $\ell_0=\underline{\ell}=\alpha(\underline{\theta})/\left[\delta+\alpha(\underline{\theta})\right]$ lending relationships, where $\underline{\theta}$ is credit market tightness at the Friedman rule. From (57), it solves:

$$(\rho+\delta)\zeta+\alpha(\underline{\theta})\zeta=\frac{\alpha(\underline{\theta})}{\underline{\theta}}\eta\left\{\lambda(1-\nu)\left[f(k^*)-k^*\right]+\underline{\theta}\zeta\right\}. \tag{62}$$

According to (30), $\underline{\theta} > 0$. We measure social welfare in the second stage of $t = 0$ before banks make entry decisions and entrepreneurs make portfolio decisions:

$$\mathcal{W} = -\zeta\theta_0(1-\ell_0) + \sum_{t=1}^{\infty} \beta^t \left\{(1-\ell_t)\lambda\nu\left[f(k_t^m) - k_t^m\right] + \ell_t\lambda\left[f(k^*) - k^*\right] - \zeta\theta_t(1-\ell_t)\right\}. \tag{63}$$

The first term on the RHS is the entry cost of banks in the initial period. The second term is the discounted sum of entrepreneurs' profits net of banks' entry cost in all subsequent periods. We consider a small deviation of the nominal interest rate from $i_1 = 0$. (The nominal interest rate is known at the time banks make entry decisions.) For $t \geq 2$, $i_t = 0$. As a result, for all $t \geq 1$, $\theta_t = \underline{\theta}$ and for all $t \geq 2$, $k_t^m = k^*$. The measure of lending relationships solves:

$$\ell_1 = (1-\delta)\underline{\ell} + \alpha(\theta_0)(1-\underline{\ell}) \tag{64}$$

$$\ell_t = \underline{\ell} + (\ell_1 - \underline{\ell})[1-\delta-\alpha(\underline{\theta})]^{t-1} \quad \text{for all } t \geq 2. \tag{65}$$

The welfare starting at time $t = 1$ if there is no deviation from the Friedman rule, $i_1 = 0$, is:

$$\begin{aligned}
\mathcal{W}_1^{FR} &= \sum_{t=1}^{\infty} \beta^{t-1}\left\{\left[(1-\ell_t)\nu + \ell_t\right]\lambda\left[f(k^*) - k^*\right] - \zeta\underline{\theta}(1-\ell_t)\right\} \\
&= \frac{\left[(1-\underline{\ell})\nu + \underline{\ell}\right]\lambda\left[f(k^*) - k^*\right] - (1-\underline{\ell})\zeta\underline{\theta}}{1-\beta} + (\ell_1 - \underline{\ell})\frac{\lambda(1-\nu)\left[f(k^*) - k^*\right] + \zeta\underline{\theta}}{1-\beta[1-\delta-\alpha(\underline{\theta})]}.
\end{aligned}$$

Welfare from $t = 0$ is:

$$\begin{aligned}
\mathcal{W} &= -\zeta\theta_0(1-\underline{\ell}) + \beta(1-\ell_1)\lambda\nu\left\{\left[f(k_1^m) - k_1^m\right] - \left[f(k^*) - k^*\right]\right\} + \beta\mathcal{W}_1^{FR} \\
&= -\zeta\theta_0(1-\underline{\ell}) + \beta(1-\ell_1)\lambda\nu\left\{\left[f(k_1^m) - k_1^m\right] - \left[f(k^*) - k^*\right]\right\} \\
&\quad + \beta\frac{\lambda\left[(1-\underline{\ell})\nu + \underline{\ell}\right]\left[f(k^*) - k^*\right] - (1-\underline{\ell})\zeta\underline{\theta}}{1-\beta} \\
&\quad + \beta(\ell_1 - \underline{\ell})\frac{\lambda(1-\nu)\left[f(k^*) - k^*\right] + \zeta\underline{\theta}}{1-\beta[1-\delta-\alpha(\underline{\theta})]}.
\end{aligned} \tag{66}$$

The second term on the RHS corresponds to the change in unbanked entrepreneurs' profits in $t = 1$ following a deviation from $i_1 = 0$. The relationship between θ_0 and k_1^m is given by (22) where we use that $\theta_1 = \underline{\theta}$, i.e.:

$$\begin{aligned}
\frac{\theta_0}{\alpha(\theta_0)} &= \beta\eta\left\{\frac{\lambda\left[f(k^*) - k^*\right] - \max_{k_1^m}\left\{-i_1 k_1^m + \lambda\nu\left[f(k_1^m) - k_1^m\right]\right\}}{\zeta}\right\} \\
&\quad -\beta(1-\eta)\underline{\theta} + \beta(1-\delta)\frac{\underline{\theta}}{\alpha(\underline{\theta})}.
\end{aligned} \tag{67}$$

Differentiating (67) we obtain:

$$\begin{aligned}
\left.\frac{\partial\theta_0}{\partial i_1}\right|_{i_1=0} &= \beta\eta\frac{\alpha(\theta_0)}{1-\epsilon(\theta_0)}\frac{k^*}{\zeta} > 0 \\
\left.\frac{\partial k_1^m}{\partial i_1}\right|_{i_1=0} &= \frac{1}{\lambda\nu f''(k^*)} < 0.
\end{aligned}$$

So a small increase of i_1 above 0 raises credit market tightness but reduces investment by unbanked entrepreneurs. From (66) the change in social welfare is

$$\left.\frac{\partial \mathcal{W}}{\partial i_1}\right|_{i_1=0} = \left\{-\zeta + \beta\alpha'(\underline{\theta})\frac{\lambda(1-\nu)\left[f(k^*)-k^*\right]+\zeta\underline{\theta}}{1-\beta[1-\delta-\alpha(\underline{\theta})]}\right\}(1-\ell)\frac{\partial\theta_0}{\partial i_1}, \tag{68}$$

where we have used that a small increase in i_1 above the Friedman rule only has a second-order effect on the profits of unbanked entrepreneurs. Using (62) to simplify the term between brackets in (68) we obtain:

$$\left.\frac{\partial \mathcal{W}}{\partial i_1}\right|_{i_1=0} = \left(\frac{\epsilon(\underline{\theta})-\eta}{\eta}\right)\zeta(1-\ell)\frac{\partial\theta_0}{\partial i_1}.$$

If $\epsilon(\underline{\theta}) > \eta$, then a deviation from the Friedman rule is optimal.

Proof of Proposition 4

We first restrict the policymaker's choice to bounded sequences $\{\theta_t\}$ that solve (31) given some initial condition, θ_{-1}. The policymaker solves:

$$\mathcal{W}(\ell_0,\theta_0) = \max_{\{k^m_{t+1}\}}\sum_{t=0}^{\infty}\beta^t\left\{-\zeta\theta_t(1-\ell_t)+\beta(1-\ell_{t+1})\lambda\nu\left[f(k^m_{t+1})-k^m_{t+1}\right]+\beta\ell_{t+1}\lambda\left[f(k^*)-k^*\right]\right\}$$
$$\text{s.t. } \{\theta_t,\ell_t\}_{t=0}^{\infty} \text{ being a solution to (31)-(37) given } \ell_0 \text{ and } \theta_0.$$

The restriction $\theta_t \in \Omega$ is justified as follows. Suppose $\theta_t > \bar{\theta}$ for some t. Then,

$$\theta_{t+1}-\bar{\theta} = \frac{\theta_t-\bar{\theta}+\bar{\alpha}\beta\eta\lambda\left(1-a\right)A\nu(k^m_{t+1})^a/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}.$$

Since $\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right] \in (0,1)$, the sequence $\{\theta_t-\bar{\theta}\}$ is unbounded, which is inconsistent with optimality as entry costs would be unbounded. Suppose next $\theta_t \in (0,\underline{\theta})$ for some t. With $\underline{\theta} > 0$,

$$\underline{\theta}-\theta_{t+1} = \frac{\underline{\theta}-\theta_t+\bar{\alpha}\beta\eta\lambda\left(1-a\right)A\nu\left[(aA)^{\frac{a}{1-a}}-(k^m_{t+1})^a\right]/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}.$$

So θ_t becomes negative in finite time, which is inconsistent with an equilibrium. The feasibility condition $\theta_{t+1} \in \Gamma\left(\theta_t\right)$ is obtained from (31) by varying k^m_{t+1} from zero to k^*. By the Principle of Optimality, $\mathcal{W}(\ell_t,\theta_t)$ solves the Bellman Equation (35), i.e., it is the fixed point of a mapping from $\mathcal{B}([0,1]\times\left[\underline{\theta},\bar{\theta}\right])$ into itself. The mapping in (35) is a contraction by Blackwell's sufficient conditions (Theorem 3.3 in Stokey and Lucas, 1989), and by the contraction mapping theorem (Theorem 3.2 in Stokey and Lucas, 1989), the fixed point exists and is unique. The correspondence Γ is continuous and the policymaker's period utility is also continuous. So $\mathcal{W}(\ell,\theta)$ is continuous by the Contraction Mapping Theorem. Given there is no initial value for θ in the original sequence problem, (29), we choose $\theta_0 \in \Omega$ to maximize $\mathcal{W}(\ell_0,\theta_0)$. Such a solution exists by the continuity of $\mathcal{W}$ and the compactness of Ω. Given θ_0, we use the policy function associated with $\mathcal{W}$ to pin down the entire trajectory for $\{\theta_t, k^m_t, \ell_t\}$.

Proof of Proposition 5

We now establish that $\theta_t \in \left(\underline{\theta}, \bar{\theta}\right)$ for all t. Suppose $\theta_T = \underline{\theta}$. Since $\Gamma(\underline{\theta}) = \{\underline{\theta}\}$, it follows that $\theta_t = \underline{\theta}$ for all $t \geq T$. Such an equilibrium is implemented under the Friedman rule. Since we imposed a condition for the Friedman rule to be suboptimal, this contradicts $\theta_T = \underline{\theta}$. Suppose next $\theta_T = \bar{\theta}$. The equilibrium is a non-monetary equilibrium, i.e., $k_t^m = 0$ and $\theta_t = \bar{\theta}$ for all t. Consider an alternative equilibrium with a constant (k^m, θ).

$$\mathcal{W} = -\zeta\theta_0(1-\ell_0) + \sum_{t=1}^{\infty} \beta^t \left\{(1-\ell_t)\lambda\nu\left[f(k^m) - k^m\right] + \ell_t\lambda\left[f(k^*) - k^*\right] - \zeta\theta(1-\ell_t)\right\}.$$

Provided that $\nu > 0$, $\partial\mathcal{W}/\partial k^m = \infty$ when evaluated at $k^m = 0$. So $\theta_T = \bar{\theta}$ cannot be an equilibrium under an optimal policy.

Proof of Proposition 6

From (22) θ_t solves $-\zeta + \alpha_t^b \beta Z_{t+1}^b \leq 0$, with an equality if $\theta_t > 0$. Using that $\alpha_t(\theta) = \bar{\alpha}_t\theta/(1+\theta)$ it can be reexpressed as

$$\frac{1+\theta_t}{\bar{\alpha}_t} \geq \beta\eta\left\{\frac{\lambda\left[f(k^*) - k^*\right] - \Delta_{t+1}}{\zeta}\right\} - \beta(1-\eta)\theta_{t+1} + \beta(1-\delta)\frac{1+\theta_{t+1}}{\bar{\alpha}_{t+1}},$$

with an equality if $\theta_t > 0$. Rearranging this inequality we obtain:

$$\theta_t \geq \beta\eta\bar{\alpha}_t\left\{\frac{\lambda\left[f(k^*) - k^*\right] - \Delta_{t+1}}{\zeta}\right\} + \beta\bar{\alpha}_t\left[\frac{1-\delta}{\bar{\alpha}_{t+1}} - (1-\eta)\right]\theta_{t+1} + \beta(1-\delta)\frac{\bar{\alpha}_t}{\bar{\alpha}_{t+1}} - 1, \tag{69}$$

with an equality if $\theta_t > 0$. If $\bar{\alpha}_t = 0$, then $\theta_t = 0$. For all $t = 0, ..., T$ the policymaker's problem simplifies to

$$\mathcal{W}_t(\ell_t) = \max_{k_{t+1}^m \in [0,k^*]} \left\{\beta\lambda\nu\left(1-\ell_{t+1}\right)\left[f\left(k^m\right) - k^m\right] + \beta\lambda\ell_{t+1}\left[f(k^*) - k^*\right] + \beta\mathcal{W}_{t+1}\left(\ell_{t+1}\right)\right\}$$

$$\text{s.t. } \ell_{t+1} = (1-\delta)\ell_t,$$

where $\mathcal{W}_{T+1}$ which is welfare function with or without commitment. In either case, the optimal policy is always $k_{t+1}^m = k^*$, which is equivalent to $i_{t+1} = 0$ for $t = 0, ..., T$.

Appendix A2: Derivations for the terms of the loan contract

Here we show how we obtain the expression for ϕ_t given by (20). First, we can write the value function of an unbanked entrepreneur with m real balances at the start of period t as

$$U_t^e(m) = \Delta_t + i_t m_t + V_t^e(m).$$

Substituting this into the expression for $W_t^e(m)$ gives

$$W_t^e(m) = m + T_t + \beta(\Delta_{t+1} + V_{t+1}^e).$$

Hence, $W_t^e(0) = T_t + \beta(\Delta_{t+1} + V_{t+1}^e)$. Assuming that $1 + \pi_t > \beta$, the value function of a banked entrepreneur with m real balances in the last stage is

$$X_t^e(m) = m + T_t - (1 + \pi_{t+1})d_{t+1} + \beta Z_{t+1}^e(d_{t+1}),$$

which uses the fact that $m_{t+1} = d_{t+1}$. This gives $X_t^e(0) = T_t - (1 + \pi_{t+1})d_{t+1} + \beta Z_{t+1}^e$.

To derive (54), we use $X_{t-1}^e(0) = T_{t-1} - (1 + \pi_t)d_t + \beta Z_t^e$ and $W_{t-1}(0) = T_{t-1} + \beta(\Delta_t + V_t^e)$, so that the surplus of a banked entrepreneur, $S_t^e = \left[X_{t-1}^e(0) - W_{t-1}^e(0)\right]/\beta$, solves

$$S_t^e = -(1 + i_t)d_t + Z_t^e - (\Delta_t + V_t^e).$$

Substituting Z_t^e by its expression in (53), we obtain

$$S_t^e = -i_t d_t + \lambda[f(k_t) - k_t - \phi_t] + T_t - (\Delta_t + V_t^e) + \beta(\Delta_{t+1} + V_{t+1}^e) + (1 - \delta)\beta S_{t+1}^e.$$

We now derive (20) as follows. Using (54) and solving for ϕ_t,

$$\phi_t = [f(k_t) - k_t] + \frac{1}{\lambda}\left[W_t^e(0) - (\Delta_t + V_t^e) + (1 - \delta)\beta S_{t+1}^e - S_t^e\right].$$

Using $S_t^e = \frac{1-\eta}{\eta} S_t^b = \frac{1-\eta}{\eta}[\lambda\phi_t + \beta(1 - \delta)S_{t+1}^b]$ and substituting above, we obtain

$$\phi_t = \eta\left[f(k^*) - k^*\right] + \frac{\eta}{\lambda}\left[W_t^e(0) - (\Delta_t + V_t^e)\right]. \tag{70}$$

A novelty in (70) is the term Δ_t which depends on the rate of return on money and in turn affects the determination of r_t. The second term on the RHS of (70) arises from the fact that an unmatched entrepreneur has the option of purchasing k_t^m with his real balances, which reduces ϕ_t. From (15) and the fact that $\beta S_{t+1}^e = X_t^e(0) - W_t^e(0)$, this outside option can be expressed as

$$\begin{aligned}
(\Delta_t + V_t^e) - W_t^e(0) &= (\Delta_t + V_t^e) + T_t - \beta(\Delta_{t+1} + V_{t+1}^e) \\
&= \Delta_t + \alpha_t\beta(T_t/\beta - W_t^e/\beta + Z_{t+1}^e) \\
&= \Delta_t + \alpha_t\beta S_{t+1}^e.
\end{aligned}$$

Since $\eta S^e_{t+1} = (1-\eta)Z^b_{t+1}$ and $Z^b_{t+1} = \zeta\theta_t/\beta\alpha_t$ from free entry of banks, the outside option reduces to

$$(\Delta_t + V^e_t) - W^e_t(0) = \frac{(1-\eta)}{\eta}\theta_t\zeta + \Delta_t. \tag{71}$$

From the RHS of (71), the entrepreneur's reservation utility consists of the option of continuing to search for a bank and the option of using internal finance. Substituting (71) into (70) gives ϕ_t as expressed in (20).

Appendix A3. Numerical procedure for optimal policy problem

The numerical method is based on the observation that the following mapping is a contraction:

$$T\mathcal{W}_{n+1}(\ell,\theta) = \max_{\theta',\ell',k^m} \left\{-\zeta\theta(1-\ell) + \beta(1-\ell')\lambda\nu\left[f(k^m)-k^m\right] + \beta\ell'\lambda\left[f(k^*)-k^*\right] + \beta\mathcal{W}_n(\ell',\theta')\right\},$$

where

$$\begin{aligned} k^m &= \nu^{\frac{-1}{a}}\left\{(aA)^{\frac{a}{1-a}} - \frac{\zeta\left[\theta-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta'-\beta(1-\delta)+1\right]}{\bar{\alpha}\beta\eta\lambda\left(1-a\right)A}\right\}^{\frac{1}{a}} \\ \ell' &= (1-\delta)\ell + \frac{\bar{\alpha}\theta}{1+\theta}(1-\ell). \end{aligned}$$

Therefore we will iterate this mapping to obtain a sequence of value functions, $\{T^n\mathcal{W}_0\}$. This sequence is Cauchy and converges to the unique fixed point of T.

1. Choose the precision of the grid for the state space: $N_\ell \in \mathbb{N}, N_\theta \in \mathbb{N}$.

   This means there are $N_\ell + 1$ values for state ℓ and $N_\theta + 1$ values for state θ.

   Denote $\varepsilon_\ell = 1/N_\ell$ and $\varepsilon_\theta = \bar{\theta}/N_\theta$. A state is a pair $\ell = \varepsilon_\ell i$ and $\theta = \varepsilon_\theta j$ where $(i,j) \in \{0, N_\ell\} \times \{0, N_\theta\}$.

2. Initialize the iterations by choosing $\mathcal{W}_0(\ell,\theta)$.

   The initial guess for the value function is:

$$\begin{aligned} W_0(\ell,\theta) &= -\zeta\theta + (1-\ell)\frac{\lambda\nu\left[f(k^m)-k^m\right]-\zeta\theta}{\rho} + \ell^{ss}\frac{\lambda\left[f(k^*)-k^*\right]}{\rho} \\ &\quad +(\ell-\ell^{ss})\frac{\lambda\left\{\left[f(k^*)-k^*\right]-\nu\left[f(k^m)-k^m\right]\right\}+\zeta\theta}{\rho+\delta+\alpha(\theta)}. \end{aligned}$$

   where

$$\begin{aligned} \ell^{ss} &= \frac{\bar{\alpha}\theta}{\delta+(\delta+\bar{\alpha})\theta} \\ k^m &= \nu^{\frac{-1}{a}}\left\{(aA)^{\frac{a}{1-a}} - \frac{\left\{1-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\right\}\theta-\beta(1-\delta)+1}{\bar{\alpha}\beta\eta\lambda\left(1-a\right)A}\zeta\right\}^{\frac{1}{a}}. \end{aligned}$$

3. Suppose $\mathcal{W}_n$ is known. To compute $\mathcal{W}_{n+1}(\ell,\theta)$, do loops over the entire state space: $i = 0...N_\ell$ and $j = 0...N_\theta$. Each loop corresponds a (θ,ℓ). For each state find $\theta' \in \Gamma(\theta)$ that maximizes the right side of the Bellman equation, i.e.,

$$-\zeta\theta(1-\ell) + \beta(1-\ell')\lambda\nu\left[f(k^m)-k^m\right] + \beta\ell'\lambda\left[f(k^*)-k^*\right] + \beta\mathcal{W}_n\left(\ell',\theta'\right).$$

   If ℓ' does not belong to the grid, do a linear interpolation to approximate the value function.

4. Once $\mathcal{W}_{n+1}(\ell,\theta)$ has been computed, return to Step 3 to compute $\mathcal{W}_{n+2}(\ell,\theta)$. The criterion to stop is: $\left(\sum_{(\ell,\theta)}\left[\mathcal{W}_{n+1}(\ell,\theta)-\mathcal{W}_n(\ell,\theta)\right]^2\right)^{\frac{1}{2}} < \epsilon$.

Appendix A4: Derivations for model with markups

Here we provide details on the key equations for the model with markups in Section 7 needed to compute the optimal monetary policy problem. A monetary equilibrium in the model with markups solves (12), (21), (22), (23), and (43) where Δ_t solves (42). Assuming $f(k) = Ak^a$ and $\alpha(\theta) = \overline{\alpha}\theta/(1+\theta)$, the second-best capital stock is $k^{**} = (Aa/\mu)^{\frac{1}{1-a}}$. Moreover, $\Delta_t = \lambda\nu\,(1-a)\,A(k_t^m)^a$. The dynamic equation for market tightness becomes

$$\theta_t = \frac{\bar{\alpha}\beta\eta\lambda A(1-a)}{\zeta}\left[\left(\frac{Aa}{\mu}\right)^{\frac{a}{1-a}} - \nu(k_{t+1}^m)^a\right] + \beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta_{t+1} + \beta(1-\delta) - 1. \tag{72}$$

The relationship between k_t^m and i_t is given by:

$$i_t = \lambda\nu\left[\frac{Aa(k_t^m)^{a-1}}{\mu} - 1\right].$$

The lowest admissible value for θ is when $k^m = k^{**}$, which gives

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta\lambda A(1-a)(1-\nu)\,(Aa/\mu)^{\frac{a}{1-a}}/\zeta + \beta(1-\delta) - 1}{1-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}.$$

The highest admissible value for θ is when $k^m = 0$, which gives

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda A(1-a)\left[(Aa/\mu)^{\frac{a}{1-a}}\right]/\zeta + \beta(1-\delta) - 1}{1-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}.$$

The planner's recursive problem is

$$\begin{aligned} W(\ell_t,\theta_t) \;=\; & \max_{\theta_{t+1}\in\Gamma(\theta_t),\ell_{t+1},k_{t+1}^m} \left\{-\zeta\theta_t(1-\ell_t) + \beta(1-\ell_{t+1})\lambda\nu\left[f(k_{t+1}^m) - k_{t+1}^m\right]\right. \\ & \left. +\beta\ell_{t+1}\lambda\left[f(k^{**}) - k^{**}\right] + \beta W(\ell_{t+1},\theta_{t+1})\right. \end{aligned}$$

where

$$\begin{aligned} k_{t+1}^m &= \nu^{\frac{-1}{a}}\left\{\left(\frac{Aa}{\mu}\right)^{\frac{a}{1-a}} - \frac{\zeta\left[\theta_t - \beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta_{t+1} - \beta(1-\delta) + 1\right]}{\bar{\alpha}\beta\eta\lambda A(1-a)}\right\}^{\frac{1}{a}} \\ \ell_{t+1} &= (1-\delta)\ell_t + \frac{\bar{\alpha}\theta_t}{1+\theta_t}(1-\ell_t), \end{aligned}$$

and the feasibility correspondence $\Gamma(\theta_t)$ is defined as

$$\Gamma(\theta_t) = \left[\frac{\theta_t + 1 - \beta(1-\delta) - \bar{\alpha}\beta\eta\lambda\,(1-a)\,A\,(aA/\mu)^{\frac{a}{1-a}}/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]},\right.$$
$$\left.\frac{\theta_t + 1 - \beta(1-\delta) - \bar{\alpha}\beta\eta\lambda\,(1-a)\,A(1-\nu)\,(aA/\mu)^{\frac{a}{1-a}}/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}\right] \cap \Omega.$$

Appendix A5: Derivations for model with transaction lenders

Here we provide the key equations for the model with transaction lenders in Section 8 needed to compute the optimal monetary policy problem.

Assuming $f(k) = Ak^a$ and $\alpha(\theta) = \overline{\alpha}\theta/(1+\theta)$, the net surplus of unbanked entrepreneurs is

$$\Delta_t = \lambda\nu\left[1-\alpha^s(1-\eta^s)\right]A(1-a)(k_t^m)^a + \lambda\nu\alpha^s(1-\eta^s)A\left(1-a\right)\left(Aa\right)^{\frac{a}{1-a}},$$

where

$$i_t = \lambda\nu\left[1-\alpha^s(1-\eta^s)\right]\left[Aa(k_t^m)^{a-1}-1\right].$$

The dynamic equation for market tightness becomes:

$$\begin{aligned}\theta_t = \frac{\bar{\alpha}\beta\eta\lambda A(1-a)}{\zeta}&\left[\left(Aa\right)^{\frac{a}{1-a}}\left[1-\nu\alpha^s(1-\eta^s)\right]-\nu\left[1-\alpha^s(1-\eta^s)\right]\left(k_{t+1}^m\right)^a\right]\\ &+\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta_{t+1}+\beta(1-\delta)-1.\end{aligned}$$

The lowest admissible value for θ is when $k^m = k^*$, which gives

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta\lambda A(1-a)\left(Aa\right)^{\frac{a}{1-a}}\left(1-\nu\right)/\zeta+\beta(1-\delta)-1}{1-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}.$$

The highest admissible value for θ is when $k^m = 0$, which gives

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda A(1-a)\left[1-\nu\alpha^s(1-\eta^s)\right]\left(Aa\right)^{\frac{a}{1-a}}/\zeta+\beta(1-\delta)-1}{1-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}.$$

We are now ready to write the planner's problem recursively as:

$$\begin{aligned}W(\ell_t,\theta_t) \quad = \quad &\max_{\theta_{t+1}\in\Gamma(\theta_t),\ell_{t+1},k_{t+1}^m}\left\{-\zeta\theta_t(1-\ell_t)\right.\\ &+\beta(1-\ell_{t+1})\lambda\nu\left\{\alpha^s\left[f(k^*)-k^*\right]+(1-\alpha^s)\left[f(k_{t+1}^m)-k_{t+1}^m\right]\right\}\\ &+\beta\ell_{t+1}\lambda\left[f(k^*)-k^*\right]+\beta W(\ell_{t+1},\theta_{t+1})\end{aligned}$$

where

$$\begin{aligned}k_{t+1}^m \quad = \quad &\left\{\nu\left[1-\alpha^s(1-\eta^s)\right]\right\}^{\frac{-1}{a}}\times\\ &\left\{\left(Aa\right)^{\frac{a}{1-a}}\left[1-\nu\alpha^s(1-\eta^s)\right]-\frac{\zeta\left[\theta_t-\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta_{t+1}-\beta(1-\delta)+1\right]}{\bar{\alpha}\beta\eta\lambda A(1-a)}\right\}^{\frac{1}{a}}\\ \ell_{t+1} \quad = \quad &(1-\delta)\ell_t+\frac{\bar{\alpha}\theta_t}{1+\theta_t}(1-\ell_t),\end{aligned}$$

and the feasibility correspondence $\Gamma(\theta_t)$ is defined as

$$\begin{aligned}\Gamma\left(\theta_t\right) = \Bigg[&\frac{\theta_t+1-\beta(1-\delta)-\bar{\alpha}\beta\eta\lambda A(1-a)\left(Aa\right)^{\frac{a}{1-a}}\left[1-\nu\alpha^s(1-\eta^s)\right]/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]},\\ &\frac{\theta_t+1-\beta(1-\delta)-\bar{\alpha}\beta\eta\lambda A(1-a)\left(Aa\right)^{\frac{a}{1-a}}\left(1-\nu\right)/\zeta}{\beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]}\Bigg]\cap\Omega.\end{aligned}$$

Appendix B: Empirical Support

According to the 2016 Small Business Credit Survey, banks were the most common source of credit for U.S. small businesses, with 86% of firms reporting loans or credit lines for business purposes on their balance sheets and 31% with credit lines but no loans. See Mach and Wolken (2006) for additional evidence on small business finance and Sufi (2009) for evidence on the use of bank lines of credit among U.S. publicly traded firms. Evidence on the costs and benefits of relationship lending is described in Petersen and Rajan (1994) and Berger and Udell (1998). Recent studies on the role of cash in corporate financing decisions include Sanchez and Yurdagul (2013) and Graham and Leary (2016).

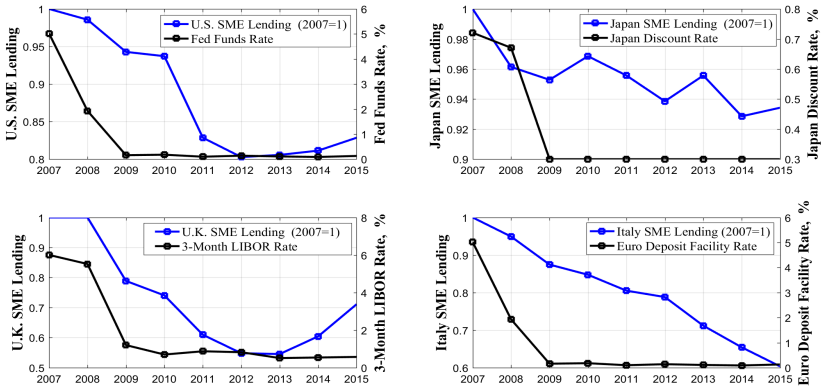

Figure 11: Bank lending to small-medium size enterprises and monetary policy across countries

Our model highlights the dynamic response of bank lending and interest rates following a destruction of lending relationships or a tightening of lending standards. Figure 11 shows the contraction in bank lending to small and medium size enterprises (SMEs) across countries from 2007 to 2015.[24] Chen, Hanson, and Stein (2017) show small business lending from banks fell sharply during the Great Recession and remained depressed even after 2010, particularly for loans from the top four U.S. banks.

Our theory also predicts a positive relationship between bank entry and the monetary policy rate, as shown in the top right panel of Figure 1 which uses bank entry and exit data from McCord and Prescott (2014). From 2007 through 2013, the number of U.S. commercial banks declined by 14%. While some of this decline was due to bank failures and exits, McCord and Prescott (2014) show nearly two thirds of the collapse was due to lack of new bank entry. Indeed, weak economic conditions in the aftermath of the crisis reduced incentives for new bank entry due in

[24] Figure 11 shows the value of new small business loans for the U.S., Japan, U.K., and Italy from the OECD Scoreboard on Financing SMEs and Entrepreneurs. See the Appendix for more details on the data.

part to low bank profitability as measured by banks' net interest margins.[25] On the relationship between monetary policy and bank margins, Borio, Gambacorta, and Hofmann (2015) find banks' net interest margins increase with short-term interest rates, for both the U.S. and across countries (Ennis, Fessenden, and Walter 2016 nonetheless finds the evidence is more mixed).

Data Description

The left panel of Figure 1 plots the volume and accumulated values of annual small business loan originations in the U.S. using Community Reinvestment Act (CRA) disclosures from Federal Financial Institutions Examination Council (FFIEC) Call Reports. Small business loans are defined as the loan amount outstanding with original amounts of less than $1 million, not secured by nonfarm nonresidential properties, and for businesses with less than $1 million annual gross revenues. The right panel of Figure 1 plots the number of U.S. commercial banks from 2007 to 2013 using bank entry and exit data from McCord and Prescott (2014). Bank data is obtained from the Federal Reserve's National Information Center and Bank Reports on Condition and Income.

Figure 11 shows the value of new small business loans for the U.S., Japan, U.K., and Italy from the OECD Scoreboard on Financing SMEs and Entrepreneurs. SME lending in the U.S. is defined by the OECD as new commercial and industrial loans between $10,000 and $1 million and includes new loans, takedowns under revolving credit agreements, notes written under credit lines, and renewals. It also includes overnight loans and construction and land development loans that are not secured by real estate. Renewals include new loans under revolving credit agreements that roll over earlier loans, including conversions of revolving credits into term loans. These data are from the Federal Reserve Board's Survey of Terms of Business Lending.

SME lending for the U.K. is defined as new term loan facilities drawn down by SMEs (source: British Banker's Association and BIS). The SME lending rate is the median rate on SME facilities from the top four U.K. lenders (source: Bank of England). SME lending for Japan is defined as business loans to SMEs from domestically licensed banks, credit associations, and credit cooperatives (source: SME agency and White Papers on Small and Medium Enterprises in Japan). SME lending for Italy is defined as performing loans including repos with maturity up to 12 months (source: Bank of Italy, supervisory returns for bank loans). The SME lending rate is the average annual percentage rate of charge for new terms loans, which includes rates charged to non-bank customers for matched loans, term loans, and revocable loans provided that the sum of these amounts equals or exceeds 75,000 euros (source: Bank of Italy, Survey of Lending Rates). The table below summarizes variable definitions from the OECD country tables.

[25] Banks' net interest margins are defined by the FFIEC as the interest income generated by banks and the amount of interest paid out to their lenders, relative to the amount of their interest-earning assets.

	SME Lending	**SME Interest Rate**
U.S.	New loans <$1 million to SMEs	Interest rate for SME loans <$1 million
Japan	New lending to SMEs from domestic banks	Prime lending rate for short-term loans
U.K.	New term loan facilities drawn down for SMEs	Median rate on SME facilities top 4 U.K. banks
Italy	Lending to SMEs from domestic banks	Interest rate on new term loans for SMEs