

# Empirical equilibrium\*

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## Abstract

We introduce *empirical equilibrium*, a refinement of Nash equilibrium. In contrast to previous refinements in the literature, empirical equilibrium is based solely on observables and does not determine as implausible all weakly dominated behavior. We show that a distribution of play is an empirical equilibrium if and only if it is the limit of regular Quantal Response Equilibria associated with quantal responses that are utility maximizing in the limit. This result provides a direct link between the practice of experimental economics and our refinement.

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*Keywords:* equilibrium refinements; behavioral game theory; regular quantal response equilibrium; empirical equilibrium.

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# 1 Introduction

*Empirical: based on, concerned with, or verifiable by observation or experience rather than theory or pure logic.*<sup>1</sup>

We introduce *empirical equilibrium*, a normal-form refinement of Nash equilibrium. In contrast to previous refinements in the literature, empirical equilibrium is based solely on observables and does not determine as implausible all weakly dominated behavior. Our main results provide a behavioral foundation for this refinement based on two cornerstone models that account for deviations from utility maximizing behavior: regular Quantal Response Equilibrium (QRE) (McKelvey and Palfrey, 1996; Goeree et al., 2005) and control costs games (van Damme, 1991).

It is well known that the set of Nash equilibria of a game may contain elements that are implausible. That is, it is somehow evident that some Nash equilibria are not likely to be observed if the game actually takes place (see Example 1). If one is interested in the positive content of this theory, this is problematic. In applications, one often performs extreme case scenario analyses based on the set of outcomes predicted for a certain game by a solution concept. If some of these outcomes are implausible, then a worst case scenario analysis can be unnecessarily pessimistic, and a best case scenario analysis can be unrealistically optimistic.

The response of game theory and economics to this problem has been to advance equilibrium refinements, i.e., selections from the Nash equilibrium solution. With only few exceptions, which have never been used in applications nor further studied (see Sec. 2), equilibrium refinements determine as implausible each equilibrium in which an agent plays a weakly dominated action with positive probability (from Selten, 1975 and Myerson, 1978, to Kohlberg and Mertens, 1986 and the latest iterations in Milgrom and Mollner, 2017 and Fudenberg and He, 2018; see van Damme, 1991 for an earlier survey). These equilibrium refinements have been widely used in applications. They are explicitly invoked to obtain comparative statics in strategic environments (e.g., Milgrom and Mollner, 2017; Fudenberg and He, 2018). They are also explicitly used to overcome the negative results that one arrives at with a worst case scenario analysis of mechanisms based on the full set of Nash equilibria (Palfrey and Srivastava, 1991; Jackson, 1992). Finally, they implicitly support the design of strategy-proof institutions, usually advocated, when available, in market design environments (e.g., Roth, 1984; Abdulkadiroğlu and Sönmez, 2003).

In what seems to be moving in a parallel universe, experimental studies have shown that weakly dominated behavior is persistently observed in strategic situations. In particular,

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<sup>1</sup>Google Dictionary search on Nov. 15, 2017.

a robust body of evidence has emerged from experiments in games that have dominant strategy equilibria induced by strategy-proof mechanisms (Coppinger et al., 1980; Kagel et al., 1987; Kagel and Levin, 1993; Attiyeh et al., 2000; Harstad, 2000; Cason et al., 2006; Chen and Sönmez, 2006; Andreoni et al., 2007; Li, 2017). More recently, the analysis of large field data sets from high stakes games, as career choice and school choice, is corroborating the findings from laboratory experiments (Hassidim et al., 2016; Artemov et al., 2017; Rees-Jones, 2017; Chen and Pereyra, 2018).

This stark contradiction of theory and data motivates us to give a fresh look at the plausibility of Nash equilibria. Our proposal is to refine this set based solely on regularities observed in data by means of the following thought experiment. We imagine that we model the unobservables that explain observable behavior in our environment.<sup>2</sup> For instance, we construct a randomly disturbed payoff model (Harsanyi, 1973; van Damme, 1991), a control cost model (van Damme, 1991), a structural QRE model (McKelvey and Palfrey, 1995), a regular QRE model (McKelvey and Palfrey, 1996; Goeree et al., 2005), etc. In order to bring our model to accepted standards of science we need to make sure it is *falsifiable*. We observe that in the most popular models for the analysis of experimental data, including the ones just mentioned, this has been done by requiring consistency with an a priori observable restriction for which there is empirical support, *weak payoff monotonicity*.<sup>3</sup> This property of the full profile of empirical distributions of play in a game requires that for each agent, differences in behavior reveal differences in expected utility. That is, between two alternative actions for an agent, say  $a$  and  $b$ , if the agent plays  $a$  with higher frequency than  $b$ , it is because given what the other agents are doing,  $a$  has higher expected utility than  $b$ . Finally, we proceed with our study and define a refinement of Nash equilibrium by means of “approachability” by behavior in our model à la Harsanyi (1973), van Damme (1991), and McKelvey and Palfrey (1996). That is, we label as implausible the Nash equilibria of our game that are not the limit of a sequence of behavior that can be generated by our model. If our model is well-specified, the equilibria that are ruled implausible by our refinement, will never be approached by observable behavior even when distributions of play approach mutual best responses.<sup>4</sup> Of course, we are not sure what the true model is. Our thought

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<sup>2</sup>Our benchmark is an experimental environment in which the researcher observes payoffs and samples frequencies of play. This observable payoffs framework is also a valuable benchmark for the foundation of Nash equilibrium (Harsanyi, 1973).

<sup>3</sup>Harsanyi (1973) does not explicitly impose weak payoff monotonicity in his randomly perturbed payoff models. The objective of his study is to show that certain properties hold for all randomly perturbed payoff models with vanishing perturbations for generic games. This makes unnecessary to discipline the model with a priori restrictions. van Damme (1991) requires permutation invariance on Harsanyi (1973)’s models. This induces weak payoff monotonicity.

<sup>4</sup>We have in mind an unmodeled evolutionary process by which behavior approaches a Nash equilibrium. Thus, we are essentially interested in the situations in which eventually a game form is a good approximation

experiment was already fruitful, however. We learned that if we were able to construct the true model and our a priori restriction does not hinder its specification, the Nash equilibria that we would identify as implausible will necessarily contain those in the complement of the closure of weakly payoff monotone behavior. This leads us to the definition of *empirical equilibrium*, a Nash equilibrium for which there is a sequence of weakly payoff monotone distributions of play converging to it. The complement of this refinement (in the Nash equilibrium set), the *empirically implausible equilibria*, are the Nash equilibria that are determined implausible by any theory that is disciplined by weak payoff monotonicity.<sup>5</sup>

An immediate question that arises is whether weak payoff monotonicity is indeed a characterizing property of empirical distributions of behavior in games. That is, to what extent our basis for plausibility is indeed supported by data? In this paper we pursue an indirect theoretical answer to this question.<sup>6</sup> We consider an equilibrium model satisfying weak payoff monotonicity that is a staple of empirical analysis of games, regular QRE. This model has been successful in fitting data and in providing a rational for the comparative statics observed in experiments in a wide range of strategic situations (see [Goeree et al., 2016](#), for a survey). This model also allows us to articulate the idea of increasing sophistication and proximity to utility maximization. We prove the following results:

1. A distribution of play is an empirical equilibrium of a game if and only if it is the limit of regular QRE associated with quantal responses that are utility maximizing in the limit (Theorem 2).

2. A distribution of play is an empirical equilibrium of a game if and only if it is the limit of equilibria in associated control cost games for vanishing spline cost functions, a form of regular QRE suitable for maximum likelihood estimation (Theorem 3).

This characterization of empirical equilibria provides a link between the practice of experimental economics and our refinement: If data will satisfactorily fit a regular QRE model, empirical distributions of play can accumulate towards a certain distribution as agents approximate best responders only if this distribution is an empirical equilibrium.

The paper proceeds as follows. Sec. 2 places our contribution in the context of the literature. Sec. 3 presents the model. Sec. 4 presents three examples that provide intuition about the ability of empirical equilibria to discriminate among weakly dominated behavior

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of the strategic situation we model, as when perturbations vanish in [Harsanyi \(1973\)](#)'s approachability theory.

<sup>5</sup>In a paper that circulated simultaneously with the first version of our paper, [Goeree et al. \(2018\)](#) study the empirical restrictions imposed by a ranking based form of payoff monotonicity that is satisfied by each Nash equilibrium. Thus, this analysis does not produce a refinement of Nash equilibrium, which is our fundamental contribution. At a technical level, our results overlap with theirs only in that they prove a result equivalent to our Lemma 2. They do not address the approximation of Nash equilibria by means of increasingly sophisticated regular QRE, nor by means of a finite dimensional form of regular QRE.

<sup>6</sup>We advance a direct analysis of data in [Brown and Velez \(2019\)](#) and [Velez and Brown \(2019a\)](#). We discuss these findings in Sec. 6.1.

and its independence from other refinements in the literature. It is worth noting that empirical equilibrium is not an example-driven definition. Its definition has not been adapted so it resolves the plausibility of equilibria in a certain class of games in a way that conforms with a particular intuition. The defining feature of this equilibrium refinement is that it seeks to base its selection of equilibria on a testable property of behavior. Sec. 5 presents our main results. Sec. 6 discusses possible violations of weak payoff monotonicity, our implicit assumption of convergence, the implications of [Harsanyi \(1973\)](#)'s purification result in empirical equilibrium analysis, and the normative content of empirical equilibrium. The Appendix contains all proofs.

## 2 Related literature

The main difference of empirical equilibrium and the refinements previously studied in the literature is that it does not automatically discard all weakly dominated behavior, i.e., it violates the so-called admissibility ([Kohlberg and Mertens, 1986](#)). Strictly speaking there are three precedents for a refinement violating this property. [van Damme \(1991\)](#)'s firm equilibria and uniformly-vanishing control costs approachable equilibria, and [McKelvey and Palfrey \(1996\)](#)'s logistic QRE approachable equilibria, also violate admissibility.<sup>7</sup> Firm equilibria and vanishing control costs approachable equilibria were developed as basic frameworks to add restrictions and provide foundations for other equilibrium refinements that do eliminate weakly dominated behavior, e.g., [Selten \(1975\)](#)'s perfect equilibria. Their potential as stand alone refinements has never been explored. Logistic QRE approachable equilibria was mentioned as a theoretical possibility by [McKelvey and Palfrey \(1996\)](#), but was never studied or used in any application. At a technical level, firm equilibria and logistic QRE approachable equilibria can be strict subsets of empirical equilibria ([Velez and Brown, 2019c](#)). Uniformly-vanishing control cost approachable equilibria are a subset of empirical equilibria. It is an open question to determine if this inclusion can be strict as well.

Economists have seldom challenged admissibility. There are three notable exceptions. [Nachbar \(1990\)](#) and [Dekel and Scotchmer \(1992\)](#) observed that weakly dominated behavior can result from the evolution of strategies that are updated by means of simple intuitive rules. Perhaps the study that is most skeptical of admissibility is [Samuelson \(1992\)](#), who shows that it has no solid epistemic foundation in all games.

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<sup>7</sup>[McKelvey and Palfrey \(1995\)](#)'s limiting equilibrium and its generalizations by [Zhang \(2016\)](#), should not be confused with the logistic QRE approachable equilibria. The former are single valued selections from the Nash equilibrium set that are defined for generic games only. They select the unique equilibrium that is approached by a central branch of the logistic QRE correspondence or a family of QRE correspondences that satisfy certain differentiability properties.

Since we are interested in determining the plausibility of a Nash equilibrium by the existence of plausible behavior arbitrarily close to it, our work is similar in spirit to [Harsanyi \(1973\)](#), who studied the plausibility of Nash equilibria based on approximation by behavior in information-wise neighboring games; and [Rosenthal \(1989\)](#), who aimed at the analysis of a stricter form of weakly payoff monotone behavior with a particular linear form in two-by-two games.

Our main result characterizes our refinement by means of proximity of regular QRE behavior. The most common form of this model in applications is [McKelvey and Palfrey \(1995\)](#)'s structural QRE ([Goeree et al., 2016](#)). As in [Harsanyi \(1973\)](#), but with some minor technical differences, behavior in the structural QRE model is generated by utility maximization in an information neighboring game. An immediate question is whether empirical equilibria can be characterized as the limits of behavior in structural QRE and [Harsanyi \(1973\)](#)'s randomly perturbed games that satisfy weak payoff monotonicity. In a companion paper we show that this is not so ([Velez and Brown, 2019c](#)). Intuitively, empirical equilibrium is a non-parametric definition, and structural QRE and [Harsanyi \(1973\)](#)'s games are constrained by the separability of payoffs and perturbations.

Finally, our paper opens up an agenda of research in which one reevaluates the applications of game theory based on our equilibrium refinement. We have made three concrete advances that show empirical equilibrium analysis is consequential for the design of economic institutions. (i) In [Velez and Brown \(2019b\)](#) we study the plausibility of weakly dominated equilibria in strategy-proof games. The significant news is that for strategy-proof mechanisms for which dominant strategies are essentially unique, weakly payoff monotone behavior can accumulate towards a bad equilibrium only in boundary information structures. This provocative prediction of empirical equilibrium analysis is consistent with behavior in experiments run a decade ago by [Andreoni et al. \(2007\)](#), and produces an alternative line of design to the stricter "obvious strategy-proofness" program of [Li \(2017\)](#). (ii) It is well known that if a worst case scenario analysis of a mechanism is done with the whole set of Nash equilibria, there are clear limits to the social choice correspondences that can be implemented. These so-called Maskin monotonicity restrictions of the Nash implementable correspondences, disappear if one limits attention to undominated equilibria ([Palfrey and Srivastava, 1991](#); [Jackson, 1992](#)). The evidence mentioned above allows one to reject undominated equilibria as a universal plausibility standard for games, however. Thus, this result is of dubious practical use. By contrast, in [Velez and Brown \(2019a\)](#) we show, by means of a full analysis of a partnership dissolution problem, that a mechanism designer who performs a worst case scenario analysis based on empirical equilibrium is also not constrained by Maskin monotonicity. Thus, it is plausible that a mechanism designer can

achieve objectives that he or she would determine impossible based on the Nash equilibrium prediction. This also means that a mechanism designer who evaluates a mechanism based on the full Nash equilibrium prediction can be designing implicitly biased mechanisms. (iii) In [Brown and Velez \(2019\)](#) we present experimental evidence in a partnership dissolution environment that supports weak payoff monotonicity and the comparative statics predicted by empirical equilibrium in this environment. That is, the implicit bias of mechanisms that is identified by empirical equilibrium, but not by the Nash equilibrium solution, is present in data.

### 3 Model

We study the plausibility of Nash equilibria in a finite normal form game  $\Gamma \equiv (N, A, u)$  where  $N \equiv \{1, \dots, n\}$  is a set of agents;  $(A_i)_{i \in N}$  are the corresponding action spaces and  $A \equiv A_1 \times \dots \times A_n$  the set of action profiles; and  $u \equiv (u_i)_{i \in N}$  is the profile of expected utility indices, i.e., functions  $u_i : A \rightarrow \mathbb{R}$ . Our interpretation of the game is standard. Agents simultaneously choose an action. Given that action profile  $a \equiv (a_i)_{i \in N}$  is chosen, agent  $i$ 's payoff is  $u_i(a)$ .

A strategy for agent  $i$  is a probability distribution on  $A_i$ , denoted generically by  $\sigma_i \in \Delta(A_i)$ . A pure strategy places probability one on a given action. We identify pure strategies with the actions themselves. A strategy is interior if it places positive probability on each possible action. A profile of strategies is denoted by  $\sigma \equiv (\sigma_i)_{i \in N} \in \Delta \equiv \Delta(A_1) \times \dots \times \Delta(A_n)$ . Given  $S \subseteq N$ , we denote a subprofile of strategies for these agents by  $\sigma_S$ . When  $S = N \setminus \{i\}$ , we simply write  $\sigma_{-i} \in \Delta_{-i} \equiv \times_{j \in N \setminus \{i\}} \Delta(A_j)$ . Consistently, we concatenate partial strategy profiles as in  $(\sigma_{-i}, \mu_i)$ . We consistently use this convention when operating with vectors, as with action profiles.

Agent  $i$ 's expected utility given strategy profile  $\sigma$  is

$$U_i(\sigma) = \sum_{a \in A} u_i(a) \sigma(a),$$

where  $\sigma(a) = \sigma_1(a_1) \dots \sigma_n(a_n)$ . Following our convention of identifying pure strategies with actions, we write  $U_i(\sigma_{-i}, a_i)$  for the utility that agent  $i$  gets from playing actions  $a_i$  when the other play  $\sigma_{-i}$ . We say that an action  $a_i \in A_i$  is weakly dominated by action  $\hat{a}_i \in A_i$  if for each  $a_{-i} \in A_{-i}$ ,  $u_i(a_{-i}, \hat{a}_i) \geq u_i(a_{-i}, a_i)$  with strict inequality for at least an element of  $A_{-i}$ . We say that  $a_i \in A_i$  is a weakly dominated action if there is another action that weakly dominates it.

The following are the basic prediction for game  $\Gamma$  and three of its most prominent

refinements.

1. (Nash, 1951) A *Nash equilibrium* of  $\Gamma$  is a profile of strategies  $\sigma$ , such that for each  $i \in N$  and each  $\mu_i \in \Delta(A_i)$ ,  $U_i(\sigma) \geq U_i(\sigma_{-i}, \mu_i)$ . We denote this set by  $N(\Gamma)$ .
2. An *undominated Nash equilibrium* of  $\Gamma$  is a Nash equilibrium of  $\Gamma$  in which no agent plays with positive probability a weakly dominated action. We denote this set by  $U(\Gamma)$ .
3. (Selten, 1975) A *perfect equilibrium* of  $\Gamma$  is a profile of strategies  $\sigma$  that is the limit of a sequence of interior strategy profiles  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  such that for each  $\lambda \in \mathbb{N}$  and each  $i \in N$ ,  $\sigma_i^\lambda$  places probability greater than  $1/\lambda$  on an a given action only if it is a best response to  $\sigma_{-i}^\lambda$ . We denote this set by  $T(\Gamma)$ .<sup>8</sup>
4. (Myerson, 1978) A *proper equilibrium* of  $\Gamma$  is a profile of strategies  $\sigma$  that is the limit of a sequence of interior strategy profiles  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  such that for each  $\lambda \in \mathbb{N}$ , each  $i \in N$ , and each pair of actions  $\{a_i, \hat{a}_i\} \subseteq A_i$ , if  $U_i(\sigma_{-i}^\lambda, a_i) > U_i(\sigma_{-i}^\lambda, \hat{a}_i)$ , then  $\sigma_i^\lambda(\hat{a}_i) \leq (1/\lambda)\sigma_i^\lambda(a_i)$ . We denote this set by  $P(\Gamma)$ .

We propose an alternative refinement of Nash equilibrium. It is based on the hypothesis that observed behavior in a game, even though noisy, satisfies the following property. This hypothesis is testable with finite data sets.

**Definition 1.** A profile of strategies  $\sigma \equiv (\sigma_i)_{i \in N}$  is *weakly payoff monotone* for  $\Gamma$  if for each  $i \in N$  and each pair of actions  $\{a_i, \hat{a}_i\} \subseteq A_i$  such that  $\sigma_i(a_i) > \sigma_i(\hat{a}_i)$ , we have that  $U_i(\sigma_{-i}, a_i) > U_i(\sigma_{-i}, \hat{a}_i)$ .

Intuitively, a profile of strategies is weakly payoff monotone for a game if differences in behavior reveal differences in expected payoffs.

As any theoretical benchmark one should not expect that Nash equilibrium will be observed in a strict sense in the strategic situations that a game represents. At best, if one samples data from agents' behavior, this data will not allow us to reject the hypothesis that behavior conforms with one of these predictions. If we have available a characterization of the behavior that is usually observed, it follows that the Nash equilibria that are separated from observable behavior will always be rejected. Thus, even though we cannot know for sure whether there will be a Nash equilibrium that will not be rejected, we confidently know that some of them will. The following definition articulates this idea when observable behavior is weakly payoff monotone.

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<sup>8</sup>Our definition of perfect equilibrium corresponds to Myerson (1978)'s characterization of Selten (1975)'s perfect equilibrium.

		Player 2	
		$b_1$	$b_2$
Player 1	$a_1$	1, 1	0, 0
	$a_2$	0, 0	0, 0

		Player 2	
		$b_1$	$b_2$
Player 1	$a_1$	2, 2	2, 1
	$a_2$	2, 3	0, 0

(a) Game  $\Gamma_1$                       (b) Game  $\Psi$

**Table 1:** (a) a game in which the set of empirical equilibria is a proper subset of the set of Nash equilibria; (b) a game in which there are empirical equilibria in which player 1 chooses a weakly dominated strategy with positive probability.

**Definition 2.** An *empirical equilibrium* of  $\Gamma$  is a Nash equilibrium of  $\Gamma$  that is the limit of a sequence of weakly payoff monotone strategies for  $\Gamma$ . We denote this set by  $E(\Gamma)$ .

The following property of a strategy profile, which implies weak payoff monotonicity, will allow us to provide an alternative characterization of empirical equilibrium.

**Definition 3.** A profile of strategies  $\sigma \equiv (\sigma_i)_{i \in N}$  is *payoff monotone* for  $\Gamma$  if for each  $i \in N$  and each pair of actions  $\{a_i, \hat{a}_i\} \subseteq A_i$ ,  $\sigma_i(a_i) \geq \sigma_i(\hat{a}_i)$  if and only if  $U_i(\sigma_{-i}, a_i) \geq U_i(\sigma_{-i}, \hat{a}_i)$ .

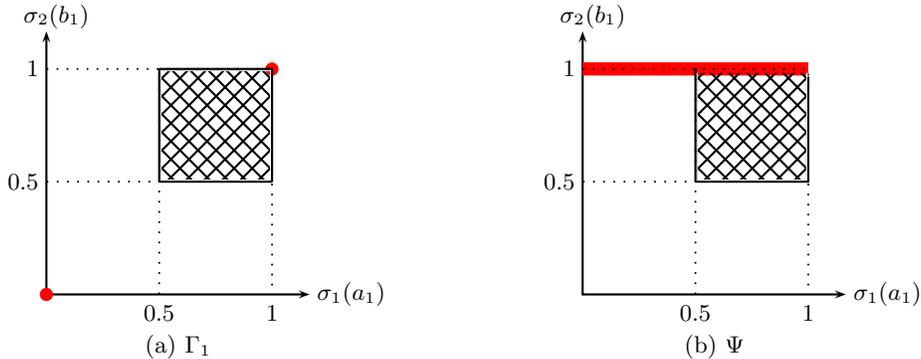
## 4 Examples

It is well known that perfect equilibria and proper equilibria are Nash equilibria. One can easily see that proper equilibria are empirical equilibria. Since for each finite game this equilibrium refinement is non-empty (Myerson, 1978), we conclude that empirical equilibria exist.

**Lemma 1.** For each  $\Gamma$ ,  $E(\Gamma) \neq \emptyset$ .

The following examples illustrate the differences of empirical equilibria with undominated equilibria, trembling hand perfect equilibria, and proper equilibria. Our first example allows us to easily see that empirical equilibrium may be a proper refinement of Nash equilibrium.

**Example 1.** Consider game  $\Gamma_1$  in Table 1 (a). This game was proposed by Myerson (1978) to illustrate that some Nash equilibria are intuitively implausible. There are two Nash equilibria in  $\Gamma_1$ ,  $(a_1, b_1)$  and  $(a_2, b_2)$ . Only  $(a_1, b_1)$  is an empirical equilibrium in this game. Indeed, for each distribution of actions of player 2, player 1’s utility from playing  $a_1$  is greater than or equal to the utility from playing  $a_2$ ; thus, in a profile of weakly payoff monotone distributions of play, agent 1 will always play  $a_1$  with probability at least 1/2 (Fig. 1 (a)); thus,  $(a_2, b_2)$  cannot be approximated by weakly payoff monotone behavior.



**Figure 1:** (a) Weakly payoff monotone distributions (shaded area) and Nash equilibria of  $\Gamma_1$ ;  $\sigma_1(a_1)$  is the probability with which agent 1 plays  $a_1$ . Equilibrium  $(a_1, b_1)$  can be approximated by weakly payoff monotone behavior. Thus, it is an empirical equilibrium of  $\Gamma_1$ . Equilibrium  $(a_2, b_2)$  cannot be approximated by weakly payoff monotone behavior. Thus, it is not an empirical equilibrium of  $\Gamma_1$ . (b) Weakly payoff monotone distributions and Nash equilibria of  $\Psi$ ; each Nash equilibrium in which agent 1 plays  $a_1$  with probability at least  $1/2$  is an empirical equilibrium.

If this game is played and agents behavior is weakly payoff monotone and approximates a Nash equilibrium, it is necessarily  $(a_1, b_1)$ .  $\square$

Each refinement that rules out weakly dominated behavior coincides with empirical equilibrium in game  $\Gamma_1$ . Undominated equilibria and empirical equilibria are independent, however.

**Example 2.** Consider game  $\Psi$  in Table 1 (b). Player 2 has a strictly dominant strategy in this game. Thus, in each Nash equilibrium  $\sigma$  of  $\Psi$ ,  $\sigma_2(b_1) = 1$ . Agent 1 is indifferent between both actions if agent 2 plays  $b_1$ . Thus, the set of Nash equilibria of this game is the distributions in which agent 1 randomizes between both actions and agent 2 plays  $b_1$ . Now, let  $\sigma$  be a weakly payoff monotone distribution for  $\Psi$ . Since  $b_1$  strictly dominates  $b_2$ ,  $\sigma_2(b_1) \geq \sigma_2(b_2)$ . If  $\sigma_2(b_2) > 0$ ,  $U_1(\sigma_2, a_1) > U_1(\sigma_2, a_2)$ . Thus, it must be the case that  $\sigma_1(a_1) \geq \sigma_1(a_2)$ . If  $\sigma_2(b_2) = 0$ ,  $U_1(\sigma_2, a_1) = U_1(\sigma_2, a_2)$ . Thus,  $\sigma_1(a_1) = \sigma_1(a_2)$ . Thus, the set of weakly payoff distributions for  $\Psi$  are those at which  $\sigma_1(a_1) \geq 1/2$  and  $\sigma_2(b_1) \geq 1/2$ , except those at which  $\sigma_2(b_1) = 1$  and  $\sigma_1(a_1) < 1/2$  (Fig. 1 (b)). The set of empirical equilibria of  $\Psi$  are the Nash equilibria in which agent 1 plays  $a_1$  with probability at least  $1/2$ . Since  $a_2$  is weakly dominated by  $a_1$  for player 1, almost all of these empirical equilibria involve one player playing a weakly dominated action with positive probability.  $\square$

Empirical equilibrium does a subtle selection from the Nash equilibrium set. It determines the plausibility of a strategy based on its relative merits with respect to the alternative actions that the agent may choose. The following example drives this point home. It illustrates it for a parametric family of games. This family is a generalization of a game proposed

		Player 2		
		$b_1$	$b_2$	$b_3$
Player 1	$a_1$	1, 1	0, 0	$-7 - c_1, -7 - c_2$
	$a_2$	0, 0	0, 0	$-7, -7$
	$a_3$	$-7 - c_1, -7 - c_2$	$-7, -7$	$-7, -7$

**Table 2:** Game  $\Gamma_2^c$  where  $c \equiv (c_1, c_2)$ ,  $c_1 > 0$ , and  $c_2 > 0$ .

by Myerson (1978) to show that it is possible to introduce weakly dominated actions in  $\Gamma_1$ , and considerably change its set of trembling hand perfect equilibria.

**Example 3.** Consider game  $\Gamma_2^c$  for some  $c \equiv (c_1, c_2)$ ,  $c_1 > 0$ , and  $c_2 > 0$  (Table 2). Standard arguments show that for each  $c > 0$ ,

$$\begin{aligned} N(\Gamma_2^c) &= \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}, \\ T(\Gamma_2^c) &= U(\Gamma_2^c) = \{(a_1, b_1), (a_2, b_2)\}, \\ P(\Gamma_2^c) &= \{(a_1, b_1)\}. \end{aligned}$$

In contrast to these refinements, the empirical equilibrium set of  $\Gamma_2^c$  depends on  $c$ . First, note that for no  $c > 0$ ,  $(a_3, b_3)$  is an empirical equilibrium of  $\Gamma_2^c$ . This is so because  $a_2$  weakly dominates  $a_3$  for player 1. Thus, in any weakly payoff monotone distribution for  $\Gamma_2^c$ , player 1 plays  $a_2$  with a probability that is at least the probability with which she plays action  $a_3$ . Thus, no sequence of weakly payoff monotone distributions for  $\Gamma_2^c$  converges to  $(a_3, b_3)$ . On the other hand for each  $c > 0$ ,  $(a_1, b_1) \in P(\Gamma_2^c) \subseteq E(\Gamma_2^c)$ .

Let us now examine the plausibility of  $(a_2, b_2)$  in  $\Gamma_2^c$ . Think of the payoffs in the game as dollar amounts. Consider first a small  $c$ , say  $c_1 \approx c_2 \approx 0.01$ . Let  $\sigma$  be an empirical distribution of play that approximates  $(a_2, b_2)$ . In such a situation,  $U_1(\sigma_2, a_1) \approx 0 > U_1(\sigma_2, a_3) \approx -7$  and  $U_2(\sigma_2, b_1) \approx 0 > U_2(\sigma_2, b_3) \approx -7$ . Thus, if expected utility guides the choices of the players, one can expect that player 1 will play  $a_1$  at least as often as  $a_3$ , and player 2 will play  $b_1$  at least as often as  $b_3$ . If this is so, action  $a_1$  will have a greater utility than action  $a_2$  for player 1, and action  $b_1$  will have a greater utility than action  $b_2$  for player 2. Thus, if expected utility guides the choices of the agents,  $\sigma$  will not be close to  $(a_2, b_2)$ . Thus, a plausible empirical distribution, i.e., one that is informed by expected utility for this game, will never be close to  $(a_2, b_2)$ .

Now, consider a large  $c$ , say  $c_1 \approx c_2 \approx 100,000$ . Again, if  $\sigma$  is an empirical distribution of play that approximates  $(a_2, b_2)$  and is guided by expected utility, player 1 will be playing  $a_1$  at least as often as  $a_3$ , and player 2 will be playing  $b_1$  at least as often as  $b_3$ . In contrast with our previous case, it does not follow that necessarily action  $a_1$  will have a greater utility than action  $a_2$  for player 1, and action  $b_1$  will have a greater utility than action  $b_2$

for player 2. This will only happen if player 1 is playing  $a_1$  at least one hundred thousand times as often as  $a_3$ , and player 2 is playing  $b_1$  at least one hundred thousand times as often as  $b_3$ . Thus, it is possible that  $\sigma$  is informed by expected utility, i.e.,  $\sigma_1(a_1) > \sigma_1(a_3)$  and  $\sigma_2(b_1) > \sigma_2(b_3)$ , and at the same time  $U_1(\sigma_2, a_2) > U_1(\sigma_2, a_1)$ ,  $U_2(\sigma_1, b_2) > U_2(\sigma_1, b_1)$ ,  $\sigma_1(a_2) \approx 1$ , and  $\sigma_2(b_2) \approx 1$ . Essentially, since the possible loss for player 1 from playing  $a_1$  is about 100,000.00, player 1 can be scared away from playing  $a_1$  if player 2 is playing  $b_3$  more than once each 100,000 times she plays  $b_1$ . This is still compatible with  $b_3$  being the worst alternative given what the other agent is doing.

These arguments can be easily formalized to show that

$$E(\Gamma_c) = \begin{cases} \{(a_1, b_1)\} & \text{if } \min\{c_1, c_2\} \leq 1, \\ \{(a_1, b_1), (a_2, b_2)\} & \text{Otherwise.} \end{cases}$$

One cannot expect that if one brings these games to a laboratory setting or has the opportunity to collect field data on them, the threshold  $\min\{c_1, c_2\} = 1$  will be a good predictor of a structural change in the behavior of the agents. However, it is reasonable that behavior in this game will depend on the size of  $c$ , as empirical equilibrium predicts, i.e., equilibrium  $(a_2, b_2)$  will be relevant only for high values of  $c$ . Undominated equilibria, perfect equilibria, and proper equilibria all miss this point. Undominated equilibrium and perfect equilibrium miss that when  $c$  is too low, actions  $a_2$  and  $b_2$  are de facto “weakly dominated” when they are played with almost certainty. That is, if they were going to be played with probability close to one, actions  $a_1$  and  $b_1$ , would be preferred for the respective players. Thus, we can rule this equilibrium out by means of the following observation. It is not reasonable that we will observe a distribution of play in which an agent is not playing her unique maximizer of utility with high probability, say more than random play.

Finally, proper equilibrium dismisses  $(a_2, b_2)$  independently of  $c$ . Think of our example with high  $c$ . For  $(a_2, b_2)$  to be a proper equilibrium of  $\Gamma_c^\epsilon$ , for large  $\lambda$  there must be a distribution of play  $\sigma^\lambda$  satisfying two conditions: (i)  $\sigma^\lambda$  is close to  $(a_2, b_2)$ , and thus  $U_1(\sigma_2^\lambda, a_1) \approx 0 > U_1(\sigma_2^\lambda, a_3) \approx -7$  and  $U_2(\sigma_1^\lambda, b_1) \approx 0 > U_2(\sigma_1^\lambda, b_3) \approx -7$ ; and (ii)  $\sigma_1^\lambda(a_1) > \lambda\sigma_1^\lambda(a_3)$  and  $\sigma_2^\lambda(b_1) > \lambda\sigma_2^\lambda(b_3)$ . For distributions where  $\lambda \geq 100,000$ ,  $a_2$  and  $b_2$  are not maximizing choices for players 1 and 2, respectively, meaning  $(a_2, b_2)$  cannot be a proper equilibrium. Thus, the reason why proper equilibrium dismisses  $(a_2, b_2)$  for high  $c$  is that it uses the same parameter for proximity to  $(a_2, b_2)$  and for the agents’ reactivity to differences in expected utility. This allows us to draw a stark difference of this refinement and empirical equilibrium. Proper equilibrium is a decision-theoretical, thought experiment in which a utility maximizing agent considers the possibility that another utility maximizing agent makes a mistake. Confronted with this thought, a utility maximizing agent will

determine a Nash equilibrium as implausible because it is impossible that agents who are infinitely reactive to expected utility make self-sustaining mistakes arbitrarily close to the equilibrium. By contrast, empirical equilibrium is an exercise performed by an observer based on weak payoff monotonicity, a testable property of behavior. The observer knows that if this property is satisfied by empirical frequencies, only empirical equilibria can be approximated by data.  $\square$

## 5 Behavioral foundations of empirical equilibrium

### 5.1 Payoff monotonicity

We start with a useful alternative characterization of empirical equilibria that simplifies the computation of this set in applications and is essential for the subsequent analysis: The set of empirical equilibria of a game can be equivalently defined by proximity of interior payoff monotone behavior.

**Theorem 1.**  $\sigma \in E(\Gamma)$  if and only if  $\sigma \in N(\Gamma)$  and there is a convergent sequence of interior payoff monotone distributions for  $\Gamma$ ,  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$ , whose limit is  $\sigma$ .

The characterization of the set of empirical equilibria by means of approximation of interior payoff monotone distributions simplifies its computation in a given game. Interior payoff monotone distributions are a strict subset of the weakly payoff monotone distributions. Their analysis is usually simpler because they do not include corner cases that can be involved and time consuming. In sufficiently general games the gain may be significant (e.g. [Velez and Brown, 2019a](#)).

Theorem 1 indicates a form of stability of empirical equilibria. Think for instance of an equilibrium in a game that is itself a non-interior weakly payoff monotone distribution, e.g., a Nash equilibrium in which each agent plays her unique best response.<sup>9</sup> One will always conclude that the equilibrium is an empirical equilibrium by taking the respective constant sequence. Theorem 1 implies that this is not the only sequence that will sustain the argument. One will always be able to find a sequence of interior payoff monotone distribution that converges to the equilibrium.

### 5.2 Regular QRE

Agents' behavior in economic experiments often deviates from Nash equilibrium. Behavior in these experiments is not random, however. A popular model for the empirical analysis of

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<sup>9</sup>These equilibria are usually referred to as *strict* (c.f., [Harsanyi, 1973](#)).

data from experiments is the *regular Quantal Response Equilibrium* model (McKelvey and Palfrey, 1996; Goeree et al., 2005). This model replaces the assumption that agents best respond to the actions of the other players, with a reduced-form noisy best response that satisfies some plausibility axioms but is otherwise unrestricted. There is extensive empirical evidence suggesting that regular QREs fit agents' behavior in experimental games (Goeree et al., 2018).<sup>10</sup>

A *quantal response function* (QRF) for agent  $i$  in game  $\Gamma$  is a function  $p_i : \mathbb{R}^{A_i} \rightarrow \Delta(A_i)$ . For each  $a_i \in A_i$  and each  $x \in \mathbb{R}^{A_i}$ ,  $p_{ia_i}(x)$  denotes the value assigned to  $a_i$  by  $p_i(x)$ . A QRF  $p_i$  is *regular* if it satisfies the following four properties (Goeree et al., 2005):

- *Interiority*:  $p_i > 0$ .
- *Continuity*:  $p_i$  is a continuous function.
- *Responsiveness*: for  $x \in \mathbb{R}^{A_i}$ ,  $\eta > 0$ , and  $a_i \in A_i$ ,  $p_{ia_i}(x + \eta 1_{a_i}) > p_{ia_i}(x)$ .<sup>11</sup>
- *Monotonicity*: for  $x \in \mathbb{R}^{A_i}$  and  $\{a_i, \hat{a}_i\} \subseteq A_i$  such that  $x_{a_i} > x_{\hat{a}_i}$ ,  $p_{ia_i}(x) > p_{i\hat{a}_i}(x)$ .

A *quantal response equilibrium* (QRE) of  $\Gamma$  with respect to a profile of QRFs,  $p \equiv (p_i)_{i \in N}$ , is a fixed point of the composition of  $p$  and the expected payoff operator in  $\Gamma$  (McKelvey and Palfrey, 1995), i.e., a profile of distributions  $\sigma \equiv (\sigma_i)_{i \in N}$  such that for each  $i \in N$ ,  $\sigma_i = p_i(U_i(\sigma_{-i}, a_i)_{a_i \in A_i})$ . We refer to a QRE for a profile of regular QRFs,  $p$ , as a *regular QRE* of  $\Gamma$  with respect to  $p$ .

Because regular QRFs are interior, monotone, and continuous, each regular QRE is an interior payoff monotone distribution for  $\Gamma$ . The converse also holds. That is, the regular QRE model can be seen as a basis for interior payoff monotone behavior.

**Lemma 2.** Let  $\sigma$  be an interior payoff monotone distribution for  $\Gamma$ . Then there is a regular QRF,  $p$ , for which  $\sigma$  is a QRE for  $\Gamma$  with respect to  $p$ .

Even though QRFs are not observable, they allow us to articulate the idea of agents' sophistication. If agents behavior is close to a Nash equilibrium we can only say that these agents' choices are close to best responses, not that these agents are almost best responders. Thus, it is interesting to identify the conditions under which we can also say that a sequence of behavior that converges to a Nash equilibrium can be associated with agents who in the limit are utility maximizers.

**Definition 4.** A sequence of regular QRFs for a game  $\Gamma$ ,  $\{p^\lambda\}_{\lambda \in \mathbb{N}}$  is *utility maximizing in the limit* if for each convergent sequence of QREs of  $\Gamma$  with respect to the corresponding QRFs, its limit is a Nash equilibrium of  $\Gamma$ .

<sup>10</sup>It may be necessary to allow for heterogeneity in quantal response functions across agents and for deviations from the common prior information structures (see Goeree et al., 2018, for a survey).

<sup>11</sup> $1_{a_i}$  denotes the vector in  $\mathbb{R}^{A_i}$  that has 1 in component  $a_i$  and 0 otherwise.

We are ready to define a refinement of Nash equilibrium in the same spirit as empirical equilibrium, but taking as basis for plausibility of behavior regular QRE for increasingly sophisticated regular QRFs.

**Definition 5.**  $\sigma \in \Delta$  is *approachable by regular QRE that are utility maximizing in the limit* if there is a sequence of regular QRF profiles for  $\Gamma$ ,  $\{p^\lambda\}_{\lambda \in \mathbb{N}}$ , which is utility maximizing in the limit, and a convergent sequence of QREs for  $\Gamma$  with respect to the corresponding QRFs,  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$ , whose limit is  $\sigma$ . We denote this set by  $R(\Gamma)$ .

Clearly, for each  $\Gamma$ ,  $R(\Gamma) \subseteq N(\Gamma)$ . The interpretation of this refinement is similar to that of empirical equilibrium. That is, if one expects a regular QRE model will satisfactorily fit data, allowing for all its parametric incarnations, then the only Nash equilibria that are plausible are those approachable by regular QRE that are utility maximizing in the limit.

The regular QRE model is an attractive basis for plausibility of equilibria in normal form games. However, if we insist on basing our refinement solely on properties falsifiable with finite data, QRFs and their properties do not pass the bar. These functions are not observable.

Fortunately, we do not have to choose between these notions of plausibility of behavior in games.

**Theorem 2.** For each  $\Gamma$ ,  $E(\Gamma) = R(\Gamma)$ .

Theorem 2 allows us to alternatively describe empirical equilibrium in a way that speaks closely to the practice in experimental economics. If one expects that data will fit a regular QRE model, the only Nash equilibria that can be approximated by data are the empirical equilibria.

### 5.3 Control costs

We learn from Theorem 2 that if behavior is weakly payoff monotone and approaches a Nash equilibrium, there is a regular QRE model that fits this behavior. The family of regular QRFs is infinitely dimensional. Thus, this theorem does not point exactly to a parametric family of models that is well specified for the analysis of experimental data.

A first candidate that comes to mind to solve this issue is the structural form of QRE. This model starts from an additive perturbation of payoffs satisfying some regularity assumptions that guarantee behavior is uniquely identified and induces a quantal response function. If one restricts perturbations to be independent, or more generally to satisfy certain symmetry properties, the induced QRF of a structural QRE model is regular. It turns out that the structural QRE model is tied to the separability of payoffs and perturbations:

For each non-trivial action space in which at least an agent has at least three actions, there are games in which some weakly monotone behavior that accumulates towards a Nash equilibrium, cannot be generated by any structural monotone QRE (Velez and Brown, 2019c).

Thus, the structural monotone QRE model, including all its parametric incarnations, e.g., the logistic form, is not flexible enough to account for some payoff monotone behavior in finite games. This brings the need to find a parametric incarnation of the regular QRE model that spans all possible payoff monotonic behavior, and thus is suitable for maximum likelihood estimation and empirical analysis. In this section we develop such a parametric model. As a byproduct we provide a constructive proof of our results in Sec. 5.2.

An alternative approach to rationalize deviations from utility maximizing behavior in normal form games is van Damme (1991)'s control costs model. Here agents are parameterized by a function that determines how difficult for the agent is to make no mistake when playing a given strategy.

A *control cost function* for player  $i$  is  $f_i : [0, 1] \rightarrow \mathbb{R} \cup \{\infty\}$  with the following properties.

- $f_i$  is strictly decreasing with  $f_i(0) = \infty$  and  $f_i(1) = 0$ .
- $f_i$  is continuously differentiable on  $(0, 1]$ .<sup>12</sup>
- $f_i$  is a strictly convex function.

Given a normal form game  $\Gamma \equiv (N, A, u)$  and a profile of control cost functions  $f \equiv (f_i)_{i \in N}$  the associated game with control costs is that in which players are  $N$ , agent  $i$ 's action space is  $\Delta(A_i)$ , and payoff of action profile  $\sigma \in \Delta$  is for each  $i \in N$ ,  $U_i(\sigma) - \sum_{a_i \in A_i} f_i(\sigma_i(a_i))$ . For each  $\sigma_{-i} \in \Delta_{-i}$ , there is a unique best response for agent  $i$  in each control cost game, which solely depends on the profile of expected utility of the different actions in  $\Gamma$  (Lemma 4.2.1 van Damme, 1991). Let  $p_i^f$  be this function. One can easily see that this function is a regular QRF (van Damme, 1991; Goeree et al., 2018).

The control cost model is less general than the regular QRE model to the extent that control cost functions impose some restrictions of behavior across different extended games, i.e., if one varies  $u$ . We now show that for a fixed  $u$ , they are behaviorally equivalent.

Let  $\sigma$  be a Nash equilibrium of the control costs game associated with  $\Gamma$  and  $f$ . By interiority of each  $p_i^f$  and continuous differentiability and strict convexity of each  $f_i$ ,  $\sigma$  can be characterized by the first order conditions (Lemma 4.2.3 van Damme, 1991): for each

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<sup>12</sup>van Damme (1991) assumes that each  $f_i$  is twice differentiable. One can easily see that Lemmas 4.2.1-4.2.5 and Theorem 4.2.6. in van Damme (1991) go through with our weaker assumption. One needs continuous differentiability in order to apply Lagrange's theorem in Lemma 4.2.3. All other results follow from convexity and continuity. The greater generality of our model allows us to easily construct control cost functions hitting some specific targets of its derivative without matching the second derivative.

$i \in N$  and  $\{a_l, a_k\} \subseteq A_i$ ,

$$U_i(\sigma_{-i}, a_l) - U_i(\sigma_{-i}, a_k) = f'_i(\sigma_i(a_l)) - f'_i(\sigma_i(a_k)).$$

Thus, we can span the whole spectrum of interior payoff monotone behavior with a family of control costs functions whose derivative can be chosen for each  $i \in N$  at  $|A_i| - 1$  given points in  $(0, 1)$ . Indeed, an asymptote at zero stitched to a second order spline is flexible enough to interpolate any increasing slope and at the same time guarantee strict convexity and continuous differentiability.

Formally, let  $\sigma$  be interior and payoff monotone for  $\Gamma$ . Suppose for simplicity that  $A_i \equiv \{a_1, \dots, a_K\}$  and  $\sigma_i(a_1) \leq \dots \leq \sigma_i(a_K)$ . Let  $m_{l+1} < 0$  and  $m_l \equiv m_{l+1} - (U_i(\sigma_{-i}, a_{l+1}) - U_i(\sigma_{-i}, a_l))$ . Since  $\sigma$  is payoff monotone for  $\Gamma$ ,  $m_l \leq m_{l+1}$ . The following function is strictly convex, strictly decreasing, and interpolates slopes  $m_l$  and  $m_{l+1}$  at the respective extremes of the interval in which it is defined: for each  $y \in [\sigma_i(a_l), \sigma_i(a_{l+1})]$ ,

$$f(y) \equiv f(\sigma_i(a_{l+1})) + m_{l+1}(y - \sigma_i(a_{l+1})) + \frac{m_{l+1} - m_l}{2(\sigma_i(a_{l+1}) - \sigma_i(a_l))}(y - \sigma_i(a_{l+1}))^2. \quad (1)$$

Let  $\varepsilon > 0$ . Define  $f_i$  on  $[\sigma_i(a_K), 1]$  as  $y \mapsto (\varepsilon/[2(1 - \sigma_i(a_K))])(y - 1)^2$ .<sup>13</sup> Then, stitch the second degree polynomials on the subsequent intervals  $[\sigma_i(a_l), \sigma_i(a_{l+1})]$  for  $l = K - 1, K - 2, \dots, 1$ , i.e., define  $f_i$  as in (1) in the respective intervals. Finally, stitch a strictly decreasing and strictly convex asymptote defined on  $(0, \sigma_i(a_1)]$ . With a view towards identifying control costs functions with further properties, one can add a calibration point  $y_i^* \in (0, 1)$  such that  $y_i^* \notin \{\sigma_i(a_1), \dots, \sigma_i(a_K)\}$  and guarantee that  $f_i(y_i^*) < f_i(\sigma_i(a_l)) + \varepsilon$  where  $\sigma_i(a_l)$  is the minimum in  $\{\sigma_i(a_1), \dots, \sigma_i(a_K)\}$  that is greater than  $y_i^*$ . Let us refer to  $f \equiv (f_i)_{i \in N}$  so constructed as a *spline calibrated by  $\sigma$ ,  $y^* \equiv (y_i^*)$ , and  $\varepsilon$* .

**Lemma 3.** Let  $\sigma$  be an interior payoff monotone distribution for  $\Gamma$ ;  $y^* \in (0, 1)^N$  such that for each  $i \in N$ ,  $y_i^* \notin \sigma_i(A_i)$ ; and  $\varepsilon > 0$ . Then  $\sigma$  is a Nash equilibrium of the control costs game associated with  $\Gamma$  and each spline calibrated by  $\sigma$ ,  $y^*$ , and  $\varepsilon$ .

The control costs model also allows us to identify sequences of regular QRFs that are utility maximizing in the limit. We say that a sequence of profiles of control costs functions,  $\{f^\lambda\}_{\lambda \in \mathbb{N}}$ , *vanishes*, if for each  $i \in N$  and each  $x \in (0, 1]$ ,  $\lim_{\lambda \rightarrow \infty} f_i^\lambda(x) = 0$ . It is well known that the behavior in games with vanishing control costs can converge only to Nash equilibria of the underlying game (van Damme, 1991, Theorem 4.3.1).<sup>14</sup> Thus, for

<sup>13</sup>This is simply a strictly decreasing, strictly convex, function such that  $f_i(1) = 0$ , which parameterizes our construction.

<sup>14</sup>Technically, Theorem 4.3.1 in van Damme (1991) applies only to vanishing sequences of control cost

a sequence of vanishing control cost functions its associated sequence of regular QRFs are utility maximizing in the limit.

Empirical equilibria can be alternatively characterized as the limits of behavior in games with vanishing control costs.

**Theorem 3.** Let  $\sigma \in \mathbb{N}(\Gamma)$  and  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  a convergent sequence of interior payoff monotone distributions for  $\Gamma$  whose limit is  $\sigma$ . Then, there is an increasing  $\kappa : \mathbb{N} \rightarrow \mathbb{N}$  and  $\{y^{\kappa(\lambda)}\}_{\lambda \in \mathbb{N}}$ , where for each  $\lambda \in \mathbb{N}$ ,  $y^{\kappa(\lambda)} \in (0, 1)^N$ , such that:

1. As  $\lambda \rightarrow \infty$  the sequence of splines calibrated by  $\sigma^{\kappa(\lambda)}$ ,  $y^{\kappa(\lambda)}$ , and  $1/\kappa(\lambda)$  vanishes.
2. For each  $\lambda \in \mathbb{N}$ ,  $\sigma^{\kappa(\lambda)}$  is a Nash equilibrium of the control costs game associated with  $\Gamma$  and the spline calibrated by  $\sigma^{\kappa(\lambda)}$ ,  $y^{\kappa(\lambda)}$ , and  $1/\kappa(\lambda)$ .

## 6 Discussion

### 6.1 Beyond weak payoff monotonicity

The equivalence of  $E(\Gamma)$  and  $R(\Gamma)$  provides a formal link between our basis to determine plausibility of behavior in a game and the standard practice in experimental economics. It is common to find experimental studies in which deviations from Nash behavior are rationalized by the logistic QRE or other parametric forms of regular QRE (see [Goeree et al., 2002](#), for a survey). Essentially, experimental economists have found that these parametric models capture part of the intuition and comparative statics observed in data. Great emphasis is usually placed on the extent to which the parametric models fit data. However, there is usually lack of formal statistical test of payoff monotonicity and the other building blocks of these models, an issue that has been pointed out by [Haile et al. \(2008\)](#).

Thus, even though a superficial reading of experimental literature suggests that the accumulated experimental evidence supports that payoff monotonicity is a plausible characterizing invariant of empirical distributions in strategic situations, the reality is that one cannot formally conclude this from the ubiquitous statements of fitness.

Testing for weak payoff monotonicity is data demanding. It is difficult to come across data sets that are powerful enough to reject this property. Essentially, it is easy to find data sets in which one can find evidence that the probability with which two actions available to an agent are played are statistically different. Weak payoff monotonicity requires that expected utilities should follow the same order. In order to have good estimates of these

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functions of the form  $\varepsilon f$  with  $\varepsilon \rightarrow 0$ . One can easily see that his argument extends for a general sequence of vanishing control cost functions as we define it because our assumption also implies that for each  $x \in (0, 1]$ ,  $(f_i^\lambda)'(x) \rightarrow 0$ . We have included an explicit proof of this result in an online Appendix.

expected utilities one needs good estimates of the whole distribution of actions. This requires large data sets for games with realistic action spaces.

Given these data limitations, it is sensible to test for less data demanding implications of weak payoff monotonicity. For instance, one can verify whether there is a positive correlation of the probability with which an agent plays an action and an unbiased estimator of its expected utility. We have done this exercise with experimental data from fair allocation experiments (Brown and Velez, 2019) and dominant strategy mechanisms (Velez and Brown, 2019b). In both cases there is strong support for the positive association of the probability with which agents choose their actions and their expected utility.

An alternative window to test for weak payoff monotonicity with limited data is offered by games in which actions are related by weak dominance. Suppose that in game  $\Gamma$ ,  $a_l$  weakly dominates action  $a_k$  for agent  $i$ . Then, in any weakly payoff monotone distribution for  $\Gamma$ , agent  $i$  should play  $a_l$  with weakly higher probability than  $a_k$ . Following this lead, Velez and Brown (2019b) documented violations of weak payoff monotonicity in second-price auction games (Andreoni et al., 2007) and pivotal mechanism games (Cason et al., 2006).

One should not be surprised that weak payoff monotonicity can be violated by empirical distributions in some games. Indeed, this property has the somehow uncomfortable strong implication that actions with equal expected utility need to be played with equal probability. Moreover, well-known phenomena as framing effects, rounding, and other regarding preferences all induce behavior that is less guided by the agents' search for a higher payoff.

One can reconcile empirical equilibrium analysis with these violations of weak payoff monotonicity. The value of this analysis is that it is a benchmark that produces policy relevant comparative statics. Thus, it is enough to make sure that these conclusions do not depend on the sharp implications of weak payoff monotonicity, and they would be retained for violations of weak payoff monotonicity that maintain some positive association of decisions with payoffs, a more robust hypothesis.

Consider the following parametric form of weak payoff monotonicity.

**Definition 6.** Let  $m \in [0, 1]$ . A profile of strategies  $\sigma \equiv (\sigma_i)_{i \in N}$  is *m-weakly payoff monotone* for  $\Gamma$  if for each  $i \in N$  and each pair of actions  $\{a_i, \hat{a}_i\} \subseteq A_i$  such that  $U_i(\sigma_{-i}, a_i) \geq U_i(\sigma_{-i}, \hat{a}_i)$ , we have that  $\sigma_i(a_i) \geq m\sigma_i(\hat{a}_i)$ .

For  $m = 1$  the property exactly corresponds to weak payoff monotonicity. For  $m > 0$ ,  $m$ -weak payoff monotonicity implies  $n$ -weak payoff monotonicity. For  $m = 0$  the property imposes no restrictions in data.

**Definition 7.** Let  $m > 0$ . An *m-empirical equilibrium* of  $\Gamma$  is a Nash equilibrium of  $\Gamma$  that

is the limit of a sequence of  $m$ -weakly payoff monotone strategies for  $\Gamma$ . We denote this set by  $E_m(\Gamma)$ .

This parametric refinement spans the whole range between empirical equilibrium ( $m = 1$ ) and Nash equilibrium ( $m = 0$ ). Using this language one can be more precise and state the specific assumptions on our characterization on behavior that supports a given comparative static. For instance, in [Velez and Brown \(2019b\)](#) all results hold for any given  $m > 0$ . In [Brown and Velez \(2019\)](#) and [Velez and Brown \(2019a\)](#) the essence of all results is sustained for any given  $m > 0$ . Moreover, in this environment, the characterization of empirical equilibria changes continuously with  $m$ , so the conclusions for  $m = 1$  are not substantially changed for  $m \approx 1$ .

## 6.2 Proximity to Nash equilibria

We have defined empirical equilibrium as a refinement of Nash equilibrium. Our construction can be understood as a reduced form evolutionary process that we leave unmodeled. That is, *if* behavior is weakly payoff monotone and approximates mutual best responses—for reasons that we do not explain, but whose effect we can empirically test—it will be an empirical equilibrium. We characterize the profiles that can result from such a process. This does not mean that we assume that in each game a Nash equilibrium will be a good prediction.

In applications of game theory one usually assumes that a Nash equilibrium will be realized. It is in this context that our definition has its most relevance. If one is already assuming that a Nash equilibrium will happen and is using this theory to evaluate policy, design a mechanism, etc., one should account for the empirical plausibility of equilibria.

Even if one is skeptical about the converge to a Nash equilibrium, there are practical reasons to advance empirical equilibrium analysis of a game. Even though the proximity to a Nash equilibrium cannot be expected universally, there is some evidence that it occurs in some games. For instance, it is common that the parameter of logistic QRE models usually increases when it is estimated for later periods in experiments (c.f., [McKelvey and Palfrey, 1995](#)).<sup>15</sup> Weak payoff monotonicity or alternative characterizations are too lax and impose few restrictions on data themselves. For instance uniformly random play by all agents is always weakly payoff monotone. Even though it is possible to obtain some comparative statics and conclusions from the analysis of these limited restrictions (c.f., [Goeree and Louis, 2018](#)), it is only when they are paired with self-sustaining behavior that they achieve more

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<sup>15</sup>It is worth noting that the interpretation of structural QRE estimates requires some care, given the potential misspecification of these models ([Velez and Brown, 2019c](#)). Non-parametric tests suggest convergence to a Nash equilibrium in some environments ([Brown and Velez, 2019](#)).

power. Remarkably proximity or evolution towards mutual best responses is also a testable hypothesis. Thus, empirical equilibrium analysis provides a more powerful benchmark to evaluate experimental and field data.

### 6.3 Empirical equilibria and the purification theorem

It is known that each so-called regular equilibrium, say  $\sigma \in N(\Gamma)$ , has the following property (see [van Damme, 1991](#), for a definition). For each sequence of randomly perturbed versions of  $\Gamma$ , with vanishing perturbations, there is a subsequence of equilibrium behavior in the perturbed games that converges to  $\sigma$  ([Harsanyi, 1973](#); [van Damme, 1991](#)). This means that each regular equilibrium is an empirical equilibrium. More strikingly, given an action space, the complement of the set of games for which all Nash equilibria are regular has Lebesgue measure zero ([Harsanyi, 1973](#)). This means that given an action space and any probability distribution on the set of all games that is absolutely continuous with respect to the Lebesgue measure, the probability that empirical equilibrium produces a strict refinement of the Nash equilibrium set is zero.

This brings an immediate question. Why should we study, characterize, and have interest on a refinement of Nash equilibrium that determines all Nash equilibria are plausible for almost all games? There is an important reason to do so. Games of interest are not randomly chosen. First, an important application of game theory is to analyze strategic situations that arise when real life institutions are operated. These institutions are the result of years or centuries of human interaction. There is no reason to think that these institutions are randomly chosen. Indeed, when games require certain incentives, their payoffs are non-generic. For instance, game  $\Gamma_{(2,2)}$  (see [Table 2](#)) was designed by [Myerson \(1978\)](#) to illustrate that a pair of weakly dominated strategies in a two-by-two game can end up being a perfect equilibrium if one enlarges the game with two weakly dominated strategies. The result is that the flat ranges of utility that this requires leads to a game that is non-generic. Second, a mechanism designer or a market designer has the privilege to choose the games to use in order to achieve the objectives he or she pursues. In some cases these objectives will lead to a single game. This game is not randomly chosen. For instance the well-known second-price auction, pivotal mechanism, VCG mechanisms, Student Optimal Differed Acceptance mechanism, Top Trading Cycles mechanism, Uniform Rule, and Median Voting are all mechanisms that are best or unique in their class (see [Velez and Brown, 2019b](#) for details). Each of these mechanisms induces direct revelation games that are non-generic and for which empirical equilibrium analysis produces significant insights ([Velez and Brown, 2019b](#)). In some other situations the mechanism designer will have a range of mechanisms that achieve the objectives he or she is looking for. In this case empirical equilibrium anal-

		Top Dog		
		10	15	20
Underdog	10	10, 30	15, 25	20, 20
	15	5, 15	15, 25	20, 20
	20	0, 20	0, 20	20, 20

**Table 3:** Game  $\Phi$ .

ysis can allow the designer to inform the choice of a mechanism by means of the empirical plausibility of their equilibria (e.g. [Velez and Brown, 2019a](#)).

## 6.4 The appeal of weakly dominated behavior

Those who view game theory as a normative theory, may not be persuaded by empirical evidence and consider the analysis of weakly dominated behavior only of the realm of behavioral economics. If game theory consists of finding the solution of a game, i.e., the best recommendations for a group of agents who will play it, it is counter intuitive at first that one can recommend an agent to choose a weakly dominated strategy. Why should an agent choose an action that is seemingly careless? The answer is that by doing so the agent may be enforcing an equilibrium outcome that is advantageous to him or her. An example will drive this point home.

Consider game  $\Phi$  in Table 3. This game can be thought of as a stylized partnership dissolution game. Two players, Underdog (U) and Top Dog (T), decide on the division of a pie of forty units. Each player has three possible actions  $\{10, 15, 20\}$ , that one can think of as bids. A higher bidder receives the pie, breaking ties in favor of T. The agent who receives the pie transfers her bid to the other agent, but when U receives the pie, it shrinks to a size of twenty.

In the only undominated Nash equilibrium of  $\Phi$ , both agents bid 10 with certainty. Game  $\Phi$  defies this prediction. Experiments show that players in strategic situations equivalent to this game persistently exhibit weakly dominated behavior ([Brown and Velez, 2016, 2019](#)). In some experimental sessions (with no repeated game effects) behavior conforms to a Nash equilibrium in which the agent who plays the role of U in  $\Phi$  plays the equivalent of 15 with positive probability, and T plays 15 ([Brown and Velez, 2019](#)).

Some introspection reveals that there is a perfectly compelling reason for U to play 15 or even 20 in  $\Phi$ . If she were to play 10 with certainty, T would play 10 with certainty. That would give U a payoff of 10. By playing 15 with positive probability, U enforces an equilibrium that gives her a payoff of 15. Bidding 10 for T is a bad idea when U is bidding 15 with enough probability. Thus, if this is happening, U will end up with a payoff of 5

after bidding 15 with low probability. In the event that this happens, U fully understands that she could have done better by playing 10, but she does not regret having played 15 in order to maintain the incentives of T in check. There is hard evidence that this is so. In the variation of this experiment in which agents are asked sequentially for their actions, usually referred to as divide-and-choose game, when T moves first, U still replies with high probability with the equivalent to action 15 when T bids 10 (Brown and Velez, 2016). Choosing a weakly dominated action for U is essential to this player in order to enforce an outcome that she finds normatively compelling in this game: That the pie is split 15-25 or 20-20, instead of 10-30.

Thus, one cannot label U’s weakly dominated behavior in this game as irrational or even boundedly rational. There is no reason to recommend that player U stick to bidding 10 when this, done with certainty, will give her a payoff of 10. Indeed, we find no reason to recommend an agent to *never* take a potentially costly action that has the purpose to maintain in check the excess of others.

There is of course self sustaining weakly dominated behavior that cannot be made sense of. Game  $\Gamma_1$  in Example 1 probably illustrates this well. Moral values and virtues that are cultivated by societies, as frugality and non-wastefulness seem expressions of a desire to prevent this type of behavior.

Empirical equilibrium achieves some balance in discriminating among weakly dominated behavior. In game  $\Gamma_1$  it rules out all weakly dominated behavior. In game  $\Phi$  empirical equilibrium predicts the pie is split 10-30 or 15-25. There is no empirical equilibrium in which the pie is split 20-20. This is consistent with our intuition in this game and still acknowledges that bidding 20 for U is perhaps too extreme.

**Lemma 4.**  $\sigma \in E(\Phi)$  if and only if either (i) both U and T bid 10 with certainty; or (ii) T bids 15 with certainty and U mixes between 10 and 15, bidding 15 with a probability in the range  $[1/3, 1/2]$ .

## 7 Concluding remarks

We present a new approach to equilibrium analysis, *empirical equilibrium*, inspired by the finding that individual data—while often quite different from Nash behavior—tends to obey the axiom of weak payoff monotonicity. In the tradition of Harsanyi (1973), we refine Nash equilibrium by requiring proximity of weakly payoff monotone behavior. We show that this refinement could be alternatively defined by requiring proximity of regular QRE (Theorem 2) and, in particular, by equilibria in control cost games (Theorem 3). Given the popularity of these models for the analysis of data from economics experiments, our results

provide a connection between the practice in this field and our theory: A researcher who expects one of these models will fit behavior in an experiment, also needs to expect that when empirical distributions approach mutual best responses, they will approach an empirical equilibrium. Remarkably, our refinement produces a delicate selection of the Nash equilibrium set that does not discard all weakly dominated Nash equilibria. Because of this, the conclusions obtained from it are not easily dismissed based on the abundant data that shows the persistence of weakly dominated behavior in many environments. Even though it is a relatively permissive refinement, it produces policy relevant testable comparative statics in applications of game theory. In particular, in mechanism design it allows one to conclude that typical invariance properties are not necessary for the so-called full implementation (Velez and Brown, 2019a); in a partnership dissolution problem, it reveals structural differences among competing mechanisms that are essentially equivalent for the Nash equilibrium prediction (Brown and Velez, 2019; Velez and Brown, 2019a); in the general problem of strategy-proof implementation, it allows one to characterize the mechanisms whose plausible equilibria are truthful equivalent (Velez and Brown, 2019b).

## 8 Appendix

*Proof of Theorem 1.* Payoff monotone distributions are weakly payoff monotone. Thus we only need to prove that an empirical equilibrium is always the limit of interior payoff monotone distributions. Let  $\mu$  be weakly payoff monotone for  $\Gamma$ . Let  $\varepsilon > 0$ . We prove that there is an interior  $\gamma$  that is payoff monotone for  $\Gamma$  such that  $\|\mu - \gamma\| < \varepsilon$ . This implies that  $\sigma \in N(\Gamma)$  is the limit of a sequence of weakly payoff monotone distributions for  $\Gamma$  if and only if it is the limit of a sequence of interior payoff monotone distributions for  $\Gamma$ .

For each  $\lambda > 0$  and each  $i \in N$ , let  $x_i \in \mathbb{R}^{A_i} \mapsto l_i^\lambda(x_i) \equiv (l_{ia_i}^\lambda(x_i))_{a_i \in A_i} \in \Delta(A_i)$  be the function defined by

$$l_{ia_i}^\lambda \equiv \frac{\exp(\lambda x_{ia_i})}{\sum_{\hat{a}_i \in A_i} \exp(\lambda x_{i\hat{a}_i})}.$$

It is well-known that this function is a regular QRF (McKelvey and Palfrey, 1995). It is usually referred to as the Logistic QRF. For each  $i \in N$ , each  $\zeta \in (0, 1)$ , and each profile of distributions  $\beta \in \Delta(A)$ , let

$$f_i^\zeta(\beta) \equiv (1 - \zeta)\mu_i + \zeta l_i^\lambda((U_i(\beta_{-i}, a_i))_{a_i \in A_i}).$$

Let  $\gamma^\zeta$  be a fixed point of  $f^\zeta$ , that exists because  $f^\zeta$  is continuous. Let  $\{a_i, \hat{a}_i\} \subseteq A_i$ .

Suppose first that  $\mu_i(a_i) = \mu_i(\hat{a}_i)$ . We know that

$$l_{a_i}^\lambda((U_i(\gamma_{-i}^\zeta, b_i))_{b_i \in A_i}) \geq l_{\hat{a}_i}^\lambda((U_i(\gamma_{-i}^\zeta, b_i))_{b_i \in A_i}),$$

if and only if  $U_i(\gamma_{-i}^\zeta, a_i) \geq U_i(\gamma_{-i}^\zeta, \hat{a}_i)$ . Thus,  $U_i(\gamma_{-i}^\zeta, a_i) \geq U_i(\gamma_{-i}^\zeta, \hat{a}_i)$  if and only if  $\gamma^\zeta(a_i) \geq \gamma^\zeta(\hat{a}_i)$ . Suppose then that  $\mu_i(a_i) > \mu_i(\hat{a}_i)$ . Since  $\mu$  is weakly payoff monotone for  $\Gamma$ ,  $U_i(\mu_{-i}, a_i) > U_i(\mu_{-i}, \hat{a}_i)$ . Since as  $\zeta \rightarrow 0$ ,  $\gamma^\zeta \rightarrow \mu$ , there is  $c > 0$  such that for each  $\zeta < c$ ,  $\gamma_i^\zeta(a_i) > \gamma_i^\zeta(\hat{a}_i)$  and  $U_i(\gamma_{-i}^\zeta, a_i) > U_i(\gamma_{-i}^\zeta, \hat{a}_i)$ . Thus, for each pair  $\{a_i, \hat{a}_i\} \subseteq A_i$ , there is  $c > 0$  such that for each  $\zeta < c$ ,  $U_i(\gamma_{-i}^\zeta, a_i) \geq U_i(\gamma_{-i}^\zeta, \hat{a}_i)$  if and only if  $\gamma^\zeta(a_i) \geq \gamma^\zeta(\hat{a}_i)$ . Since  $\Gamma$  has finite action spaces, there is  $c > 0$  such that for each  $\zeta < c$ , each  $i \in N$ , and each pair  $\{a_i, \hat{a}_i\} \subseteq A_i$ ,  $U_i(\gamma_{-i}^\zeta, a_i) \geq U_i(\gamma_{-i}^\zeta, \hat{a}_i)$  if and only if  $\gamma^\zeta(a_i) \geq \gamma^\zeta(\hat{a}_i)$ .  $\square$

*Proof of Lemma 2 and Theorem 2.* Since best response operators in games with control costs are regular QRFs, Lemma 2 is a corollary of Lemma 3. When a sequence of control cost functions vanishes, the corresponding best response operators are infinitely sophisticated in the limit (See footnote 14). Thus, Theorem 2 is a corollary of Theorem 3.  $\square$

*Proof of Lemma 3.* Let  $f$  be a control cost function. Since for each  $f_i(0) = \infty$ , any equilibrium of the game associated with  $\Gamma$  and  $f$  is interior (van Damme, 1991, Lemma 4.2.1). Moreover,  $\sigma$  is an equilibrium of the control costs game associated with  $\Gamma$  and  $f$  if and only if for each  $i \in N$ , and each pair  $\{a_l, a_k\} \subseteq A_i$ , (van Damme, 1991, Theorem 4.2.6.)

$$U_i(\sigma_{-i}, a_l) - U_i(\sigma_{-i}, a_k) = f'_i(\sigma_i(a_l)) - f'_i(\sigma_i(a_k)). \quad (2)$$

Thus, given  $\sigma$ , one can construct  $f$  for which  $\sigma$  is an equilibrium of the game associated with  $\Gamma$  and  $f$  if for each  $i$  one construct  $f_i$  for which (2) holds.

Let  $\sigma$  be interior and payoff monotone for  $\Gamma$ . Let  $i \in N$ . Assume that  $A_i \equiv \{a_1, \dots, a_K\}$ . For each  $k = 1, \dots, K$ , let  $u_k \equiv U_i(\sigma_{-i}, a_k)$ . Suppose without loss of generality that  $u_1 \leq \dots \leq u_K$ . Since  $\sigma$  is payoff monotone for  $\Gamma$ ,  $\sigma_i$  is ordinally equivalent to  $u$ . In particular,  $\sigma_i(a_1) \leq \dots \leq \sigma_i(a_K)$ . Let  $\{y_1, \dots, y_L\} \equiv \{y \in (0, 1) : \exists k \in \{1, \dots, K\}, \sigma_i(a_k) = y\}$  be the set of values that  $\sigma_i$  takes. For each  $l = 1, \dots, L$ , let  $u_l \equiv U_i(\sigma_{-i}, a_k)$  for  $a_k$  such that  $\sigma_i(a_k) = y_l$ . Suppose without loss of generality that  $0 < y_1 < \dots < y_L < 1$ . Since  $\sigma_i$  is ordinally equivalent to  $u$ ,  $u_1 < \dots < u_L$ . Fix  $\varepsilon > 0$  and  $0 < y_0 < y_1$  and let  $y_{L+1} = 1$  ( $y_0$  is not necessary for the proof of the lemma; we introduce it in order to use the argument later in the proof of Theorem 3). Let  $m_{L+1} \equiv 0$ ;  $m_L \equiv -\varepsilon$ ; for each  $l = 1, \dots, L - 1$ , let  $m_l \equiv m_{l+1} - (u_{l+1} - u_l)$ ; and  $m_0 = m_1 - \varepsilon$ .

Consider  $f_i$  that assigns to each  $y \in (0, 1]$  the value

$$f_i(y) \equiv \begin{cases} m_{L+1}(y - y_{L+1}) + \frac{m_{L+1} - m_L}{2(y_{L+1} - y_L)}(y - y_{L+1})^2 & \text{if } y \in [y_L, y_{L+1}]. \\ f_i(y_{l+1}) + m_{l+1}(y - y_{l+1}) + \frac{m_{l+1} - m_l}{2(y_{l+1} - y_l)}(y - y_{l+1})^2 & \text{if } y \in [y_l, y_{l+1}), l = 0, \dots, L - 1. \\ f_i(y_0) - m_0 y_0^2 \frac{1}{y} & \text{if } y \in (0, y_0) \end{cases}$$

The function  $f_i$  is a second order spline stiched to a hyperbole in the interval  $(0, y_0)$ . It is continuous and has continuous derivative on  $(0, 1]$ . Indeed, it coincides with infinitely differentiable functions in  $(0, 1]$  with the exception of  $y_0, \dots, y_L$ . In these points, the derivative from the left and from the right match, so its derivative on  $(0, 1]$  is well defined and continuous. The function is strictly decreasing, for its derivative is always negative on  $(0, 1)$ . It satisfies  $\lim_{y \rightarrow 0} f_i(y) = \infty$ . The second derivative of  $f_i$  is well defined and positive on  $(0, 1] \setminus \{y_0, \dots, y_L\}$ . Thus, the function is strictly convex.

Finally, let  $\{a_s, a_k\} \subseteq A_i$ . Let  $l$  and  $t$  be such that  $\sigma_i(a_l) = y_l$  and  $\sigma_i(a_k) = y_t$ . Suppose without loss of generality that  $l < t$ . Then,

$$\begin{aligned} f'_i(\sigma_i(a_s)) - f'_i(\sigma_i(a_k)) &= m_l - m_r \\ &= (m_l - m_{l+1}) + (m_{l+1} - m_{l+2}) + \dots + (m_{r-1} - m_r) \\ &= (u_l - u_{l+1}) + (u_{l+1} - u_{l+2}) + \dots + (u_{r-1} - u_r) \\ &= u_l - u_r \\ &= U_i(\sigma_{-i}, a_s) - U_i(\sigma_{-i}, a_k). \end{aligned}$$

□

*Proof of Theorem 3.* Let  $\sigma \in E(\Gamma)$  and  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  a sequence of interior payoff monotone distributions converging to  $\sigma$ . Let  $i \in N$ ,  $A_i \equiv \{a_1, \dots, a_K\}$ . By passing to a subsequence if necessary we can suppose without loss of generality that for each  $\lambda \in \mathbb{N}$ ,  $\sigma_i^\lambda(a_1) \leq \dots \leq \sigma_i^\lambda(a_K)$ . By convergence of  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  we have that  $\sigma_i(a_1) \leq \dots \leq \sigma_i(a_K)$ . Let  $k$  be the lowest index for which  $U_i(\sigma_{-i}, a_k) = U_i(\sigma_{-i}, a_K)$ . Then, there is  $\eta > 0$  such that for each  $l < k$ ,  $U_i(\sigma_{-i}, a_k) - U_i(\sigma_{-i}, a_l) > \eta$ .

Let  $\{f^\lambda\}$  be a sequence of control cost functions constructed as in the proof of Lemma 3 for parameters  $\varepsilon < 1/\lambda$  and  $y_0 < 1/\lambda$ . Let  $\{a_r, a_s\} \subseteq \{a_k, \dots, a_K\}$ . Suppose that  $\sigma_i(a_k) = 0$ . By passing to a subsequence if necessary we can assume that for each  $\lambda \in \mathbb{N}$ ,  $\sigma_i^\lambda(a_k) < 1/\lambda$  and  $f_i^\lambda(\sigma_i^\lambda(a_k)) < 2/\lambda$  (this subsequence is constructed by first selecting a distribution for which the utility among the best responses of agent  $i$  to  $\sigma_{-i}$  have utility differences at most  $1/\lambda$  and the probabilities are in the required ranges; then for that distribution construct the function  $f_i^\lambda$  with  $\varepsilon < 1/\lambda$  and  $y_0 < 1/\lambda$ ). Since  $\lim_{\lambda \rightarrow \infty} \sigma_i^\lambda(a_k) = 0$ , for each  $y \in (0, 1]$ ,  $\lim_{\lambda \rightarrow \infty} f_i^\lambda(y) = 0$ . Repeating the argument for each agent we construct a subsequence

$\{f^\lambda\}$  that vanishes.

Suppose that  $\sigma_i(a_k) > 0$  and  $k > 1$ . By passing to a subsequence if necessary we can assume that for each  $\lambda \in \mathbb{N}$ ,  $\sigma_i^\lambda(a_{k-1}) < 1/\lambda$ ,  $(f_i^\lambda)'(\sigma_i^\lambda(a_{k-1})) = (f_i^\lambda)'(\sigma_i^\lambda(a_k)) + U_i(\sigma_{-i}^\lambda, a_{k-1}) - U_i(\sigma_{-i}^\lambda, a_k) < 4/\lambda$ ,  $(f_i^\lambda)'(\sigma_i^\lambda(a_k)) = 1/\lambda + U_i(\sigma_{-i}^\lambda, a_k) - U_i(\sigma_{-i}^\lambda, a_K) < 2/\lambda$ . Then for each  $\lambda$  we can construct a control cost function  $g^\lambda$  for which  $\sigma^\lambda$  is an equilibrium of the game associated with  $\Gamma$  and  $g^\lambda$ , as in the proof of Lemma 3 for parameters  $\varepsilon < 1/\lambda$  and  $y_0 < 1/\lambda$ , and including a calibration point  $y = 1/\lambda$  strictly which is strictly in between  $\sigma_i^\lambda(a_{k-1})$  and  $\sigma_i^\lambda(a_k)$  and for which we can set a slope of  $g_i^\lambda$  equal to  $3\lambda$ . Then, the sequence of control cost functions  $\{g^\lambda\}_{\lambda \in \mathbb{N}}$  is such that for each  $y \in (0, 1]$ ,  $\lim_{\lambda \rightarrow \infty} g_i^\lambda(y) = 0$ . The result concludes as in the previous case.

Finally, suppose that  $\sigma_i(a_k) > 0$  and  $k = 1$ . Note that the slope of each  $f_i^\lambda$  in the set  $[y_0^\lambda, 1]$  is bounded above by  $2\varepsilon + U_i(\sigma_{-i}^\lambda, a_1) - U_i(\sigma_{-i}^\lambda, a_K)$ . Since  $\lim_{\lambda \rightarrow \infty} y_0^\lambda = 0$  and  $\lim_{\lambda \rightarrow \infty} U_i(\sigma_{-i}^\lambda, a_1) - U_i(\sigma_{-i}^\lambda, a_K) = 0$ , then for each  $y \in (0, 1]$ ,  $\lim_{\lambda \rightarrow \infty} f_i^\lambda(y) = 0$ . The result concludes as in the previous case.  $\square$

*Proof of Lemma 4.* Each of the distributions described in the statement of the lemma are weakly payoff monotone themselves. Thus, by definition they are empirical equilibria of  $\Phi$  sustained by the respective constant sequences. Suppose that  $\sigma \in E(\Phi)$ . Let  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  be a sequence of interior and payoff monotone distributions for  $\Phi$  that converges to  $\sigma$  (this sequence exists by Theorem 1). Let  $\lambda \in \mathbb{N}$ . Then,  $U_U(\sigma_T^\lambda, 10) > U_U(\sigma_T^\lambda, 15) > U_U(\sigma_T^\lambda, 20)$ . Thus,  $\sigma_U^\lambda(10) > \sigma_U^\lambda(15) > \sigma_U^\lambda(20)$ . Thus,  $\sigma_U^\lambda(20) < 1/3$ . If  $\lim_{\lambda \rightarrow \infty} \sigma^\lambda = \sigma$ , then  $\sigma_U(20) \leq 1/3$ . Thus,  $U_T(\sigma_U^\lambda, 15) > U_T(\sigma_U^\lambda, 20)$  and consequently  $\sigma_T(20) = 0$ . Thus,  $U_U(\sigma_T^\lambda, 10) > U_U(\sigma_T^\lambda, 20)$ . Thus,  $\sigma_U(20) = 0$ . Thus,  $\sigma_T(20) = 0$ . Since for each  $\lambda$ ,  $U_U(\sigma_T^\lambda, 10) > U_U(\sigma_T^\lambda, 15)$ , we have that  $\sigma_U(10) \geq \sigma_U(15)$ . Thus,  $\sigma_U(15) \leq 1/2$ . Suppose that  $\sigma_T(15) > 0$ . Then,  $\sigma_U(15) \geq 1/3$ , for otherwise  $U_T(\sigma_U, 15) < U_T(\sigma_U, 10)$ . If  $\sigma_T(10) > 0$ , then  $U_U(\sigma_T, 10) > U_U(\sigma_T, 15) > U_U(\sigma_T, 20)$  and  $\sigma_U(15) = \sigma_U(20) = 0$ . Thus,  $\sigma_T(10) = 1$ .  $\square$

## References

- Abdulkadiroğlu, A., Sönmez, T., 2003. School choice: A mechanism design approach. *Amer Econ Review* 93 (3), 729–747.  
URL <http://www.jstor.org/stable/3132114>
- Andreoni, J., Che, Y.-K., Kim, J., 2007. Asymmetric information about rivals' types in standard auctions: An experiment. *Games Econ Behavior* 59 (2), 240 – 259.  
URL <http://dx.doi.org/10.1016/j.geb.2006.09.003>

- Artemov, G., Che, Y.-K., He, Y., 2017. Strategic 'mistakes': Implications for market design research, Mimeo.
- Attiyeh, G., Franciosi, R., Isaac, R. M., Jan 2000. Experiments with the pivot process for providing public goods. *Public Choice* 102 (1), 93–112.  
URL <https://doi.org/10.1023/A:1005025416722>
- Brown, A. L., Velez, R. A., 2016. The costs and benefits of symmetry in common-ownership allocation problems. *Games Econ Behavior* 96, 115–131.  
URL <http://dx.doi.org/10.1016/j.geb.2016.01.008>
- Brown, A. L., Velez, R. A., 2019. Empirical bias and efficiency of alpha-auctions: experimental evidence, mimeo Texas A&M University.
- Cason, T. N., Saijo, T., Sjöström, T., Yamato, T., 2006. Secure implementation experiments: Do strategy-proof mechanisms really work? *Games Econ Behavior* 57 (2), 206 – 235.  
URL <http://dx.doi.org/10.1016/j.geb.2005.12.007>
- Chen, L., Pereyra, J. S., 2018. Self selection in school choice, Mimeo.
- Chen, Y., Sönmez, T., 2006. School choice: an experimental study. *Journal of Economic Theory* 127 (1), 202 – 231.  
URL <http://www.sciencedirect.com/science/article/pii/S0022053104002418>
- Coppinger, V. M., Smith, V. L., Titus, J. A., 1980. Incentives and behavior in english, dutch and sealed-bid auctions. *Economic Inquiry* 18 (1), 1–22.  
URL <https://doi.org/10.1111/j.1465-7295.1980.tb00556.x>
- Dekel, E., Scotchmer, S., 1992. On the evolution of optimizing behavior. *J Econ Theory* 57 (2), 392 – 406.  
URL [https://doi.org/10.1016/0022-0531\(92\)90042-G](https://doi.org/10.1016/0022-0531(92)90042-G)
- Fudenberg, D., He, K., 2018. Player-compatible equilibrium, mimeo, Accessed on October 4th, 2018.  
URL <http://economics.mit.edu/files/15442>
- Goeree, J., Holt, C. A., Louis, P., Palfrey, T. R., Rogers, B., 2018. Rank-dependent choice equilibrium: A non-parametric generalization of qre. In: *The Handbook of Research Methods and Applications in Experimental Economics*. Forthcoming.

- Goeree, J., Louis, P., 2018. M equilibrium: A dual theory of beliefs and choices in games.  
URL <https://arxiv.org/pdf/1811.05138.pdf>
- Goeree, J. K., Holt, C. A., Palfrey, T. R., 2002. Quantal response equilibrium and overbidding in private-value auctions. *Journal of Economic Theory* 104 (1), 247 – 272.  
URL <http://www.sciencedirect.com/science/article/pii/S002205310192914X>
- Goeree, J. K., Holt, C. A., Palfrey, T. R., 2005. Regular quantal response equilibrium. *Experimental Economics* 8 (4), 347–367.  
URL <http://dx.doi.org/10.1007/s10683-005-5374-7>
- Goeree, J. K., Holt, C. A., Palfrey, T. R., 2016. *Quantal Response Equilibrium: A Stochastic Theory of Games*. Princeton Univ. Press, Princeton, NJ.
- Haile, P. A., Hortasu, A., Kosenok, G., 2008. On the empirical content of quantal response equilibrium. *Amer Econ Review* 98 (1), 180–200.  
URL <http://www.jstor.org/stable/29729968>
- Harsanyi, J. C., Dec 1973. Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points. *Int J Game Theory* 2 (1), 1–23.  
URL <https://doi.org/10.1007/BF01737554>
- Harstad, R. M., Dec 2000. Dominant strategy adoption and bidders' experience with pricing rules. *Experimental Economics* 3 (3), 261–280.  
URL <https://doi.org/10.1007/BF01669775>
- Hassidim, A., Romm, A., Shorrer, R. I., May 26 2016. 'strategic' behavior in a strategy-proof environment, mimeo.  
URL <https://ssrn.com/abstract=2784659>
- Jackson, M. O., 1992. Implementation in undominated strategies: A look at bounded mechanisms. *The Review of Economic Studies* 59 (4), 757–775.  
URL <http://www.jstor.org/stable/2297996>
- Kagel, J. H., Harstad, R. M., Levin, D., 1987. Information impact and allocation rules in auctions with affiliated private values: A laboratory study. *Econometrica* 55 (6), 1275–1304.  
URL <http://www.jstor.org/stable/1913557>
- Kagel, J. H., Levin, D., 1993. Independent private value auctions: Bidder behaviour in first-, second- and third-price auctions with varying numbers of bidders. *The Economic*

- Journal 103 (419), 868–879.  
URL <http://www.jstor.org/stable/2234706>
- Kohlberg, E., Mertens, J.-F., 1986. On the strategic stability of equilibria. *Econometrica* 54 (5), 1003–1037.  
URL <http://www.jstor.org/stable/1912320>
- Li, S., November 2017. Obviously strategy-proof mechanisms. *Amer Econ Review* 107 (11), 3257–87.  
URL <http://dx.doi.org/10.1257/aer.20160425>
- McKelvey, R. D., Palfrey, T. R., 1995. Quantal response equilibria for normal form games. *Games and Economic Behavior* 10 (1), 6–38.  
URL <http://dx.doi.org/10.1006/game.1995.1023>
- McKelvey, R. D., Palfrey, T. R., 1996. A statistical theory of equilibrium in games. *Japanese Econ Review* 47 (2), 186–209.
- Milgrom, P., Mollner, J., 2017. Extended proper equilibrium, mimeo.  
URL [https://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=3035565](https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3035565)
- Myerson, R. B., Jun 1978. Refinements of the nash equilibrium concept. *International Journal of Game Theory* 7 (2), 73–80.  
URL <https://doi.org/10.1007/BF01753236>
- Nachbar, J. H., Mar 1990. “evolutionary” selection dynamics in games: Convergence and limit properties. *Int J Game Theory* 19 (1), 59–89.  
URL <https://doi.org/10.1007/BF01753708>
- Nash, J., 1951. Non-cooperative games. *Annals of Mathematics* 54 (2), 286–295.  
URL <http://www.jstor.org/stable/1969529>
- Palfrey, T. R., Srivastava, S., 1991. Nash implementation using undominated strategies. *Econometrica* 59 (2), 479–501.  
URL <http://www.jstor.org/stable/2938266>
- Rees-Jones, A., 2017. Suboptimal behavior in strategy-proof mechanisms: Evidence from the residency match. *Games Econ Behavior*.  
URL <http://www.sciencedirect.com/science/article/pii/S0899825617300751>

- Rosenthal, R. W., Sep 1989. A bounded-rationality approach to the study of noncooperative games. *Int J Game Theory* 18 (3), 273–292.  
URL <https://doi.org/10.1007/BF01254292>
- Roth, A. E., 1984. The evolution of the labor market for medical interns and residents: A case study in game theory. *J Political Econ* 92 (6), 991–1016.  
URL <https://doi.org/10.1086/261272>
- Samuelson, L., 1992. Dominated strategies and common knowledge. *Games Econ Behavior* 4 (2), 284 – 313.  
URL [https://doi.org/10.1016/0899-8256\(92\)90020-S](https://doi.org/10.1016/0899-8256(92)90020-S)
- Selten, R., Mar 1975. Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory* 4 (1), 25–55.  
URL <https://doi.org/10.1007/BF01766400>
- van Damme, E., 1991. *Stability and Perfection of Nash Equilibria*. Springer Berlin Heidelberg, Berlin, Heidelberg.  
URL <https://link.springer.com/book/10.1007/978-3-642-58242-4>
- Velez, R. A., Brown, A. L., 2019a. Empirical bias of extreme-price auctions: analysis, mimeo, Texas A&M University.
- Velez, R. A., Brown, A. L., 2019b. Empirical strategy-proofness, mimeo, Texas A&M University.
- Velez, R. A., Brown, A. L., 2019c. The paradox of monotone structural QRE, mimeo, Texas A&M University.
- Zhang, B., 2016. Quantal response methods for equilibrium selection in normal form games. *J Math Econ* 64 (Supplement C), 113 – 123.  
URL <http://www.sciencedirect.com/science/article/pii/S0304406816300118>

## Appendix not for publication

**Lemma 5** (van Damme, 1991). Let  $\{f^\lambda\}_{\lambda \in \mathbb{N}}$  be a sequence of control costs functions that vanishes and  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  a corresponding convergent sequence of the control cost game associated with  $\Gamma$  and  $f^\lambda$ . Then,  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  converges to a Nash equilibrium of  $\Gamma$ .

*Proof of Lemma 5.* Let  $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$  be a convergent sequence such that for each  $\lambda \in \mathbb{N}$ ,  $\sigma^\lambda$  is an equilibrium of the control cost game associated with  $\Gamma$  and  $f^\lambda$ . Let  $\sigma \equiv \lim_{\lambda \rightarrow \infty} \sigma^\lambda$ . Let  $i \in N$  and  $a_i \in A_i$  be a best response to  $\sigma_{-i}$  for agent  $i$  in  $\Gamma$ . Suppose that  $a_k \in A_i$  is not a best response to  $\sigma_{-i}$  for agent  $i$ . We prove that  $\sigma_i(a_k) = 0$ . Since as  $\lambda \rightarrow \infty$ ,  $\sigma^\lambda \rightarrow \sigma$ , we also have that  $\sigma_i^\lambda(a_i) \rightarrow \sigma_i(a_i)$ ,  $\sigma_i^\lambda(a_k) \rightarrow \sigma_i(a_k)$ ,  $U_i(\sigma_{-i}^\lambda, a_i) \rightarrow U_i(\sigma_{-i}, a_i)$ , and  $U_i(\sigma_{-i}^\lambda, a_k) \rightarrow U_i(\sigma_{-i}, a_k)$ . Thus, there is  $\Lambda \in \mathbb{N}$  such that for each  $\lambda \geq \Lambda$ ,  $\sigma_i^\lambda(a_k) \leq \sigma_i^\lambda(a_i)$ . Suppose first that  $\sigma_i(a_i) = 0$ . Since  $\sigma_i^\lambda(a_i) \rightarrow 0$ ,  $\sigma_i^\lambda(a_k) \rightarrow 0$ . Suppose then that  $\sigma_i(a_i) > 0$ . By (van Damme, 1991, Theorem 4.2.6), for each  $\lambda \in \mathbb{N}$ ,

$$U_i(\sigma_{-i}^\lambda, a_l) - U_i(\sigma_{-i}^\lambda, a_k) = (f^\lambda)'_i(\sigma_i^\lambda(a_l)) - (f^\lambda)'_i(\sigma_i^\lambda(a_k)).$$

The left side of the expression above converges to a positive number. Since  $\sigma_i^\lambda(a_l) \rightarrow \sigma_i(a_l) > 0$  and  $\{f^\lambda\}_{\lambda \in \mathbb{N}}$  vanishes,  $(f^\lambda)'_i(\sigma_i^\lambda(a_l)) \rightarrow 0$ . Thus,  $(f^\lambda)'_i(\sigma_i^\lambda(a_k)) \rightarrow 0$ , for otherwise there is a subsequence of  $\{\sigma_i^\lambda(a_k)\}_{\lambda \in \mathbb{N}}$  that converges in the interior of  $(0, 1]$ . If this is so the right side of the equation above converges to zero. This is a contradiction.  $\square$

The following proposition formally states our claims in Example 3

**Proposition 1.** Consider the game  $\Gamma_c$  in Table 2. Then, for each  $c > 0$ ,

$$\begin{aligned} N(\Gamma_c) &= \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}, \\ T(\Gamma_c) = U(\Gamma_c) &= \{(a_1, b_1), (a_2, b_2)\}, \\ P(\Gamma_c) &= \{(a_1, b_1)\}. \end{aligned}$$

Moreover,

$$E(\Gamma_c) = \begin{cases} \{(a_1, b_1)\} & \text{if } \min\{c_1, c_2\} \leq 1, \\ \{(a_1, b_1), (a_2, b_2)\} & \text{Otherwise.} \end{cases}$$

*Proof.* We first prove that  $N(\Gamma_c) = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ . Let  $c \equiv (c_1, c_2)$  such that  $c_1 > 0$  and  $c_2 > 0$ . One can easily see that the action profiles  $(a_1, b_1)$ ,  $(a_2, b_2)$ , and  $(a_3, b_3)$  are the only pure strategy Nash equilibria of  $\Gamma_c$ . Now, let  $\sigma \in N(\Gamma_c)$ . If  $\sigma_1(a_2) > 0$ , then  $\sigma_2(b_3) = 0$ . Then,  $\sigma_1(a_3) = 0$ . It follows that either  $\sigma$  is equal to  $(a_1, b_1)$  or  $(a_2, b_2)$ . Symmetry implies the same is true when  $\sigma_2(b_2) > 0$ . Thus, suppose that  $\sigma_1$  is not a pure strategy. Suppose that

$\sigma_1(a_1) > 0$ ,  $\sigma_1(a_2) = 0$ , and  $\sigma_2(b_2) = 0$ . Then  $\sigma_2(b_3) = 0$ . Thus,  $\sigma = (a_1, b_1)$ . A symmetric argument shows that if  $\sigma_2(b_1) > 0$ ,  $\sigma_1(a_2) = 0$ , and  $\sigma_2(b_2) = 0$ , then  $\sigma = (a_1, b_1)$ .

It is well known that at each perfect equilibrium no agent plays a weakly dominated strategy. Clearly,  $a_3$  and  $b_3$  are weakly dominated for players 1 and 2, respectively. Thus,  $T(\Gamma_c) \subseteq \{(a_1, b_1), (a_2, b_2)\}$ . Now, let  $t \equiv \min\{c_1, c_2\}$ ,  $\varepsilon \equiv \min\{t, t/(3c_1), t/(3c_2), 1/3\}$ , and for each  $\lambda \in \mathbb{N}$ ,  $\sigma^\lambda$  be the strategy profile for which  $\sigma_1^\lambda(a_1) \equiv \varepsilon c_2/(2\lambda t)$  and  $\sigma_1^\lambda(a_3) \equiv \varepsilon/(\lambda t)$ ; and  $\sigma_2^\lambda(b_1) \equiv \varepsilon c_1/(2\lambda t)$  and  $\sigma_2^\lambda(b_3) \equiv \varepsilon/(\lambda t)$ . Then,

$$\begin{aligned} U_1(\sigma_2^\lambda, a_1) &= \varepsilon c_1/(2\lambda t) - (7 + c_1)\varepsilon/(\lambda t) = -7\varepsilon/(\lambda c_1) - \varepsilon c_1/(2\lambda t), \\ U_1(\sigma_2^\lambda, a_2) &= -7\varepsilon/(\lambda c_1), \\ U_1(\sigma_2^\lambda, a_3) &= -(7 + c_1)\varepsilon/(2\lambda) - 7(1 - \varepsilon/\lambda - \varepsilon/(\lambda c_1)) - 7\varepsilon/(\lambda c_1). \end{aligned}$$

Thus,  $a_2$  is the unique best response to  $\sigma_2^\lambda$  for agent 1. Symmetry implies that  $b_2$  is the unique best response to  $\sigma_1^\lambda$  for agent 2. Since  $\sigma_1^\lambda$  places probability at most  $1/\lambda$  in both  $a_1$  and  $a_3$ ;  $\sigma_2^\lambda$  places probability at most  $1/\lambda$  in both  $b_1$  and  $b_3$ ; and as  $\lambda \rightarrow \infty$ ,  $\sigma^\lambda \rightarrow (a_2, b_2)$ , we have that  $(a_2, b_2) \in T(\Gamma_c)$ .

Let  $\Lambda > 2$  be such that for each  $\lambda \geq \Lambda$ ,  $1 - 1/(2\lambda) - 1/(3\lambda^2) > \max\{c_1/(3\lambda^2), c_2/(3\lambda^2), 1/\lambda\}$ . Let  $\lambda \geq \Lambda$  and  $\sigma^\lambda$  be the symmetric profile of strategies such that  $\sigma_1^\lambda(a_2) \equiv 1/(2\lambda)$  and  $\sigma_1^\lambda(a_3) \equiv 1/(3\lambda^2)$ . Thus,  $U_1(\sigma_2^\lambda, a_1) - U_1(\sigma_2^\lambda, a_2) = 1 - 1/(2\lambda) - 1/(3\lambda^2) - c_1/(3\lambda^2) > 0$ . Clearly,  $U_1(\sigma_2^\lambda, a_2) > U_1(\sigma_2^\lambda, a_3)$ . Similarly,  $U_2(\sigma_1^\lambda, b_1) - U_2(\sigma_1^\lambda, b_2) > 0$  and  $U_2(\sigma_1^\lambda, b_2) > U_2(\sigma_1^\lambda, b_3)$ . Since  $\sigma_1^\lambda(a_1) > \sigma_1^\lambda(a_2)/\lambda$ ,  $\sigma_1^\lambda(a_2) > \sigma_1^\lambda(a_3)/\lambda$ ,  $\sigma_2^\lambda(b_1) > \sigma_2^\lambda(b_2)/\lambda$ , and  $\sigma_2^\lambda(b_2) > \sigma_2^\lambda(b_3)/\lambda$ ; and as  $\lambda \rightarrow \infty$ ,  $\sigma^\lambda \rightarrow (a_1, b_1)$ , we have that  $(a_1, b_1) \in P(\Gamma_c) \subseteq T(\Gamma_c)$ .  $\square$