OPTIMAL DIVIDEND PAYOUTS UNDER JUMP-DIFFUSION RISK PROCESSES

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This article considers the dividend optimization problem for an insurer with a jump-diffusion risk process in the presence of fixed and proportional transaction costs. Due to the presence of a fixed transaction cost, the mathematical problem becomes an impulse stochastic control problem. Using a stochastic impulse control approach, we transform the stochastic control problem into a quasi-variational inequality for a second-order nonlinear integro-differential equation. Under a risk-neutral assumption for the insurer, we solve this problem explicitly and construct the value function together with the optimal policy. Finally, we discuss the expected time to the first dividend payment when the optimal strategy is employed.

Keywords Dividend payouts; Jump-diffusion processes; Stochastic impulse control; Quasi-variational inequalities.

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1. INTRODUCTION

In recent years there has been increasing attention towards the utilization of stochastic control theory to insurance-related problems. This is due to the fact that a company, such as a property-liability insurance company, or a pension-fund management company, can control reinsurance strategies or investment strategies and can pay dividends to maximize (or minimize) a certain objective function under different constraints. This leads to a stochastic control problem. Roughly, the
literature on this topic can be divided into two groups. The first is the classical risk model. For example, see Buhlmann\[4\], Hipp and Plum\[10\], Taskar\[16\], Azcue and Muler\[3\], and Schmidli\[17\]. The second is the diffusion surplus risk model. In this model, the liquid asset processes of the corporation are driven by Brownian motion with constant drift and diffusion coefficients. The drift term corresponds to the expected profit per unit time, while the diffusion term is interpreted as risk. The classic studies on this subject are those by Jeanblanc-Picque and Shiryaev\[12\], Asmussen and Taksar\[2\], Hojgaard and Taksar\[11\], Paulsen\[14,15\], and Cadenillas et al.\[5\]. It is well known that in the absence of fixed transaction costs, optimality is achieved by using a barrier strategy, i.e., there is a level $b^*$ so that whenever surplus goes above $b^*$, the excess is paid out as dividends. However, due to the fixed transaction cost, the barrier strategy is not the optimal policy. Jeanblanc-Picque and Shiryaev\[12\] consider the surplus process modelled by an Ito process in which there is a fixed cost each time the dividends are paid out. They apply impulse control theory to obtain the optimal policy. More recently, Cadenillas et al.\[5\] consider the similar problem, but allow proportional reinsurance and assume that the surplus process follows a diffusion process. Paulsen\[15\] considers a general model when payments are subject to both fixed and proportional costs.

The present article considers the optimal dividend pay-outs for an insurer in the presence of fixed and proportional transaction costs under a jump-diffusion process. As we all know, the claims process is an important part of the surplus process for the insurance company. When a claim occurs, the corresponding surplus process for the insurer is a jump. Thus it is more appropriate to add the claims process into the diffusion surplus process. In addition, the jumps can be interpreted as the important news for the company, for example, a company involved in the credit crunch during 2008. By its very nature, important information arrives only at a discrete point in time. This component is modeled by a “jump” process reflecting the nonmarginal impact of the information. Therefore, we extend the diffusion model considered by Jeanblanc-Picque and Shiryaev\[12\] to a class of jump-diffusion risk process for an insurer, which was first introduced by Gerber and Shiu\[9\]. In our model, the net amount of money received by the shareholders during $[0, t)$ equals $\sum_{i=1}^{\infty} \zeta_i I_{[\tau_i < t]}$, where $\tau_i$ are stopping times that represent the moments in time when the dividends are paid out, and $\zeta_i$ are the amounts of the dividend payments. We assume that the $i$th dividend payment corresponding to total transaction cost payment by the insurer is $c + \lambda \zeta_i$, where the constant $c$ is called the fixed part and the quantity $\lambda \zeta_i$ is called the proportional part, respectively, of the transaction cost. The sequence of stopping times $\tau_i$, as well as the random variables $\zeta_i$, are controllable. The dividend amount is $\zeta_i$, which can take on any value not exceeding the amount of liquid
asset available at time $\tau_i$. Further, we assume that the claim arrival point process follows a Poisson process. The primary aim of this article is to find the payout-scheme that maximizes the expected cumulative discounted dividend pay-outs up to the time to ruin in a jump-diffusion surplus model. The company is bankrupt when the surplus process first exists in the solvency region, which is imposed on the company by the regulatory authorities. Without loss of generality, we assume that the surplus process first attains zero, standing for the company bankruptcy. Applying the stochastic impulse control approach, we obtain the optimal value function and the optimal policy.

2. THE MODEL

Let $(\Omega, F, P)$ be a complete probability space containing all the random variables we meet in the following. Suppose that if the decision-maker makes no interventions, the insurer surplus $U(t)$ is given by

$$dU(t) = \mu dt + \sigma dB(t) - \int_{R^+} zN(dt, dz), \quad U(0) = x > 0,$$

(2.1)

where $\mu$ is the premium received continuously, $\sigma$ is a positive constant, and $B(t)$ is a standard Brownian motion, representing the uncertainty income for the insurer. The last integral term is a compound Poisson process standing for the total discontinuous changes in the surplus process, which is posited to the composition of two types of changes: (i) the claim size process and (ii) important events about the company that cause “abnormal” changes in the surplus process. We assume that the sequence of jumps are independent with common distribution function $Q(x)$. Let $p_1$ be the mean of this distribution. $N(dt, dz)$ is the compound Poisson measure with Poisson intensity $\lambda$. For simplicity, we view the important changes in the surplus as claims. Hereafter, we call the last term in (2.1) the aggregate claims process.

In addition, we assume that $B(t)$ and the compound Poisson process defined by (2.1) are independent. For a motivation of this model see, for example, Dufresne and Gerber[7]. Now, we denote the $i$th time of dividends payment and amount of dividends by $\tau_i$ and $\zeta_i$, respectively.

**Definition 2.1.** A sequence

$$\pi := (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_n, \ldots)$$

(2.2)

is said to be an admissible control or admissible policy if $0 < \tau_1 < \tau_2 < \cdots$ are $\mathcal{F}_t$ stopping times, and the random variable $\zeta_i, i = 1, 2, \ldots$ is $\mathcal{F}_{\tau_i}$ measurable with $0 < (1 + \lambda)\zeta_i + c \leq U(\tau_i^-)$. The last condition means that
when the dividends are distributed, the total amount withdrawn cannot exceed the reserve available at that time. We call the $\tau_i$ the intervention times and $\zeta_i$ the impulses, i.e., the amount of dividends withdrawn from the reserve. The class of all admissible controls is denoted by $\Pi$.

**Definition 2.2.** The measure $\nu$ on $R^+$, defined by

$$
\nu(A) = E[\#\{t \in [0, 1] : \Delta U_t \neq 0, \Delta U_t \in A\}], \quad A \in \mathcal{B}(R^+),
$$

is called the Levy measure of $U : \nu(A)$ is the expected number, per unit time of jumps whose size belongs to $A$.

We assumed that there is a positive safety loading:

$$
\mu > \nu(Q).
$$

(2.3)

Note that $\nu(dx) = \lambda Q(dx)$; then (2.3) is equivalent to

$$
\mu > \lambda \rho_1.
$$

We assume that the insurer will pay the proportional transaction cost $\lambda \zeta_i$ plus a constant transaction cost $c$ for each dividend payment. Therefore, until time $t$, the total amounts of liquid assets withdrawn can be represented as

$$
L^\pi(t) = \sum_{i=1}^{+\infty} I_{[\tau_i < t]}((1 + \lambda)\zeta_i + c).
$$

Then for a given admissible policy $\pi$, the corresponding surplus process $U^{(\pi)}(t)$ is given by

$$
dU^{(\pi)}(t) = \mu dt + \sigma dB(t) - \int_{R^+} zN(dt, dz) - dL^\pi(t), \quad U^{(\pi)}(0) = x > 0.
$$

(2.4)

For simplicity, we suppose that the insurer is risk-neutral. Define the time to ruin for the insurer by

$$
\tau = \inf\{t > 0; U^{(\pi)}(t) \leq 0\}.
$$

(2.5)

With each admissible control $\pi$, the value function is defined by

$$
J^{(\pi)}(x) = E\left[\sum_{\tau_j \leq \tau} e^{-\rho_1\zeta_j}\right],
$$

(2.6)
where \( \rho > 0 \) is a constant (the discounted rate). Our goal is to seek the optimal value function defined by

\[
\Phi(x) = \sup_{\pi \in \Pi} J^{(\pi)}(x) = J^{(\pi^*)}(x),
\]

and the optimal policy \( \pi^* = (\tau_1^*, \tau_2^*, \ldots; \zeta_1^*, \zeta_2^*, \ldots) \in \Pi \) such that the following equality is true:

\[
\Phi(x) = J^{(\pi^*)}(x).
\]

**Definition 2.3.** Let \( H \) be the space of all measurable functions \( h : [0, +\infty) \to \mathbb{R} \). The intervention operator \( M : H \to H \) is defined by

\[
Mh(x) = \sup \left\{ h(x - (1 + \lambda)\zeta - c) + \zeta : 0 < \zeta < \frac{x - c}{1 + \lambda} \right\}.
\]

The intervention operator can be interpreted as follows. Suppose that for each initial position \( x \) there exists an optimal policy. If the surplus process starts at \( x \) and is controlled by the optimal policy, then the expected utility associated with this optimal policy is \( \Phi(x) \). On the other hand, suppose the payment of dividends occurs at time 0. If the amount of the liquid assets used to pay dividends is \( \zeta \); then the initial reserve decreases from \( x \) to \( x - (1 + \lambda)\zeta - c \). If after that, the optimal policy is followed, then the total expected utility associated with this policy is \( \Phi(x - (1 + \lambda)\zeta - c) + \zeta \). If the initial payout \( \zeta \) is chosen to be the one which brings the maximum value to \( \Phi(x - (1 + \lambda)\zeta - c) + \zeta \), then the total expected utility under such a policy would be \( M\Phi(x) \). Since the first policy is optimal, its associated expected utility is greater or equal to that of the expected utility associated with the second policy. Hence, \( \Phi(x) \geq M\Phi(x) \) with equality being true if \( x \) is the position process where it is optimal to intervene.

Define the operator \( \mathcal{A} \) by

\[
\mathcal{A}f(x) = -\rho f(x) + \mu f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{R^+} (f(x - z) - f(x)) v(dz).
\]

Such an operator is called infinitesimal operator. For more information about the infinitesimal operator, see Applebaum (Ref.\[1\], Chapter 3).

**Definition 2.4.** A function \( \phi : [0, +\infty) \to [0, +\infty) \) is said to satisfy the quasi-variational inequalities of the control problem if for every \( x \in [0, +\infty) \)

\[
\mathcal{A}\phi(x) \leq 0,
\]

\[
\phi(x) \geq M\phi(x),
\]
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\[
(\phi(x) - M\phi(x))(\mathbb{A}\phi(x)) = 0, \quad (2.12)
\]
\[
\phi(0) = 0. \quad (2.13)
\]

We observe that a solution \( \phi \) of the QVI splits \((0, +\infty)\) into two regions: a continuation region

\[
D := \{ x \in (0, +\infty) : \phi(x) > M\phi(x) \text{ and } \mathbb{A}\phi(x) = 0 \}
\]

and an intervention region

\[
\Sigma := \{ x \in (0, +\infty) : \phi(x) = M\phi(x) \text{ and } \mathbb{A}\phi(x) \leq 0 \}.
\]

**Definition 2.5.** The control \( \pi^\phi = (\tau_1^\phi, \tau_2^\phi, \ldots; \xi_1^\phi, \xi_2^\phi, \ldots; \xi_n^\phi, \ldots) \) is called the QVI control associated with \( \phi \) if the associated state process \( U^\phi(t) \) given by (2.4) satisfies

\[
\tau_1^\phi = \inf \{ t \geq 0 : \phi(U^\phi(t)) = M\phi(U^\phi(t)) \},
\]
\[
\zeta_1^\phi := \arg \max_{\{\xi > 0, \xi \leq U^\phi(t^\phi_1)\}} \{ \phi(U^\phi(t^\phi_1)) - (1 + \lambda)\zeta - c \},
\]

and for every \( n \geq 2 \)

\[
\tau_n^\phi = \inf \{ t > \tau_{n-1} : \phi(U^\phi(t)) = M\phi(U^\phi(t)) \},
\]
\[
\zeta_n^\phi := \arg \max_{\{\xi > 0, \xi \leq U^\phi(t^\phi_n)\}} \{ \phi(U^\phi(t^\phi_n)) - (1 + \lambda)\zeta - c \},
\]

with \( \tau_0^\phi := 0, \zeta_0^\phi := 0. \)

### 3. A VERIFICATION THEOREM

In this section, we show that the optimal value function can be obtained by solving the QVI (2.10)–(2.13) if the QVI control \( \pi^\phi \) associated with \( \phi \) is admissible.

**Theorem 3.1.** Let \( \phi \in C^1(0, \infty) \) be a solution of the QVI (2.10)–(2.13). Suppose there exists \( M > 0 \) such that \( \phi \) is twice continuously differentiable on \((0, M)\) and \( \phi \) is linear on \([M, \infty)\). Then, for every \( x \in (0, +\infty) \)

\[
\Phi(x) \leq \phi(x).
\]
Furthermore, if the QVI control associated with $\phi$ is admissible, then $\phi$ coincides with the optimal value function and the QVI control policy associated with $\phi$ is the optimal policy, i.e.,

$$
\Phi(x) = \phi(x).
$$

**Proof.** Let $\varepsilon$ be a small positive number and $\tau^\varepsilon$ be the first hitting time of $[0, \varepsilon]$ by the surplus process $U^{(x)}$. We note that $\phi'(x)$ is a continuous function on $[\varepsilon, +\infty)$ and it is a constant for $x \geq M$. Thus, we get $\phi'$ is bounded on $[\varepsilon, +\infty)$ and for $x \in [\varepsilon, +\infty)$,

$$
\sigma^2 E^x \left[ \int_0^{\tau^\varepsilon} \left\{ e^{-\rho t} \phi'(U^{(x)}(t)) \right\}^2 dt \right] < \infty. \quad (3.2)
$$

The linearity of $\phi$ for $x \geq M$, the dominated convergence theorem, and the fact that $\phi(0) \leq \phi(\varepsilon)$ imply that

$$
\lim_{t \to +\infty} E^x \left[ e^{-\rho(t_\varepsilon \wedge \tau^\varepsilon)} \phi(U^{(x)}(t \wedge \tau^\varepsilon)) \right] = E^x \left[ e^{-\rho \tau^\varepsilon} \phi(U^{(x)}(\tau^\varepsilon)) \right] \leq \phi(\varepsilon). \quad (3.3)
$$

Since $U^{(x)}(t)$ is a semi-martingale on the stochastic interval $[\tau_j, \tau_{j+1})$, and $\phi$ is twice continuously differentiable on $(0, +\infty)$, with a possible exception of point $M$, where $\phi$ and $\phi'$ are continuous, and $\phi''$ might have a discontinuity of the first order, we can apply Ito’s formula (see Proposition 8.19 and Remark 8.4 in Cont and Tankov) to get

$$
E^x \left[ e^{-\rho(t_\varepsilon \wedge \tau^\varepsilon \wedge \tau)} \phi(U^{(x)}(t_\varepsilon \wedge \tau^\varepsilon \wedge \tau)) \right] - E^x \left[ e^{-\rho(t_{j-1}(\tau^\varepsilon \wedge \tau)} \phi(U^{(x)}(t_{j-1}(\tau^\varepsilon \wedge \tau))) \right] = E^x \left[ \int_{t_{j-1}(\tau^\varepsilon \wedge \tau, \tau_{j}(\tau^\varepsilon \wedge \tau))} e^{-\rho s} \phi(U^{(x)}(s)) ds \right]. \quad (3.4)
$$

Moreover, for $j \geq 1$, we have

$$
e^{-\rho(t_\varepsilon \wedge \tau^\varepsilon \wedge \tau)} \phi(U^{(x)}(t_\varepsilon \wedge \tau^\varepsilon \wedge \tau)) - \phi(x) = \sum_{j=1}^n \left\{ e^{-\rho(t_j(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)} \phi(U^{(x)}(t_j(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)) - e^{-\rho(t_{j-1}(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)} \phi(U^{(x)}(t_{j-1}(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)) \right\}$$

$$= \sum_{j=1}^n \left\{ e^{-\rho(t_j(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)} \phi(U^{(x)}(t_j(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)) - e^{-\rho(t_{j-1}(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)} \phi(U^{(x)}(t_{j-1}(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau)) \right\}$$

$$+ \sum_{j=1}^n I_{[\tau_j(\tau^\varepsilon \wedge \tau) \wedge \tau^\varepsilon \wedge \tau]} e^{-\rho t_j} \left\{ \phi(U^{(x)}(t_j)) - \phi(U^{(x)}(t_j^-)) \right\}. \quad (3.5)
Substituting (3.4) into (3.5), together with (2.10), we get
\begin{align*}
\phi(x) + \sum_{j=1}^{n} E^x & \left[ I_{[\tau_j \leq \tau^\varepsilon]} \right] e^{-\rho \tau_j} \left[ \phi(U^{(\pi)}(\tau_j)) - \phi(U^{(\pi)}(\tau^-_j)) \right] \\
& - E^x \left[ e^{-\rho (\tau_n \wedge \tau^\varepsilon \wedge t)} \phi(U^{(\pi)}(\tau_n \wedge \tau^\varepsilon \wedge t)) \right] \\
& = -E^x \left[ \sum_{j=1}^{n} \int_{[\tau_j \leq \tau^\varepsilon \wedge \tau_n \wedge t]} e^{-\rho \tau} \phi(U^{(\pi)}(s)) ds \right] \geq 0. \quad (3.6)
\end{align*}

This inequality becomes an equality for the QVI control \( \pi^\phi \) associated with \( \phi \). By (2.11), we obtain
\begin{align*}
\phi(U^{(\pi)}(\tau^-_j)) & \geq M \phi(U^{(\pi)}(\tau_j)) \geq \phi(U^{(\pi)}(\tau^-_j)) - (1 + \lambda) \zeta_j - \zeta_j \\
& = \phi(U^{(\pi)}(\tau_j)) + \zeta_j. \quad (3.7)
\end{align*}

Therefore
\begin{align*}
\phi(U^{(\pi)}(\tau_j)) - \phi(U^{(\pi)}(\tau^-_j)) \leq -\zeta_j, \quad (3.8)
\end{align*}

and
\begin{align*}
\phi(x) - E^x \left[ e^{-\rho (\tau_n \wedge \tau^\varepsilon \wedge t)} \phi(U^{(\pi)}(\tau_n \wedge \tau^\varepsilon \wedge t)) \right] \geq E^x \left[ \sum_{j=1}^{n} I_{[\tau_j \leq \tau^\varepsilon \wedge \tau_n \wedge t]} e^{-\rho \tau_j} \zeta_j \right]. \quad (3.9)
\end{align*}

According to (3.3),
\begin{align*}
\lim_{t \to +\infty} \left\{ \phi(x) - E^x \left[ e^{-\rho (\tau_n \wedge \tau^\varepsilon \wedge t)} \phi(U^{(\pi)}(\tau_n \wedge \tau^\varepsilon \wedge t)) \right] \right\} \\
= \phi(x) - E^x \left[ e^{-\rho (\tau_n \wedge \tau^\varepsilon)} \phi(U^{(\pi)}(\tau_n \wedge \tau^\varepsilon)) \right]. \quad (3.10)
\end{align*}

For such controls, it is easy to show that \( \tau_n \to +\infty \) a.s. (otherwise, \( J^{(\pi)}(x) = -\infty \)). Therefore, \( P(\tau_n \to \infty) = 1 \) implies
\begin{align*}
\lim_{n \to \infty} \left\{ \phi(x) - E^x \left[ e^{-\rho (\tau_n \wedge \tau^\varepsilon)} \phi(U^{(\pi)}(\tau_n \wedge \tau^\varepsilon)) \right] \right\} = \phi(x) - E^x \left[ e^{-\rho \tau^\varepsilon} \phi(U^{(\pi)}(\tau^\varepsilon)) \right]. \quad (3.11)
\end{align*}

From (3.9)–(3.11), we get
\begin{align*}
\phi(x) - E^x \left[ e^{-\rho \tau^\varepsilon} \phi(U^{(\pi)}(\tau^\varepsilon)) \right] \geq E^x \left[ \sum_{j=1}^{\infty} I_{[\tau_j \leq \tau^\varepsilon]} e^{-\rho \tau_j} \zeta_j \right]. \quad (3.12)
\end{align*}
Hence, let $\varepsilon \to 0$ on both sides of Equation (3.12), for every admissible control $\pi \in \Pi$,

$$\phi(x) \geq J^{(x)}(x).$$

(3.13)

This inequality also becomes an equality for the QVI control associated with $\pi^0$. Thus we get $\phi(x) = \Phi(x)$.

4. THE SMOOTH SOLUTION TO THE QVI AND THE OPTIMAL POLICY

In this section, we are along the standard line of the optimization problem (see Schmidli\cite{17} or Framstad\cite{8}): first “guess” a solution; then we establish that the proposed solution is correct. To solve the QVI (2.10)–(2.13), inspired by Jeanblanc-Picque and Shiryave\cite{12}, we try a function of the form

$$\phi(x) = e^{rx}$$

for some constant $r \in \mathbb{R}$, and as a candidate for the continuous region $D$ we conjecture that

$$D = \{ x : 0 < x < x^* \},$$

(4.2)

where $x^* \in \mathbb{R}^+$ remains to be determined. Then from (2.12), it follows that

$$\mathcal{A}\phi(x) = -\rho \phi(x) + \mu \phi'(x) + \frac{1}{2} \sigma^2 \phi''(x) + \int_{\mathbb{R}^+} (\phi(x - z) - \phi(x))\nu(dz) = 0,$$

(4.3)

for $x \in D$. Plugging (4.1) into (4.3), we deduce that $r$ must solve the following equation:

$$h(r) := -\rho + \mu r + \frac{1}{2} \sigma^2 r^2 + \int_{\mathbb{R}^+} (e^{-rz} - 1)\nu(dz) = 0,$$

(4.4)

where $\nu$ is the Levy measure associated with the surplus process $U(t)$. Since $h(0) = -\rho < 0$, $\lim_{r \to +\infty} h(r) = +\infty$, and $\lim_{r \to -\infty} h(r) = +\infty$, we see that there exist two solutions $r_1$, $r_2$ of $h(r) = 0$ such that

$$r_2 < 0 < r_1.$$

(4.5)

Furthermore we have the following result.
Proposition 4.1. The roots of \( r_1 \) and \( r_2 \) satisfy

\[ |r_2| > r_1. \]  \hspace{1cm} (4.6)

Proof. Rewrite Equation (4.4) as

\[ h(r) := -\rho + (\mu - v(Q)) r + \frac{1}{2} \sigma^2 r^2 + \int_{R^+} (e^{-rz} - 1 + rz) v(dz) = 0, \]  \hspace{1cm} (4.7)

Note the assumption (2.3) and \( e^{-rz} - 1 + rz \geq 0 \) for \( z, r \), we have \( |r_2| > r_1 \).

With such choice of \( r_1, r_2 \), we try

\[ \phi(x) = A_1 e^{r_1 x} + A_2 e^{r_2 x}, \quad A_i \text{ constants.} \]  \hspace{1cm} (4.8)

Since \( \phi(0) = 0 \), we have \( A_1 + A_2 = 0 \). Thus Equation (4.8) can be represented as

\[ \phi(x) = A_1 (e^{r_1 x} - e^{r_2 x}), \quad 0 < x < x^*. \]  \hspace{1cm} (4.9)

Define

\[ \phi_0(x) = A_1 (e^{r_1 x} - e^{r_2 x}) \quad \text{for all } x > 0. \]  \hspace{1cm} (4.10)

By (2.11) we know that for \( x \geq x^* \), we have

\[ \phi(x) = M\phi_0(x). \]  \hspace{1cm} (4.11)

Motivated by Jeanblanc-Picque and Shiryaev\[12\], and Cadenillas et al.\[5\], we choose two thresholds \( \tilde{x}, x^* \) subject to \( \tilde{x} < x^* \), and whenever \( U(x) \) reaches \( x^* \), pay dividend \( x^* - \tilde{x} \). It remains to determine three constants \( A_1, x^*, \tilde{x} \). According to the definition of intervention operator:

\[ M\phi_0(x) = \sup \left\{ \phi_0(x - c - (1 + \lambda) \zeta) + \zeta : 0 < \zeta < \frac{x - \zeta}{1 + \lambda} \right\}. \]  \hspace{1cm} (4.12)

To study \( M\phi_0(x) \), we first let

\[ h_1(\zeta) = \phi_0(x - c - (1 + \lambda) \zeta) + \zeta. \]  \hspace{1cm} (4.13)

The first-order condition for a maximum point \( \zeta^* = \zeta^*(x) \) for \( h_1(\zeta) \) is that

\[ \phi_0'(x - c - (1 + \lambda) \zeta^*) - \frac{1}{1 + \lambda} = 0. \]
Suppose that there exists a unique point $\bar{x} \in (0, x^*)$ subject to
\[ \phi'_0(\bar{x}) = \frac{1}{1 + \lambda}. \]  
(4.14)

Then $\bar{x} = x - c - (1 + \lambda)\xi^*$ and from (4.11) we deduce that
\[ \phi(x) = \phi_0(\bar{x}) + x - \bar{x} - c \quad \text{for } x = x^*. \]  
(4.15)

In particular,
\[ \phi'(x^*) = \frac{1}{1 + \lambda} \]  
(4.16)

and
\[ \phi(x^*) = \phi_0(\bar{x}) + \frac{x^* - \bar{x} - c}{1 + \lambda}. \]  
(4.17)

Thus, we conclude that
\[ \phi(x) = \begin{cases} A_1(e^{\eta x} - e^{\eta \bar{x}}), & 0 < x < x^*, \\ \phi_0(\bar{x}) + \frac{x - \bar{x} - c}{1 + \lambda}, & x = x^*. \end{cases} \]  
(4.18)

where $\bar{x}, x^*, A_1$ are determined by Equations (4.14), (4.16), and (4.17).

Next, we will show that the system of three equations in three unknowns:
\[ A_1(r_1 e^{\eta \bar{x}} - r_2 e^{\eta x}), \]  
(4.19)
\[ A_1(r_n e^{\eta x^*} - r_2 e^{\eta \bar{x}}), \]  
(4.20)
\[ A_1(e^{\eta x^*} - e^{\eta \bar{x}}) = A_1(e^{\eta \bar{x}} - e^{\eta \bar{x}}) + \frac{x^* - \bar{x} - c}{1 + \lambda}, \]  
(4.21)
has a solution $A_1, \bar{x}, x^*$ such that $A_1 > 0$ and $0 < \bar{x} < x^*$. Let
\[ g(x) = A_1(r_1 e^{\eta x} - r_2 e^{\eta \bar{x}}) - \frac{1}{1 + \lambda}; \]  
(4.22)

then
\[ g'(x) = A_1(r_1^2 e^{\eta x} - r_2^2 e^{\eta \bar{x}}). \]  
(4.23)
We see that equation $g'(x) = 0$ has a unique solution:

$$\tilde{x} = \frac{2(\ln|z_2| - \ln|z_1|)}{r_1 - r_2}.$$  
\hfill (4.24)

Since

$$g''(x) = A_1 \left( r_1 \, e^{r_1 x} - r_2 \, e^{r_2 x} \right) > 0,$$  
\hfill (4.25)

and $\lim_{x \to -\infty} g(x) = +\infty$, it follows that $\tilde{x}$ is a minimum point for $g(x)$. If $g(\tilde{x}) < 0$, then $g(x) = 0$ has exactly two roots $\bar{x}$ and $x^*$ such that $0 < \bar{x} < \tilde{x} < x^*$. Note that $g'(\tilde{x}) = 0$, then

$$g(\tilde{x}) < 0 \iff A_1 (r_1 e^{r_1 \tilde{x}} - r_2 e^{r_2 \tilde{x}}) - \frac{1}{1 + \lambda} < 0$$

$$\iff A_1 < K := \frac{1}{1 + \lambda} \frac{1}{r_1 (1 - \frac{r_1}{r_2}) e^{r_1 \tilde{x}}}.$$  
\hfill (4.26)

From this we conclude that (4.19) and (4.20) have exactly two solutions: $\tilde{x} = \tilde{x}(A_1)$ and $x^* = x^*(A_1)$ such that

$$0 < \tilde{x}(A_1) < \tilde{x} < x^*(A_1),$$

if and only if $0 < A_1 < K$.

From now on we assume that $0 < A_1 < K$. To show that there exists $A_1 \in (0, K)$ such that the system (4.19), (4.20), and (4.21) holds, substituting (4.20) into (4.21), gives

$$\left( e^{r_1 x^*(A_1)} - e^{r_2 x^*(A_1)} \right) = e^{r_1 \tilde{x}(A_1)} - e^{r_2 \tilde{x}(A_1)}$$

$$+ (x^*(A_1) - \tilde{x}(A_1)) \left( r_1 e^{r_1 x^*(A_1)} - r_2 e^{r_2 x^*(A_1)} \right).$$  
\hfill (4.27)

Next, we show that Equation (4.27) wrt variable $A_1$ has a solution, subject to $A_1 \in (0, K)$.

**Proposition 4.2.** For all $c > 0$, there exists $A^* \in (0, K)$ such that (4.27) holds. With such a choice of $A = A^*$, the triple $A = A^*$, $\tilde{x} = \tilde{x}(A^*)$, $x^* = x^*(A^*)$ is a solution of the system (4.19)–(4.21).

**Proof.** Put

$$W(A_1) = e^{r_1 \tilde{x}(A_1)} - e^{r_2 \tilde{x}(A_1)} + (x^*(A_1) - \tilde{x}(A_1)) \left( r_1 e^{r_1 x^*(A_1)} - r_2 e^{r_2 x^*(A_1)} \right)$$

$$- \left( e^{r_1 x^*(A_1)} - e^{r_2 x^*(A_1)} \right).$$  
\hfill (4.28)
To study the solution of (4.27), we first consider the function $W(A_1)$ on $(0, K)$. Dividing (4.19) by (4.20), we find that

$$e^{r_1 \bar{x}(A_1)} - e^{r_1 x^*(A_1)} = \frac{r_2}{r_1} (e^{r_2 \bar{x}(A_1)} - e^{r_2 x^*(A_1)}).$$

(4.29)

Rearranging (4.28), we get

$$W(A_1) = \left(\frac{r_2}{r_1} - 1\right)(e^{r_2 \bar{x}(A_1)} - e^{r_2 x^*(A_1)}) + (x^*(A_1) - \bar{x}(A_1) - c)(r_1 e^{r_1 x^*(A_1)} - r_2 e^{r_2 x^*(A_1)}).$$

(4.30)

From construction of $\phi_0(x)$ and the mean value theorem with $\bar{x} = \bar{x}(A_1)$, $x^* = x^*(A_1)$, it follows that

$$\phi_0(x^*) - \phi_0(\bar{x}) = \phi_0(\xi(A_1))(x^* - \bar{x}) = \frac{x^* - \bar{x} - c}{1 + \lambda},$$

(4.31)

where $\xi(A_1) \in (\bar{x}(A_1), x^*(A_1))$. Since $\phi_0'(x) = r_1 e^{r_1 x} - r_2 e^{r_2 x} > 0$ and $x^*(A_1) - \bar{x}(A_1) > 0$ together with (4.31) we get that

$$x^*(A_1) - \bar{x}(A_1) - c > 0 \text{ for all } A_1 > 0.$$

From (4.19), (4.20), and (4.26), it is easy to see that

$$\lim_{A_1 \to 0} \bar{x}(A_1) \to +\infty, \quad \lim_{A_1 \to 0} x^*(A_1) \to +\infty,$$

(4.32)

$$\lim_{A_1 \to K} \bar{x}(A_1) = \lim_{A_1 \to K} x^*(A_1) = \bar{x}.$$

(4.33)

Combining (4.30), (4.32), and (4.33), we have that

$$\lim_{A_1 \to 0} W(A_1) = +\infty, \quad \lim_{A_1 \to K} W(A_1) = -c(r_1 e^{r_1 \bar{x}} - r_2 e^{r_2 \bar{x}}) < 0.$$  

(4.34)

Thus, for all $c > 0$, there exists $A^* \in (0, K)$ such that $W(A^*) = 0$.

Now, we can summarize this as follows.

**Theorem 4.1.** The optimal value function is

$$\Phi(x) = \begin{cases} A^*(e^{r_1 x} - e^{r_2 x}), & 0 < x < x^*, \\ \phi_0(\bar{x}) + \frac{x - \bar{x} - c}{1 + \lambda}, & x \geq x^*, \end{cases}$$

(4.35)
where \( A^*, \tilde{x}, x^* \) are given by Proposition 4.2, \( r_1 \) and \( r_2 \) are determined by Equation (4.4). And the control

\[
\pi^* = (\tau^*_1, \tau^*_n, \zeta^*_1, \zeta^*_n),
\]

defined by

\[
\tau^*_1 := \inf\{t \geq 0 : U^*_t = x^*\}, \quad (4.36)
\]
\[
\zeta^*_1 := x^* - \tilde{x}, \quad (4.37)
\]

and for every \( n \geq 2 \):

\[
\tau^*_n := \inf\{t > \tau_{n-1} : U^*_t = x^*\}, \quad (4.38)
\]
\[
\zeta^*_n := x^* - \tilde{x}, \quad (4.39)
\]

where

\[
U^*(t) = x + \mu t + \sigma B(t) - \int_0^{+\infty} \int_{R^+} zN(dt, dz) - (x^* - \tilde{x}) \sum_{n=1}^{\infty} I_{\{\tau^*_n < t\}}, \quad (4.40)
\]

is optimal.

**Proof.** From (4.18) we find that

\[
\phi'(x) = \begin{cases} A_1(n e^{t \cdot x} - r_2 e^{t \cdot \tilde{x}}), & 0 < x < x^*, \\ \frac{1}{1 - \lambda}, & x \geq x^*, \end{cases} \quad (4.41)
\]

and

\[
\lim_{x \to (x^*)^-} \phi'(x) = \lim_{x \to (x^*)^+} \phi'(x) = \frac{1}{1 - \lambda}. \]

We see that the function \( \phi(x) \) given by (4.18) belongs to \( C^1(0, +\infty) \). And from the above analysis, \( \phi(x) \) satisfies all the conditions of Theorem 3.1. Thus, it follows that

\[
\Phi(x) \leq \phi(x).
\]

In addition, it is easy to see that the control \( \pi^* \) is admissible and associated with \( \phi \). Therefore, applying Theorem 3.1, we conclude that \( \phi \) coincides with the optimal value function and \( \pi^* \) is the optimal policy.
5. THE EXPECTED TIME TO THE FIRST DIVIDEND PAYMENT

We have seen in Section 4 that the optimal surplus process is given by (4.40). This implies that the optimal reserve process behaves like a jump-diffusion in the region \((0, x^*)\). Thus the ultimate ruin probability occurs with certain, i.e., \(P(\tau < +\infty) = 1\). We have also seen in Section 4 that the optimal dividend payment to be distributed each time is equal to \(x^* - \bar{x}/\rho\). In general, it is of interest to know not only the amount of the dividend payments but also their interarrival times. Here we consider the expected time to the first dividend payment. If at time 0, the value of the cash reserve is \(x \in (0, x^*)\), then the expected time to the first dividend payment is given by

\[
E^x[ e^{-\rho \tau_1}],
\]

where

\[
\tau_1 = \inf\{t \in [0, +\infty) : U^*(t) \notin (0, x^*)\}.
\]

For simplicity, let \(g(x) = E^x[ e^{-\rho \tau_1}]\). The next result will give a sufficient condition to find \(g(x)\).

**Theorem 5.1.** Let \(g(x)\) be bounded and twice continuously differentiable on \((0, x^*)\) with a bounded first order derivative there. If \(g(x)\) solves

\[
\mathbb{A} g(x) = 0 \quad \text{on } x \in D
\]

(5.1)

together with the following boundary condition

\[
g(x^*) = 1,
\]

then \(g(x)\) can be represented as \(E^x[ e^{-\rho \tau_1}]\).

**Proof.** Note that under the positive safety loading assumption (2.3), \(\tau_1\) is a.s. finite, i.e., \(P(\tau_1 < +\infty) = 1\). By Itô’s formula (see Section 3.3, Karatzas) integrating from 0 to \(t \land \tau_1\) and taking expectation, gives

\[
E[ e^{-\rho (t \land \tau_1)} g(U^*(t \land \tau_1))] = g(x) + E\left[ \int_0^{t \land \tau_1} e^{-\rho s} \mathbb{A} g(U^*(s)) ds \right]
\]

\[
+ E\left[ \int_0^{t \land \tau_1} e^{-\rho s} \sigma g'(U^*(s)) dB(s) \right] \quad (5.2)
\]

and there is no dividend payout when \(x < x^*\); hence

\[
E\left[ \int_0^{t \land \tau_1} e^{-\rho s} \sigma g'(U^*(s)) dB(s) \right] = 0.
\]
Letting $t \to 0$ on both sides of (5.2), we have
\[
g(x) = E_x[e^{-\rho t}].
\]

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