1. Mastering Math - High School Math Review

This math review is intended to prepare students for the first-year math used in the core engineering courses. This material is accompanied by a corresponding review video posted on the First Year Engineering YouTube channel. This worksheet will cover a review of cross product, an engineering application, and a practice problem set. This review was developed by John O'Keefe along with the Elsie MacGill Learning Centre.

1.1 Cross Product

Cross products are specific to a 3-D space, and can not be performed in 2-D. If two 3-D vectors are considered: $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$. The 3-D cross product can be calculated as follows:

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k} \tag{1.1}$$

Where $\vec{A} \times \vec{B}$ denotes the cross product between the vectors $\vec{A}$ and $\vec{B}$. Typically the method of determinant is used to determine the order of multiplication but this will be covered in a later section. The cross product produces a vector that is orthogonal to both of the original vectors per the right hand rule as shown below:

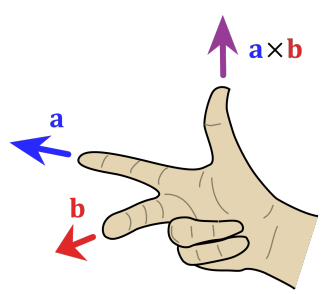

Figure 1.1: Right Hand Rule [5]

Where the direction of $\vec{A}$ is placed along the index finger, $\vec{B}$ along the middle finger, and the thumb produces the resultant direction. It's important to remember that a cross product produces a vector, whereas a dot product produces a scalar value. The cross product can be related to the magnitude of the vectors and the angle between the two vectors as follows:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|sin(\theta) \tag{1.2}$$

Where $|\vec{A}|$ and $|\vec{B}|$ are the magnitudes of each vector and θ is the angle between the two vectors. The cross product is a maximum when $\theta = 90°$ as $\sin(\theta) = 1$. The cross product is equivalent to zero when the two vectors are parallel since $\sin(\theta) = 0$ when $\theta = 0°$. This expression provides the magnitude of the cross product, to determine the direction more information is required. The direction of the resultant vector must be calculated through the right hand rule. The cross product can be visualized as the area of a parallelogram as shown below:

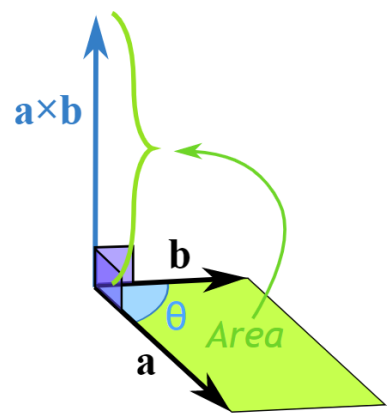

Figure 1.2: Area of a parallelogram [2]

Where $\vec{a}$ and $\vec{b}$ create the side lengths of the parallelogram. The area is equivalent to the cross product, $\vec{a} \times \vec{b}$. If the vectors are parallel, the area of the parallelogram is zero. If the vectors are perpendicular they form a rectangle which maximizes the area.

1.1.1 Examples

1. Calculate the cross product between the 2-D vectors $\vec{A} = 2\hat{i}+3\hat{j}+0\hat{k}$ and $\vec{B} = -\hat{i}+2\hat{j}+0\hat{k}$.

Solution: The cross product can not be used to evaluate a 2-D vectors, so we must represent them in a 3-D space. We can apply Eq. 1.1 to calculate the cross product:

$$\vec{A} \times \vec{B} = (3)(0) - (0)(2))\hat{i} + ((0)(-1) - (2)(0))\hat{j} + ((2)(2) - (3)(-1)))\hat{k}$$
$$\vec{A} \times \vec{B} = 7\hat{k}$$

2. Calculate the cross product between the vectors: $\vec{A} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = 2\hat{i} + \hat{j} + 5\hat{k}$.

Solution: We can apply Eq. 1.1 to calculate the cross product:

$$\vec{A} \times \vec{B} = ((3)(5) - (1)(1))\hat{i} + ((1)(2) - (1)(5))\hat{j} + ((1)(1) - (3)(2))\hat{k}$$
$$\vec{A} \times \vec{B} = 14\hat{i} - 3\hat{j} - 5\hat{k}$$

3. Calculate the cross product of two vectors with magnitudes $|\vec{F}| = 200$ and $|\vec{d}| = 5$, with an angle of 30° between them.

Solution: We can apply Eq. 1.2 to calculate the magnitude of the cross product:

$$|\vec{F} \times \vec{d}| = (200)(5)\sin(30)$$
$$|\vec{F} \times \vec{d}| = 500$$

More information is required to calculate the direction.

4. Calculate the cross product between $\vec{C} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{D} = -2\hat{i} - 6\hat{j} + 4\hat{k}$.

Solution: We can apply Eq. 1.1 to calculate the cross product:

$$\vec{C} \times \vec{D} = ((3)(4) - (-2)(-6))\hat{i} + ((-2)(-2) - (1)(4))\hat{j} + ((1)(-6) - (3)(-2))\hat{k}$$
$$\vec{C} \times \vec{D} = 0$$

The result is zero since these two vectors are parallel.

1.1.2 Engineering Examples

The main application of the cross product is to determine the moment produced by a force in Statics (ECOR 1045). A moment is a technical term for torque and will be used throughout engineering. To calculate the moment (torque) of a force the following equation is used:

$$\vec{M} = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}|\sin(\theta) \tag{1.3}$$

Where $\vec{M}$ is the moment produced, $\vec{r}$ is the vector connecting the force to the moment, and $\vec{F}$ is the force vector.

1. Determine the moment produced on the nut, P, by the wrench.

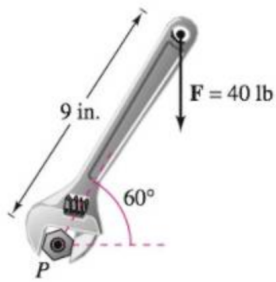

Figure 1.3: Engineering example 1 [4]

Solution: We are given the magnitudes of the two vectors and the angle between them is $\theta = 30°$. We can calculate the moment by either $\vec{M} = \vec{r} \times \vec{F}$ or $\vec{M} = |\vec{r}||\vec{F}|\sin(\theta)$. Both approaches will be taken, starting with the first. We must convert both r and F into vectors:

$$\vec{r} = 9(\cos(60)\hat{i} + \sin(60)\hat{j})$$
$$\vec{r} = 4.5\hat{i} + 7.8\hat{j}$$

$$\vec{F} = -40\hat{j}$$

Then the moment can be calculated as:

$$\vec{M} = \vec{r} \times \vec{F}$$
$$\vec{M} = [(4.5)(0) - (0)(-40)]\hat{i} + [(0)(0) - (4.5)(0)]\hat{j} + [(4.5)(-40) - (7.8)(0)]\hat{k}$$
$$\vec{M} = -180\hat{k}\ lb\ in$$

Since the direction is $-\hat{k}$, this is a vector into the page which produces a clockwise moment of 180 lb in. Next the second approach is taken using the magnitudes of the vectors and the angle between them as follows:

$$|\vec{M}| = |\vec{r}||\vec{F}|\sin(\theta)$$
$$|\vec{M}| = (9)(40)\sin(30)$$
$$|\vec{M}| = 180\ lb\ in$$

The direction can be calculated using the right hand rule as going into the page. The direction vector for this is $(-\hat{k})$, therefore $\vec{M} = -180\hat{k}\ lb\ in$.

2. Determine the moment produced by the 100 N force about point O.

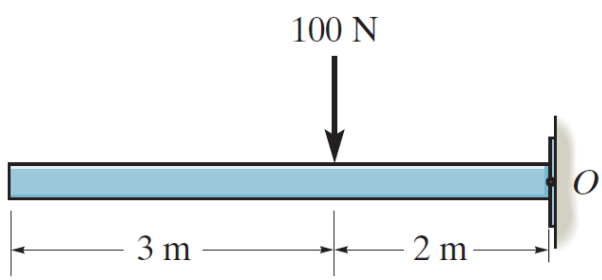

Figure 1.4: Engineering example 2 [1]

Solution: We can calculate the cross product by either method again. Starting off using the vectors of $\vec{r}$ and $\vec{F}$:

$$\vec{r} = -2\hat{i}$$

$$\vec{F} = -100\hat{j}$$

Then the moment can be calculated as:

$$\vec{M} = \vec{r} \times \vec{F}$$
$$\vec{M} = [(0)(0) - (0)(-100)]\hat{i} + [(0)(0) - (-2)(0)]\hat{j} + [(-2)(-100) - (0)(0)]\hat{k}$$
$$\vec{M} = 200\hat{k}\ N\ m$$

Since the direction is $\hat{k}$, this is a vector out of the page, this produces a counterclockwise moment of $200Nm$. Next the second approach is taken using the magnitudes of the vectors and the angle between them as follows:

$$|\vec{M}| = |\vec{r}||\vec{F}|\sin(\theta)$$
$$|\vec{M}| = (2)(100)\sin(90)$$
$$|\vec{M}| = 200\ N\ m$$

The direction can be calculated using the right hand rule as going out of the page. The direction vector for this is $(\hat{k})$, therefore $\vec{M} = 200\hat{k}\ N\ m$.

1.1.3 Practice Problems

1. Calculate the cross product of: $\vec{A} = \hat{i} + 3\hat{j}$ and $\vec{B} = -2\hat{i} - 2\hat{j}$.

2. Calculate the cross product of: $\vec{C} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{D} = 2\hat{i} - \hat{j} + \hat{k}$.

3. Calculate the cross product between $\vec{M} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{N} = 2\hat{i} + 2\hat{j} + \hat{k}$.

4. Calculate the magnitude of the cross product with vectors $|\vec{F}| = 25$ and $|\vec{X}| = 3$ and an angle of $\theta = 70°$ between the two vectors.

5. Calculate the magnitude of the moment (torque) if $|\vec{r}| = 2$ m, $|\vec{F}| = 30$ N and $\theta = 30°$.

6. Calculate the moment of: $\vec{r} = 2\hat{i}\ m$ and $\vec{F} = 26\hat{i} + 15\hat{j}\ N$.

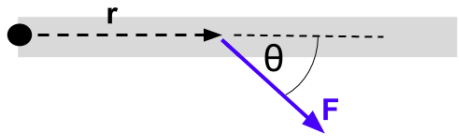

Figure 1.5: Problem 6 [3]

7. Calculate the moment about O.

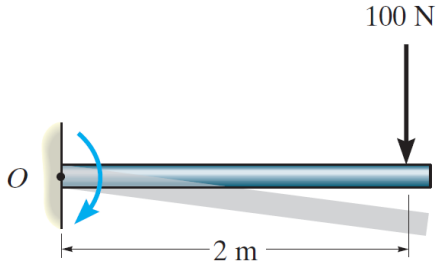

Figure 1.6: Problem 7 [1]

1.1.4 Answers

1. $-8\hat{k}$

2. $3\hat{i} - 6\hat{k}$

3. $-3\hat{i} + 6\hat{k}$

4. 70.5

5. $30\ Nm$

6. $-30\hat{k}\ Nm$

7. $-200\hat{k}\ Nm$

Bibliography

[1] R C Hibbeler. *Engineering Mechanics Statics 14th Edition*. Pearson, 2016.

[2] Math is Fun. Cross product. https://www.mathsisfun.com/algebra/vectors-cross-product.html, 2017. Accessed on 2020-06-12.

[3] Chris Kirkpatrick. *Calculus and Vectors*. Nelson, 2016.

[4] James Stewart. *Calculus, Early Transcendental Functions - 7th edition*. Cengage Learning, 2019.

[5] Wikimedia. Right hand cross product. https://commons.wikimedia.org, July 2008. Accessed on 2020-06-12.