MAAE 3004
Dynamics of Machinery

Lecture Slide Set 1

Introduction and

Free Vibration of Single Degree of Freedom Systems

Department of Mechanical and Aerospace Engineering
Carleton University

Outline

Introduction

Objective:

"The theory of machines and mechanisms is an applied science which is used to understand the relationship between the geometry and motions of the parts of a machine or mechanism and the forces which produce these motions"

Uicker, Pennock, and Shigley, *Theory of Machines and Mechanisms*, 5^{th} edition, Oxford, 2017.

Evaluation:

All quizzes and Labs must be performed in your registered Lab Section

Homework problems assigned weekly on Brightspace	0%
Math quiz online **Opens Friday, September 22, 8:30 am** **Closes Monday, September 25, 12:00 pm**	5%
Two course content quizzes:	
1. Week of October 16-20, Vibration	10%
2. Week November 27 - December 1, Kinematics Quizzes will take place in Lab (PA Session)	10%
Lab 1: vibration experiments, in Lab (PA Session) October 2-6 **Report due online, Friday October 20, 12:00 pm**	10%
Lab 2: kinematics, in Lab (PA Session) October 30 - November 3 **Report due online, Friday December 1, 12:00 pm, and** **5 minute presentation in Lab (PA session) November 13-17**	10%
Final Exam, **Exam Period December 10-22, 2023**	55%
Total	100%

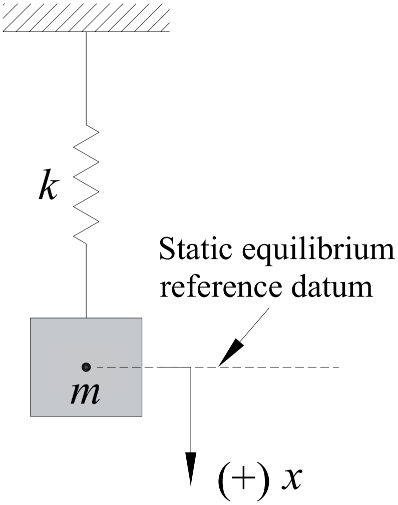
References

1. S.S. Rao, *Mechanical Vibrations,* 6th *Edition*, Pearson, 2017
2. W.T. Thompson, M.D. Dahleh, *Theory of Vibration with Applications*, 5th Edition, Prentice-Hall, 1998
3. M.L James, G.M. Smith, J.C. Wolford, P.W. Whaley, *Vibration of Mechanical and Structural Systems*, Harper & Row, 1989
4. J.J. Uicker, G.R. Pennock, and J.E. Shigley, *Theory of Machines and Mechanisms*, 5th Edition, Oxford, 2017
5. J.L Meriam, L.G. Kraige, J.N. Bolton, *Engineering Mechanics: Dynamics*, 9th Edition, Wiley, 2018
6. R.C. Hibbeler, *Engineering Mechanics: Dynamics*, 14th Edition, Pearson, 2016
7. J.M. McCarthy, G.S. Soh, *Geometric Design of Linkages*, 2nd Edition, Springer, 2010

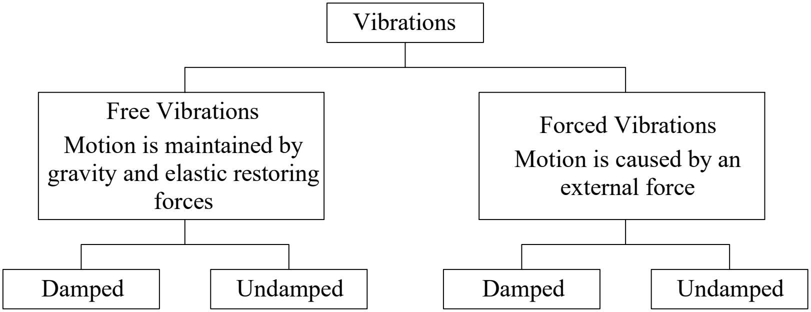
Mass, Damper, Coil Spring

Vibration

- Vibration refers to the periodic motion of a mechanical system of connected bodies about the system's equilibrium position.
- The frequency at which a mechanical system vibrates when displaced from it's equilibrium position and the released is called *natural frequency*.
- All mechanical systems contain some inherent property that dissipates energy, referred to as *damping*.
- The magnitude of the damping has no effect on the natural frequency.



Vibration

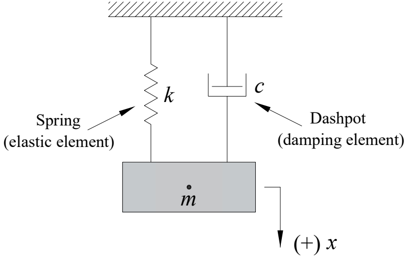

Vibrations
Free Vibrations
Motion is maintained by gravity and elastic restoring forces
Forced Vibrations
Motion is caused by an external force
Damped
Undamped
Damped
Undamped

Vibration

- As we will see, a general differential equation of motion for a system of masses, elastic, and damping elements is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad (1)$$

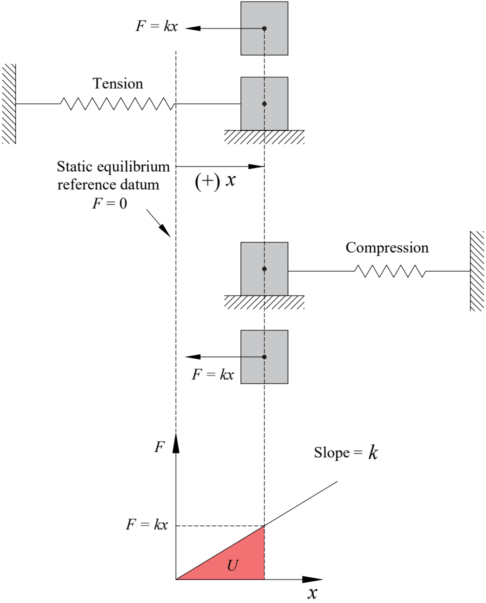
where



x	$=$	displacement of the mass from the static equilibrium position,
$\dot{x}$	$=$	velocity of the mass,
$\ddot{x}$	$=$	acceleration of the mass,
ζ	$=$	the damping ratio,
ω_n	$=$	the undamped natural circular frequency.

- The damping ratio ζ depends on the damping mechanism(s) and mechanical system parameters such as mass and geometry.
- The natural circular frequency ω_n depends on the mechanical system parameters mass and stiffness.

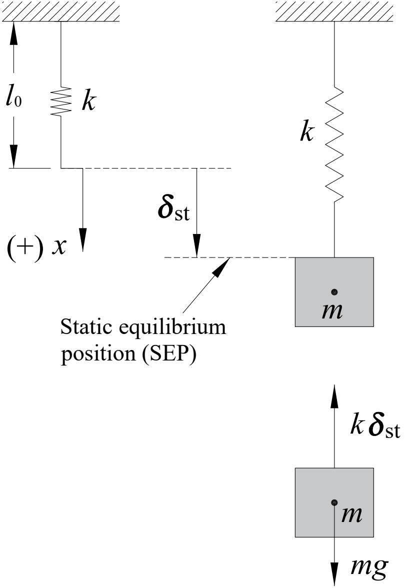
Basic Concepts



Spring Elements (Linear)

- The restoring force of a spring is always directed towards the static equilibrium position.
- Spring constant: $k \left[\frac{\text{N}}{\text{m}}\right]$
- Force: $F = kx$ [N]
- Work: $U = \frac{1}{2}kx^2$ [Nm] or [J] (work, strain, or potential energy)

Static Deflection



From the free-body diagram Newton's second law gives

$$mg - k\delta_{\mathrm{st}} = m\ddot{x}$$

At the static equilibrium position $x = 0$ the force sum must be zero, so that

$$mg - k\delta_{\mathrm{st}} = 0 \Rightarrow \delta_{\mathrm{st}} = \frac{mg}{k}$$

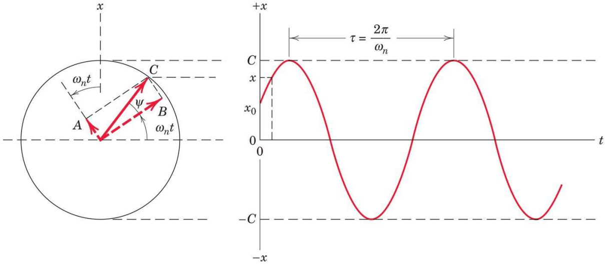
and the equation of motion is

$$m\ddot{x} + kx = 0$$

which is generally written as

$\ddot{x} + \omega_n^2 x = 0$, where $\omega_n = \sqrt{k/m}$.

Undamped Natural Frequency



- The undamped natural frequency f_n can be approximated empirically, but what is measured is the damped natural frequency f_d, but $f_n \approx f_d$ and

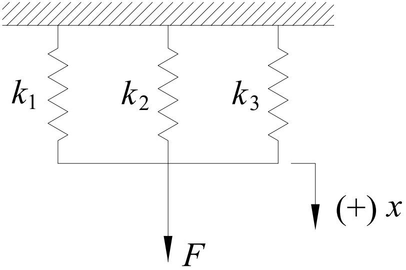
$$f_n = \frac{\omega_n}{2\pi} = \frac{\sqrt{k/m}}{2\pi} \left[\frac{\text{rad/s}}{\text{rad}}\right] = \left[\frac{\text{cycles}}{\text{s}}\right] = [\text{Hz}]$$

- The undamped natural period τ_n is

$$\tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{k/m}} \left[\frac{\text{s}}{\text{cycle}}\right]$$

Equivalent Springs

Springs in parallel



- The springs in a mechanical system can be in parallel, series, or in combination.
- When springs are in parallel, the deformation of each spring is the same for a given applied force.
- The reaction forces of the three springs are

$$\begin{aligned} F_1 &= k_1 x \\ F_2 &= k_2 x \\ F_3 &= k_3 x \end{aligned}$$

- The sum of these three forces must be equal in magnitude to the applied force, therefore

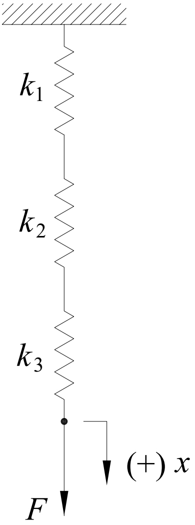
$$F = k_1 x + k_2 x + k_3 x = (k_1 + k_2 + k_3) x = k_{eq} x$$

- For n springs in parallel the equivalent spring constant is

$$k_{eq} = \sum_{i=1}^{n} k_i$$

Equivalent Springs

Springs in series



- When springs are in series, the force in each spring is the same as the given applied force.
- The total deformation x of the springs is the sum of the individual deformations.
- Thus, with

$$F = k_1 x_1 = k_2 x_2 = k_3 x_3$$

and

$$x = x_1 + x_2 + x_3$$

we find that

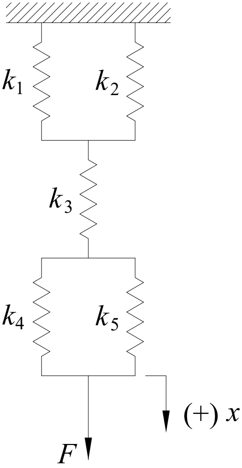
$$x = F\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)$$

- The equivalent spring constant for springs in series is

$$k_{eq} = \frac{1}{\sum_{i=1}^{n} \frac{1}{k_i}}, \text{ or } \frac{1}{k_{eq}} = \sum_{i=1}^{n} \frac{1}{k_i}$$

Equivalent Springs

Parallel and series combination

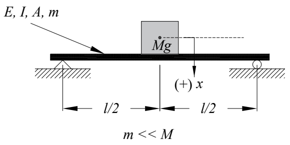


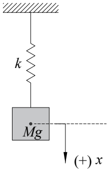
- When springs are in combinations of series and parallel, a general procedure for determining the equivalent stiffness is to first determine k_{eq} for parallel combinations in the mechanical system, and then combine them with the series elements to obtain yet another k_{eq}.
- For the parallel/series combination in the figure, the equivalent spring constant is

$$\frac{1}{k_{eq}} = \frac{1}{k_1 + k_2} + \frac{1}{k_4 + k_5} + \frac{1}{k_3}$$

Elastic Elements as Springs

Cantilevered Elastic Beams





actual system

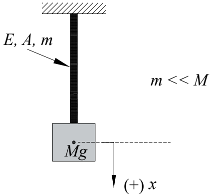
$$\delta_{st} = \frac{Mg\mathrm{l}^3}{48EI}$$

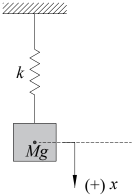
model

$$\delta_{st} = \frac{Mg}{k}$$

$$k = \frac{Mg}{\delta_{st}} = \frac{48EI}{\mathrm{l}^3}$$

Elastic Bars





actual system

$$\delta_{st} = \frac{Mg\mathrm{l}}{AE}$$

model

$$\delta_{st} = \frac{Mg}{k}$$

$$k = \frac{Mg}{\delta_{st}} = \frac{AE}{\mathrm{l}}$$

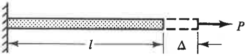
δ_{st} [m]:	static deflection
M [kg]:	applied mass
m [kg]:	beam mass
l [m]:	cantilever length
E [Pa]:	Young's modulus of elasticity (σ/ϵ)
I [m^4]:	second moment of cross-section area
A [m^2]:	cross-section area

Elastic Elements as Springs

TABLE 2-1 SPRING CONSTANTS AND DEFLECTION EQUATIONS OF ELASTIC ELEMENTS

A = area of cross section
E = modulus of elasticity
I = area moment of inertia about neutral axis
G = modulus of rigidity
J = polar moment of inertia

Axial (rods, cables, etc.)

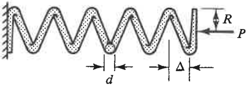


(a)

$$\Delta = \frac{Pl}{AE}$$

$$k = \frac{P}{\Delta} = \frac{AE}{l}$$

Coil spring



(b)

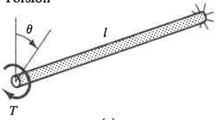
$$\Delta = \frac{64PnR^3}{Gd^4}$$

$$k = \frac{P}{\Delta} = \frac{Gd^4}{64nR^3}$$

n = number of active coils

R = mean helix radius

Torsion



(c)

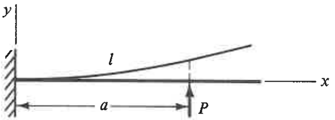
$$\theta = \frac{Tl}{GJ}$$

$$k = \frac{T}{\theta} = \frac{GJ}{l}$$

$$J = \frac{\pi d^4}{32} \ (d = \text{dia.})$$

Elastic Elements as Springs

Cantilever beam



(d)

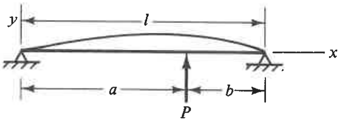
$$k = \frac{P}{y|_{x=a}}$$

$$k|_{a=l} = \frac{3EI}{l^3}$$

$$y = \frac{P}{6EI}(3ax^2 - x^3) \qquad x \le a$$

$$y = \frac{P}{6EI}(3a^2x - a^3) \qquad x \ge a$$

Simply supported beam (pinned-pinned)*



(e)

$$k = \frac{P}{y|_{x=a}}$$

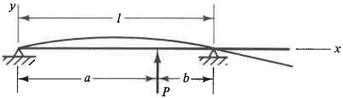
$$k|_{a=l/2} = \frac{48EI}{l^3}$$

$$y = \frac{Pbx}{6EIl}(l^2 - x^2 - b^2) \qquad x \le a$$

$$y = \frac{Pb}{6EIl}\left[(l^2 - b^2)x - x^3 + \frac{l}{b}(x-a)^3\right] \qquad x \ge a$$

Elastic Elements as Springs

Pinned-pinned beam with overhang*

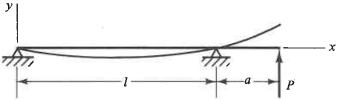


(f)

$$y = \frac{Pa}{6EIl}(a^2 - l^2)(x - l) \qquad x \geq l$$

Pinned-pinned beam with overhang (P at $x = l + a$)*

$$k = \frac{P}{y|_{x=a+l}} = \frac{3EI}{a^2(a + l)}$$

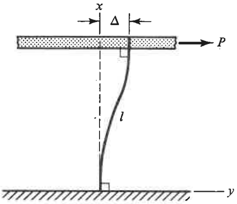


(g)

$$y = \frac{Pax}{6EIl}(x^2 - l^2) \qquad x \leq l$$

$$y = \frac{P}{6EIl}[ax(x^2 - l^2) - (l + a)(x - l)^3] \qquad x \geq l$$

Fixed-fixed beam with lateral displacement



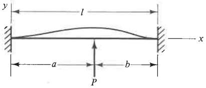
$$\Delta = \frac{Pl^3}{12EI}$$

$$k = \frac{12EI}{l^3}$$

$$y = \frac{P}{12EI}(3lx^2 - 2x^3)$$

Elastic Elements as Springs

Fixed-fixed beam*



(i)

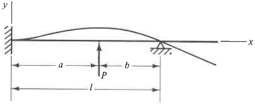
$$k = \frac{P}{y|_{x=a}}$$

$$k|_{a=l/2} = \frac{192EI}{l^3}$$

$$y = \frac{Pb^2}{6EIl^3}[(2b - 3l)x^3 + 3l(l - b)x^2] \qquad (x \le a)$$

$$y = \frac{Pb^2}{6EIl^3}\left[(2b - 3l)x^3 + 3l(l - b)x^2 + \frac{l^3}{b^2}(x - a)^3\right] \qquad (x \ge a)$$

Fixed-pinned beam with overhang*



(j)

$$k = \frac{P}{y|_{x=a}}$$

$$k|_{a=l/2} = \frac{768EI}{7l^3}$$

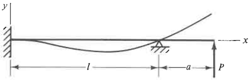
$$y = \frac{P}{12EI}\left[3b\left(1 - \frac{b^2}{l^2}\right)x^2 - \frac{b}{l}\left(3 - \frac{b^2}{l^2}\right)x^3\right] \qquad x \le a$$

$$y = \frac{P}{12EI}\left[3b\left(1 - \frac{b^2}{l^2}\right)x^2 - \frac{b}{l}\left(3 - \frac{b^2}{l^2}\right)x^3 + 2(x - a)^3\right] \qquad a \le x \le l$$

$$y = \frac{-pba^2}{4EIl}(x - l) \qquad x \ge l$$

Fixed-pinned beam with overhang (P at $x = l + a$)*

$$k = \frac{P}{y|_{x=l+a}} = \frac{12EI}{a^2(3l + 4a)}$$



(k)

$$y = \frac{Pa}{4EIl}(x^3 - lx^2) \qquad x \le l$$

$$y = \frac{Pa}{4EIl}\left[x^3 - lx^2 - \left(\frac{2l}{3a} + 1\right)(x - l)^3\right] \qquad x \ge l$$

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* Axial extensions due to axial end constraints considered negligible.

Example 1.1

Given the hoisting drum that is mounted at the end of a rectangular cross-section cantilever beam and carrying a steel wire cable, determine the k_{eq} of the system. The cable length $= \mathrm{l}$ and the beam and cable have a Young's modulus $=$ E.

For a cantilever beam:

$$\delta_{max} = \frac{Wb^3}{3EI} \Rightarrow k_b = \frac{W}{\delta_{max}}$$

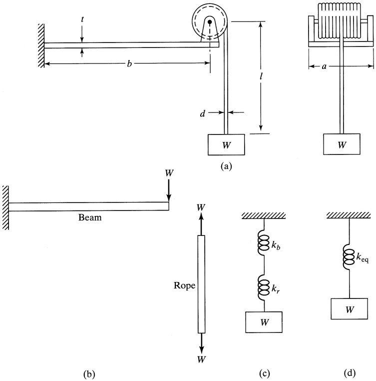
$$k_b = \frac{3EI}{b^3} = \frac{3E}{b^3}\left(\frac{1}{12}at^3\right) = \frac{Eat^3}{4b^3}$$

For a cable: $k_c = \frac{AE}{\mathrm{l}} = \frac{\pi d^2 E}{4\mathrm{l}}$

k_b and k_c are in series,

$$\frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_c} = \frac{4b^3}{Eat^3} + \frac{4\mathrm{l}}{\pi d^2 E}$$

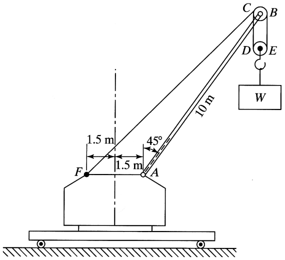
Therefore, $k_{eq} = \frac{E}{4}\left(\frac{\pi a t^3 d^2}{\pi d^2 b^3 + \mathrm{l}at^3}\right)$



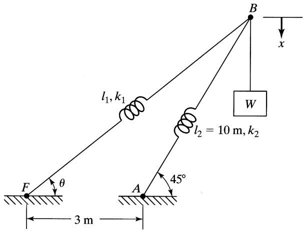
Example 1.2

Consider the crane as shown.
Boom AB: uniform steel bar ($E = 207 \times 10^9$ Pa) with $A_2 = 2500\ \mathrm{mm}^2$
Cable FCBED: steel ($E = 207 \times 10^9$ Pa), $A_1 = 100\ \mathrm{mm}^2$
Effects of cable CBED: negligible

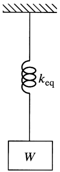
Determine k_{eq} in the vertical direction.



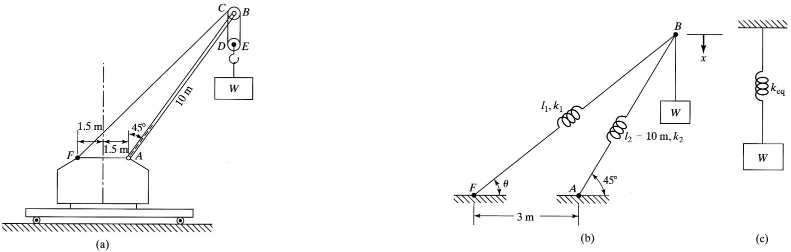
(a)



(b)



(c)



Use equivalence of potential energy of the actual system and the model

$$\mathrm{l_1}^2 = 3^2 + 10^2 - 2(3)(10)\cos 135^o \quad \Rightarrow \quad \mathrm{l_1} = 12.31 \text{ m}$$

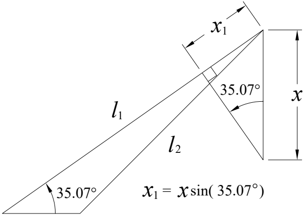
Also,

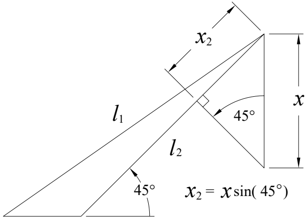
$$10^2 = (12.31)^2 + 3^2 - 2(12.31)(3)\cos\theta \quad \Rightarrow \quad \theta = 35.07^o$$

$$k_1 = \frac{A_1 E_1}{\mathrm{l_1}} = \frac{(100 \times 10^{-6} \text{ m}^2)(207 \times 10^9 \text{ N/m}^2)}{12.31 \text{ m}} = 1.68 \times 10^6 \text{ N/m}$$

$$k_2 = \frac{A_2 E_2}{\mathrm{l_2}} = \frac{(2500 \times 10^{-6} \text{ m}^2)(207 \times 10^9 \text{ N/m}^2)}{10 \text{ m}} = 5.175 \times 10^7 \text{ N/m}$$

U = Potential Energy of the system for displacement x in the vertical direction





Therefore,

$U = \frac{1}{2}k_1{x_1}^2 + \frac{1}{2}k_2{x_2}^2$ [Nm]

$U = \frac{1}{2}(1.68 \times 10^6)(xsin35.07^o)^2 + \frac{1}{2}(5.175 \times 10^7)(xsin45^o)^2$

Also for the model

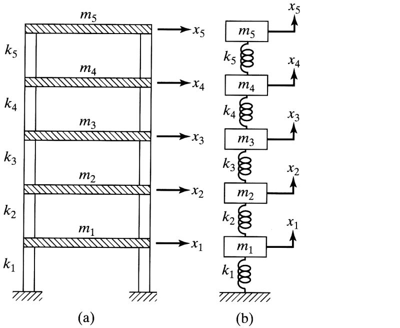
$U = \frac{1}{2}k_{eq}x^2$

$k_{eq} = 1.68 \times 10^6 sin^2 35.07^o + 5.175 \times 10^7 sin^2 45^o$
$k_{eq} = 26.43 \times 10^6 \text{ N/m}$

Mass or Inertia Elements

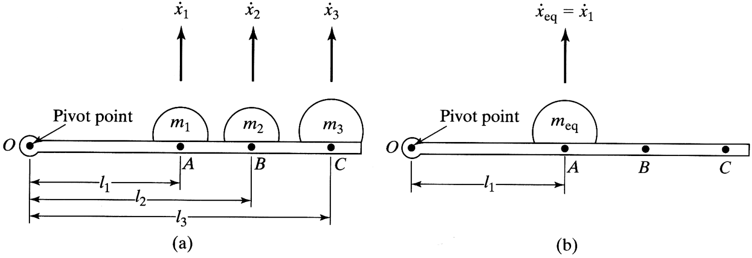
Mass and inertia elements are rigid bodies that gain or lose kinetic energy.

- Combination of masses:
  - several possible models can exist
  - appropriate model is often determined by the purpose of analysis
  - equivalent mass, m_{eq}, is determined by equating the kinetic energy of the actual system with the model



S.S. Rao.*Mechanical Vibrations*. Pearson Education Inc., New Jersey, United States, 4th edition, 2004.

Case 1: Translational masses (connected by a rigid massless bar)



S.S. Rao. *Mechanical Vibrations*. Pearson Education Inc., New Jersey, United States, 4th edition, 2004.

Kinetic energy of the system = T [Nm]

$T = \frac{1}{2} m_1 \dot{x}_1{}^2 + \frac{1}{2} m_2 \dot{x}_2{}^2 + \frac{1}{2} m_3 \dot{x}_3{}^2$

Assume we need m_{eq} at A; then

$T = \frac{1}{2} m_{eq} \dot{x}_1{}^2$

Then

$\frac{1}{2} m_{eq} \dot{x}_1{}^2 =$

$\frac{1}{2} m_1 \dot{x}_1{}^2 + \frac{1}{2} m_2 \dot{x}_2{}^2 + \frac{1}{2} m_3 \dot{x}_3{}^2$

but $\dot{x}_2 = \frac{l_2}{l_1} \dot{x}_1$ and $\dot{x}_3 = \frac{l_3}{l_1} \dot{x}_1$

Therefore

$m_{eq} = m_1 + (\frac{l_2}{l_1})^2 m_2 + (\frac{l_3}{l_1})^2 m_3$

Case 2: Coupled translational and rotational masses

a) Equivalent translational mass: m_{eq}

The rotational mass moment of inertia is $\bar{I}$ [Nm]

Actual system: $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2$

Model: $T = \frac{1}{2}m_{eq}\dot{x}^2$

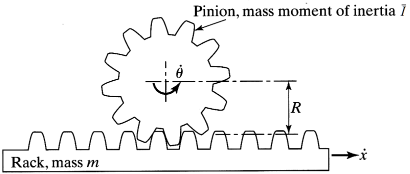
Therefore,

$$\frac{1}{2}m_{eq}\dot{x}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2$$

but $\dot{\theta} = \dfrac{\dot{x}}{R}$

Therefore,

$$m_{eq} = m + \frac{\bar{I}}{R^2}$$



S.S. Rao. *Mechanical Vibrations*. Pearson Education Inc., New Jersey, United States, 4th edition, 2004.

b) Equivalent rotational mass moment: $\overline{I}_{eq}$

Model: $T = \frac{1}{2}\overline{I}_{eq}\dot{\theta}^2$

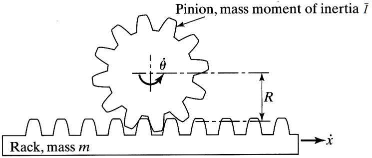
Therefore,

$$\frac{1}{2}\overline{I}_{eq}\dot{\theta}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\overline{I}\dot{\theta}^2$$

but $\dot{x} = R\dot{\theta}$

Therefore,

$$\overline{I}_{eq} = mR^2 + \overline{I}$$



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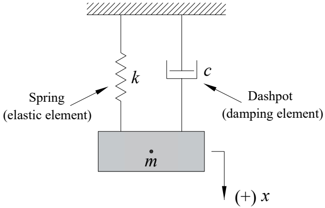
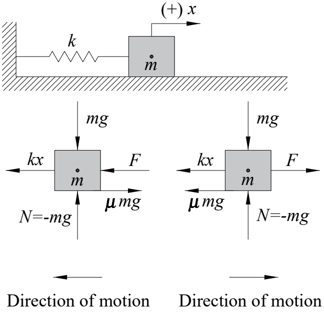
Damping Elements

Convert the vibrational energy into heat or sound in a gradual manner.

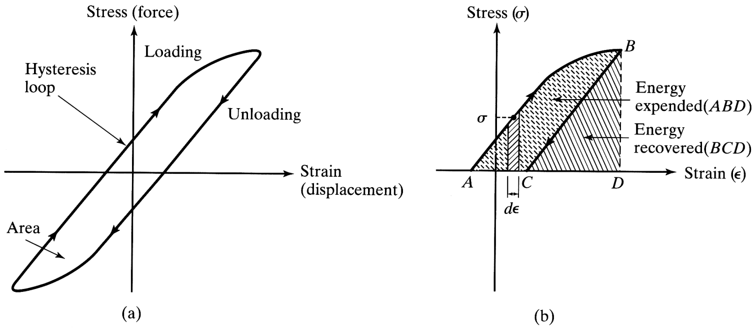
Damping Models

- Viscous Damping
  - when vibrating in a fluid medium the damping is scaled by the velocity
  - $F_d = cv = c\dot{x}$
  - The viscous damping coefficient is $c \left[\frac{Ns}{m}\right]$
  - examples:
    - fluid film between sliding surfaces
    - fluid flow around a piston in a cylinder
- Coulomb or Dry Friction Damping
  - caused by kinetic friction, μ
  - $F_d = \mu mg$: constant but changes direction





- Hysteretic (Material or Solid) Damping
  - due to energy absorbed/dissipated by deforming materials
  - caused by friction between the sliding internal planes
  - Hysteretic behavior of $\sigma - \epsilon$



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Undamped Systems

Direct Equilibrium Method

Newton's 2^{nd} Law states that *the rate of change of momentum of any mass m is equal to the resultant of the forces acting on it*:

$$\vec{f}(t) = \frac{d}{dt}\left(m\frac{d\vec{x}}{dt}\right) = \frac{dm}{dt}\frac{d\vec{x}}{dt} + m\frac{d^2\vec{x}}{dt^2} \tag{2}$$

Assuming constant mass, $\frac{dm}{dt} = 0$. Therefore,

$$\vec{f}(t) = m\frac{d^2\vec{x}}{dt^2} = m\ddot{x}(t) \tag{3}$$

or

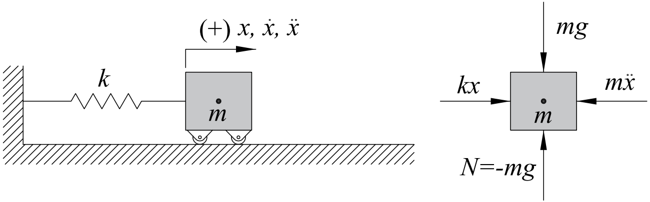
$$\vec{f}(t) - m\ddot{\vec{x}}(t) = 0 \tag{4}$$

Recall $-m\ddot{\vec{x}}$ is the inertia force.

d'Alembert's Principle states that *any mass m subjected to an acceleration develops an inertia force proportional to its acceleration and opposing the acceleration*.

$$F(t) - m\ddot{x}(t) = 0$$

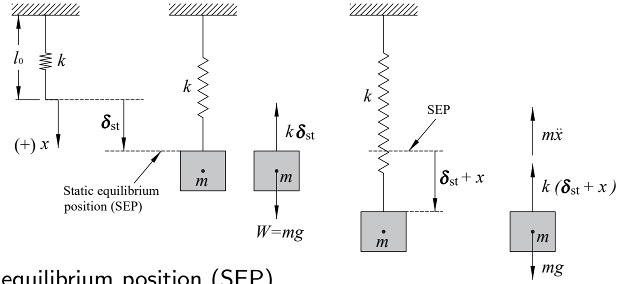
This allows equations of motion to be formulated as equations of dynamic equilibrium. Consider the following system.



Dynamic equilibrium: $\Sigma F_x = 0 \quad \Rightarrow F(t) = kx = -m\ddot{x}$, so

$$\begin{aligned} -kx - m\ddot{x} &= 0, \quad \text{or} \\ m\ddot{x} + kx &= 0 \end{aligned}$$

Consider the effect of gravity on the spring-mass system shown below where l_o is the free length of the spring



At static equilibrium position (SEP)

$$W = mg = k\delta_{st}$$

At dynamic equilibrium under d'Alembert's Principle

$$m\ddot{x} + k(\delta_{st} + x) - W = 0$$
$$m\ddot{x} + k\delta_{st} + kx - k\delta_{st} = 0$$
$$m\ddot{x} + kx = 0$$

Note that the equation of motion expressed with reference to the static equilibrium position of the dynamic system is not affected by gravitational forces.

Solution of the Equation of Motion

- Recall:

$$m\ddot{x} + kx = \ddot{x} + \omega_n^2 x = 0, \tag{5}$$

where

$$\omega_n = \sqrt{\frac{k}{m}} \tag{6}$$

- The physical significance of this substitution will shortly be made clear.
- In the absence of damping, the displacement x of the mass under the restoring force of the spring will be a periodic function called *simple harmonic motion*.
- The equation for simple harmonic motion is a homogeneous, second-order, linear differential equation with constant coefficients having the well known solution:

$$x = A\cos(\omega_n t) + B\sin(\omega_n t) \tag{7}$$

- Using the trigonometric identity

$$\sin(\theta + \psi) = \sin(\theta)\cos(\psi) + \cos(\theta)\sin(\psi)$$

Equation (7) can be re-written as

$$x = C\sin(\omega_n t + \psi) \tag{8}$$

- The coefficients of integration A and B from Equation (7), and C and ψ from Equation (8) are typically determined by specified initial conditions for displacement and velocity of the mass at time $t = 0$.
- The first time derivative of Equation (7) is

$$\dot{x} = -A\omega_n \sin(\omega_n t) + B\omega_n \cos(\omega_n t) \tag{9}$$

- Evaluating Equations (7) and (9) at time $t = 0$ leads to

$$x_0 = A \quad \text{and} \quad \dot{x}_0 = B\omega_n$$

- Substituting these values for A and B into Equation (7) yields

$$x = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} \sin(\omega_n t) \tag{10}$$

- The constants C and ψ from Equation (8) can be determined from initial conditions in a similar way by first determining it's first time derivative:

$$\dot{x} = C\omega_n \cos(\omega_n t + \psi) \tag{11}$$

- Evaluating Equations (8) and (11) at time $t = 0$ leads to

$$x_0 = C \sin(\psi) \quad \text{and} \quad \dot{x}_0 = C\omega_n \cos(\psi)$$

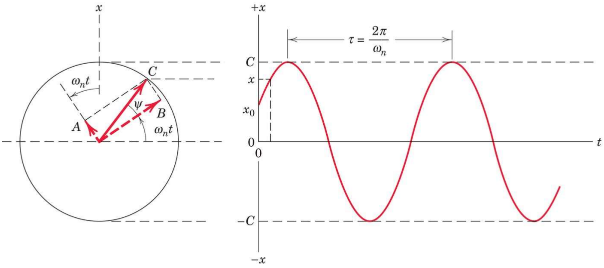
- Solving simultaneously for C and ψ leads, after some algebra, to

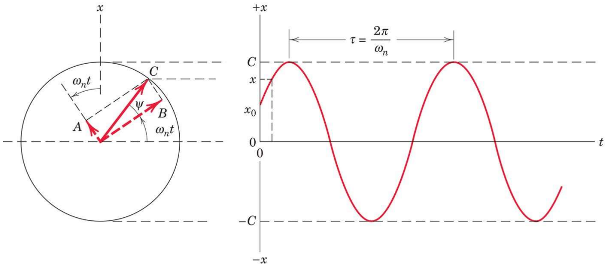
$$C = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \quad \text{and} \quad \psi = \tan^{-1}\left(\frac{x_o\omega_n}{\dot{x}_0}\right)$$

- Comparing these two coefficients to A and B, we immediately see that

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \psi = \tan^{-1}\left(\frac{A}{B}\right)$$

- The motion x is seen to be projected onto the vertical axis of the rotating vector having length C and phase angle ψ with respect to rotating vector B.

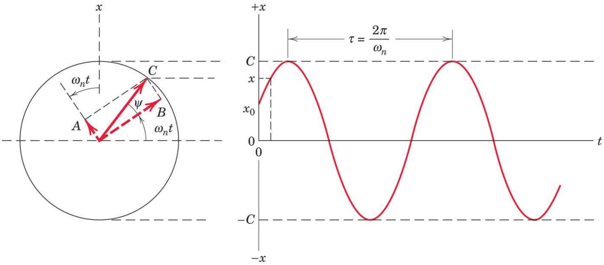




- Vectors A, B, and C all rotate with a constant angular velocity which is called the *natural circular frequency* having units of radians per second, again defined to be

$$\omega_n = \sqrt{\frac{k}{m}}$$

- Vector C is the amplitude of orthogonal components A and B, and is therefore the amplitude of the harmonic oscillation.



- The number of complete cycles per unit time is the *natural frequency* expressed in hertz (Hz), where 1 Hz = 1 cycle per second:

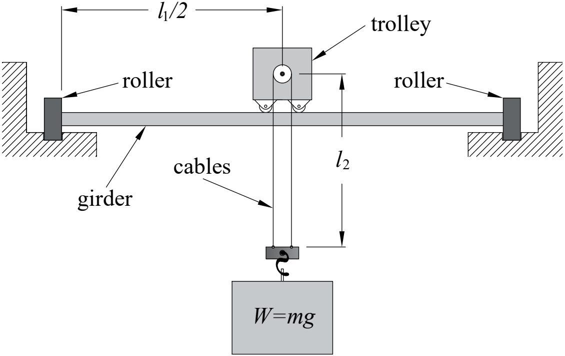
$$f_n = \frac{\omega_n}{2\pi}$$

- The time required for C to make one complete rotation is the *natural period* and has units of seconds:

$$\tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

Example 1.3

Given the overhead trolley crane and specified parameters, determine ω_n under the applied load $W = mg$



Mass of trolly, cables, etc. is negligible

Girder: E_g, I, l_1

Cables: E_c, diameter $= d$, l_2

Applied load: $W = mg$

Solution:

The spring constant for the deflection of the centre of a simply supported (pinned-pinned) beam (girder) under the applied load is: $k_g = \dfrac{48 E_g I}{l_1^3}$

The spring constant for a cable subjected to axial loading is:
$k_c = \dfrac{AE_c}{l_2} = \dfrac{\pi d^2 E_c}{4 l_2}$

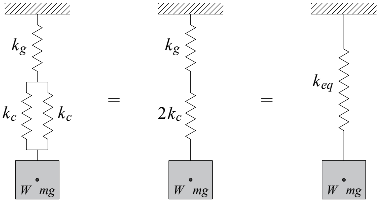
The two cables are arranged in parallel but together are in series with the girder, and hence

$$\frac{1}{k_{eq}} = \frac{1}{k_g} + \frac{1}{2k_c}$$

Therefore,

$$k_{eq} = \frac{2k_c k_g}{k_g + 2k_c} = \frac{48\pi d^2 E_g I E_c}{96 l_2 E_g I + \pi d^2 l_1^3 E_c}$$

Then $$\omega_n = \left(\frac{k_{eq}}{m}\right)^{1/2} = \left(\frac{k_{eq} g}{W}\right)^{1/2}$$



Example 1.4

Given the frictionless pulleys (sheaves), determine ω_n and f_n. The pulleys have negligible mass.

The two pulleys are considered frictionless and massless

There is constant tension in the cable

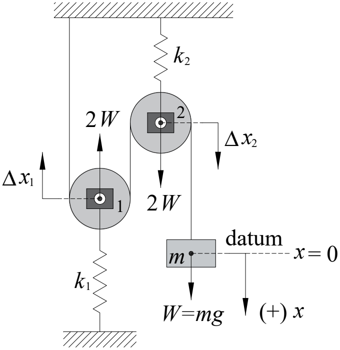
The cable length is constant

Pulley 1 moves up by a distance:

$$\Delta x_1 = \frac{2W}{k_1}$$

Pulley 2 moves down by a distance:

$$\Delta x_2 = \frac{2W}{k_2}$$



The cable on either side of the pulley is free to move the mass downward a distance x.

But the length of cable that rolls over the pulley must be distance x on each side of the pulley.

Therefore, $x = 2\Delta x_1 + 2\Delta x_2 = 2(\Delta x_1 + \Delta x_2) = 2\left[\frac{2W}{k_1} + \frac{2W}{k_2}\right]$

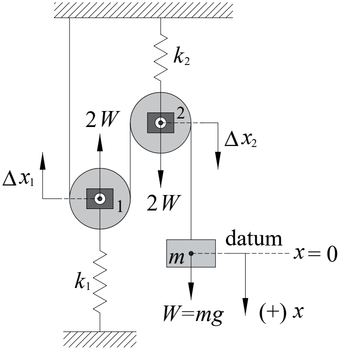
Then $x = \frac{W4(k_1 + k_2)}{k_1 k_2} = \frac{W}{k_{eq}}$

Therefore $k_{eq} = \frac{k_1 k_2}{4(k_1 + k_2)}$

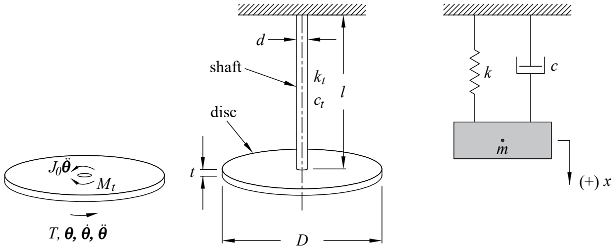
$$\omega_n = \left(\frac{k_{eq}}{m}\right)^{1/2} = \frac{1}{2}\left[\frac{k_1 k_2}{m(k_1 + k_2)}\right]^{1/2} \quad \text{rad/s}$$

or

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{4\pi}\left[\frac{k_1 k_2}{m(k_1 + k_2)}\right]^{1/2} \quad \text{cycles/s}$$



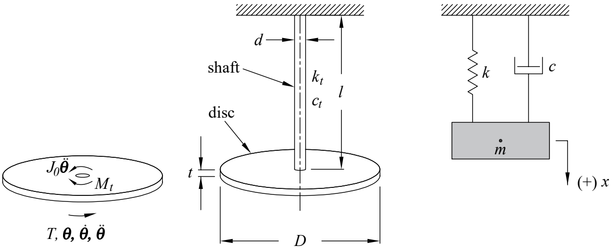
Torsional Stiffness and Viscous Damping



- Mass, m, is a measure of an object's resistance to linear acceleration
- Mass moment of inertia, $\bar{I}$, is a measure of an object's resistance to angular acceleration
- Polar mass moment of inertia, J_0, is a measure of an object's resistance to torque

$$\bar{I} = \int r^2 dm \ [kg \ \ m^2] \qquad\qquad J_0 = \int r^2 dm \ [kg \ \ m^2]$$

Torsional Stiffness and Viscous Damping



- The torsional stiffness and damping are analogous to the linear coefficients

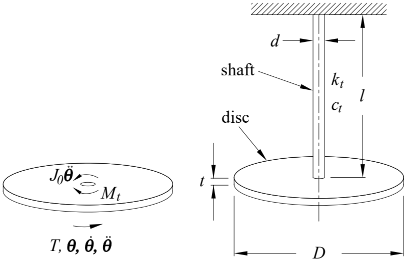
$$k_t = \frac{T}{\Delta\theta}\ \left[\frac{Nm}{rad}\right] \qquad c_t = \frac{T}{\dot{\theta}}\ \left[\frac{Nms}{rad}\right]$$

- The linear coefficients are

$$k = \frac{F}{\Delta x}\ \left[\frac{N}{m}\right] \qquad c = \frac{F}{\dot{x}}\ \left[\frac{Ns}{m}\right]$$

Example 1.5

Torsional Vibration: angular oscillation of a rigid body about a specific axis.
What is the natural period, τ_n, and equation of motion for this system?



Displacements: Angular coordinate, θ
Applied moments result from:

i) torsion of an elastic member
ii) inertia moment

$\sum \vec{M}_0 = 0$ (including inertia torque)
or $M_t + J_0\ddot{\theta} = 0$

Torsional Pendulum Solution

$J_0 = \frac{1}{2}mR^2 = \frac{1}{2}\left(\pi\left(\frac{D}{2}\right)^2 t\rho\right)\left(\frac{D}{2}\right)^2 = \frac{\rho t \pi D^4}{32}$ = polar mass moment of inertia of the disc

$M_t = \frac{GJ\theta}{l}$, where M_t is the torque required to produce θ, G is the shear modulus, and J is the polar area moment of the shaft

θ = Angular rotation of the disc = angle of twist of the shaft.

By theory of torsion of circular shafts:

$J = \frac{\pi d^4}{32}$ = polar area moment of inertia of the cross-section of the shaft

If the disc is displaced by θ from its equilibrium, the shaft acts as a torsional spring providing a restoring torque of magnitude M_t. Therefore for the torsional spring constant, we have:

$$k_t = \frac{M_t}{\theta} = \frac{GJ}{l} = \frac{\pi G d^4}{32l}$$

Then by dynamic equilibrium:

$M_t + J_0\ddot{\theta} = 0$, where M_t is the shaft restoring torque and $J_0\ddot{\theta}$ is the inertial couple

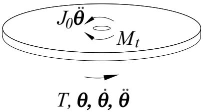
or by virtue of $M_t = k_t\theta$

$J_0\ddot{\theta} + k_t\theta = 0$
$m\ddot{x} + kx = 0$ (analogous linear system)

Therefore $\omega_n = \left(\frac{k_t}{J_0}\right)^{1/2}$ (natural circular frequency)

and $f_n = \frac{1}{2\pi}\left(\frac{k_t}{J_0}\right)^{1/2}$

$\tau_n = 2\pi\left(\frac{J_0}{k_t}\right)^{1/2}$



- The general solution of the second-order linear differential equation with constant coefficients

$$\ddot{\theta} + \omega_n^2 \theta = 0$$

has the well known solution

$$\theta(t) = A\cos(\omega_n t) + B\sin(\omega_n t) \tag{12}$$

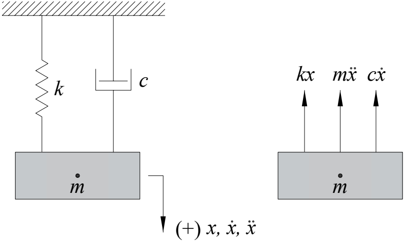
- The constants of integration A and B are determined from initial conditions at time $t = 0$
  - Evaluating Equation (12) at $t = 0$ we can immediately see that $\theta_0 = A$
  - Evaluating the first time derivative of Equation (12) reveals that $\dot{\theta}_0 = B\omega_n$
- The equation of motion for the torsional pendulum is therefore

$$\theta(t) = \theta_0 \cos(\omega_n t) + \frac{\dot{\theta}_0}{\omega_n}\sin(\omega_n t)$$

Free Vibration with Viscous Damping

Equations of Motion: Direct Equilibrium Method

Consider the following viscously damped linear system



Dynamic Equilibrium:

$\Sigma F = 0$ (including the inertia force)

$m\ddot{x} + c\dot{x} + kx = 0$

Since $m\ddot{x} + c\dot{x} + kx = 0$ is second order linear differential equation with constant coefficients, it is safe to assume the solution to the equation of motion to have the form

$$x(t) = e^{st}$$

Then substitute $x(t) = e^{st}$, $\dot{x}(t) = se^{st}$, and $\ddot{x}(t) = s^2 e^{st}$ into the equation of motion to obtain

$$\left(ms^2 + cs + k\right) e^{st} = 0 \qquad (13)$$

For the assumed solution to satisfy the differential equation, the expression in parentheses must equal zero since $e^{st} \neq 0$ for finite values of t.

The expression in parentheses is referred to as the *characteristic equation*, and its solution yields the characteristic roots, which are known as *eigenvalues*

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (14)$$

The word eigenvalue comes from the German word *Eigenwert*, in which Eigen means characteristic, or intrinsic, and wert means value.

Since two arbitrary constants are required in the solution of a second-order ordinary differential equation, the general solution is

$$x(t) = Ae^{s_1 t} + Be^{s_2 t} \tag{15}$$

where A and B are constants to be determined from the initial conditions of the system.

The system response falls in one of three categories depending on the amount of damping present: critically damped, underdamped, and overdamped.

(i) Critically Damped Systems:

In this case $\zeta = 1$ and

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = 0$$

Then, *the critical damping value* c_c is obtained as:

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

define the *damping ratio* ζ as:

$$\zeta = \frac{c}{c_c}, \qquad \text{Then:} \qquad \frac{c}{2m} = \frac{c}{c_c}\frac{c_c}{2m} = \zeta\omega_n$$

The eigenvalues can be written as:

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

At critical damping, i.e. at $\zeta = 1$, we get:

$$s_{1,2} = -\frac{c_c}{2m} = -\omega_n$$

Because the roots of the characteristic equation are real and repeated for $\zeta = 1$, the general solution to the differential equation is:

$$x(t) = (A + Bt)e^{-\omega_n t}$$

Using initial conditions:

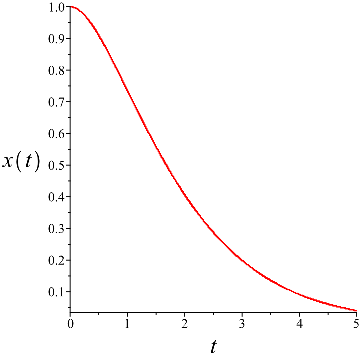
$$\text{for } t = 0 \qquad \begin{array}{ll} x = x_o \Rightarrow & A = x_o \\ \dot{x} = Be^{-\omega_n t} - \omega_n(A + Bt)e^{-\omega_n t} = \dot{x}_o \Rightarrow & B = \dot{x}_o + \omega_n x_o \end{array}$$

And we have:

$$x(t) = [x_o + (\dot{x}_o + \omega_n x_o)t]e^{-\omega_n t} \tag{16}$$

$$\dot{x}(t) = (\dot{x}_o + \omega_n x_o)e^{-\omega_n t} - \omega_n(x_o + (\dot{x}_o + \omega_n x_o)t)e^{-\omega_n t} \tag{17}$$

Hence, for critical damping, the motion is aperiodic; the plot shows the typical response for initial conditions $x_0 \neq 0$ and $\dot{x}_0 = 0$.



Note: c_c is the smallest amount of damping for which the free response of the system is aperiodic.

Example 1.6: Critically Damped Recoil Mechanism

- Consider a critically damped recoil mechanism in a piece of artillery
- Let the recoil mechanism consist of a spring to store energy during recoil, and a dashpot damper to provide the critical damping
- When fired, the barrel of the artillery piece instantaneously acquires an initial velocity $\dot{x}_0 \neq 0$ while still in its initial position of $x_0 = 0$
- For these conditions $A = 0$ while $B = \dot{x}_0$
- The displacement of the gun is then given by

$$x(t) = \dot{x}_0 t e^{-\omega_n t} \tag{18}$$

- The maximum barrel displacement occurs when

$$\frac{d}{dt}\left(\dot{x}_0 t e^{-\omega_n t}\right) = 0 = \dot{x}_0 e^{-\omega_n t} - \dot{x}_0 \omega_n t e^{-\omega_n t} = 1 - \omega_n t$$

- Therefore

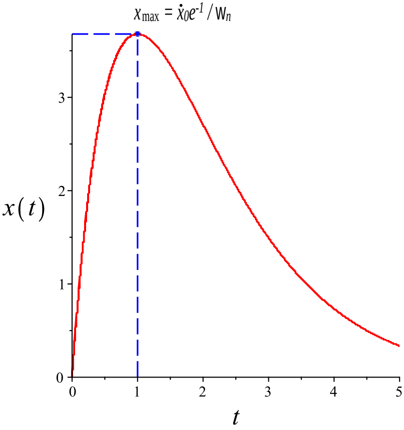
$$x_{\max} = \frac{\dot{x}_0 e^{-1}}{\omega_n} \text{ when } t = \frac{1}{\omega_n}$$

Example 1.6: Critically Damped Recoil Mechanism

For example, let $x_0 = 0$, $\dot{x}_0 = 10$ m/s, $\omega_n = 1$ rad/s, and $\zeta = 1$

We obtain

$$x_{\max} = 3.68 \text{ m} \quad \text{when } t = 1 \text{ s}$$



(ii) Over-damped Systems:

In this case $\zeta > 1$, which means $c > c_c$ and $\dfrac{c}{2m} > \sqrt{\dfrac{k}{m}}$

then $\sqrt{\zeta^2 - 1} > 0$ and s_1 and s_2 are always real, distinct, and negative, i.e.

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$
$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

and $s_2 < s_1$

Then:

$$x(t) = Ae^{s_1 t} + Be^{s_2 t}$$

such that for initial conditions at $t = 0$: $x = x_0$ and $\dot{x} = \dot{x}_0$

$$A = \frac{-x_0 s_2 + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}, \qquad B = \frac{-x_0 s_1 - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

Motion is aperiodic and since both $s_1 < 0$ and $s_2 < 0$, the response diminishes exponentially.

A periodic system response is only possible when $\zeta < 1$ and both roots of the characteristic equation are complex congugates

When $\zeta > 1$ both eigenvalues are real and aperiodic motion is possible

Note that it was shown that $s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$

where

$$\zeta\omega_n = \frac{c}{2m}, \text{ and } \omega_n = \sqrt{\frac{k}{m}}$$

The differential equation of motion can thus be expressed in terms of ζ and ω_n as follows:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \Rightarrow \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

Leading to $\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = 0$

This form allows direct comparison of the coefficients of a governing differential equation to efficiently obtain ζ, ω_n and ω_d, the *damped* natural circular frequency

(iii) Underdamped Systems:

For this case $\zeta < 1$.
Recall the eigenvalues for damped free vibration are

$$s_{1,2} = \omega_n \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

The quantity under the square root of the eigenvalue equation can only be negative if:

$\zeta < 1$, or $\dfrac{c}{2m} < \sqrt{\dfrac{k}{m}}$, or , $c < c_c$

Then

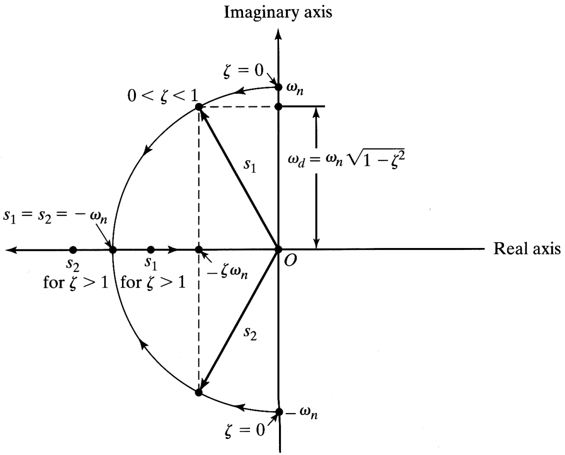
$$s_1 = \left(-\zeta + i\sqrt{1 - \zeta^2}\right)\omega_n = -\zeta\omega_n + i\omega_d$$

$$s_2 = \left(-\zeta - i\sqrt{1 - \zeta^2}\right)\omega_n = -\zeta\omega_n - i\omega_d$$

where the *damped natural circular frequency* of the vibration is defined to be

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Observations: Complex (imaginary) plane can be used to show the nature of roots s_1 and s_2 with respect to $\zeta > 1$, $\zeta = 1$, and $\zeta < 1$, by plotting $s_1 = -\zeta\omega_n + i\omega_d$, and $s_2 = -\zeta\omega_n - i\omega_d$ as points.



S.S. Rao. *Mechanical Vibrations*. Pearson Education Inc., New Jersey, United States, 4th edition, 2004.

The complex eigenvalues lead to the trigonometric functions associated with oscillatory, periodic vibration: we can re-write Equation (15) as

$$\begin{aligned} x(t) &= Ae^{-\zeta\omega_n t + i\omega_d t} + Be^{-\zeta\omega_n t - i\omega_d t} = e^{-\zeta\omega_n t}\left[Ae^{i\omega_d t} + Be^{-i\omega_d t}\right] \\ &= e^{-\zeta\omega_n t}\left[A cos\omega_d t + B sin\omega_d t\right] \\ &= Ce^{-\zeta\omega_n t} cos\left[\omega_d t - \phi^*\right] \\ &= Ce^{-\zeta\omega_n t} sin\left[\omega_d t + \phi\right] \end{aligned}$$

using initial conditions at $t = 0$,

$$x = x_0 \Rightarrow \quad A = x_0, \qquad \dot{x} = \dot{x}_0 \Rightarrow B = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}, \qquad C = \sqrt{A^2 + B^2}$$

$$\phi = tan^{-1}(A/B) = \frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}$$

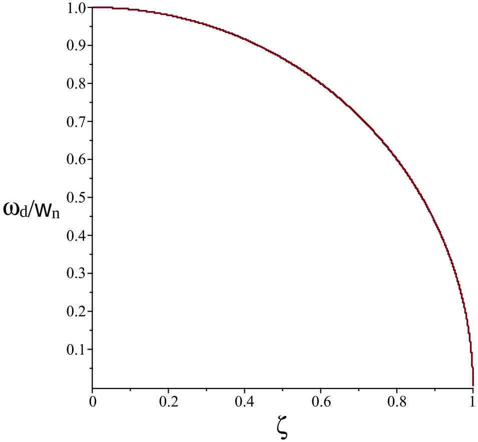
$$\phi^* = tan^{-1}(B/A) = \frac{\dot{x}_0 + \zeta\omega_n x_0}{x_0\omega_d} = \frac{\pi}{2} - \phi$$

Therefore:

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} sin\omega_d t\right] \tag{19}$$

Observations:

- $\omega_d < \omega_n$
- $\dfrac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$ plots as a quarter of a unit circle, showing that as damping increases, the damped natural frequency decreases.

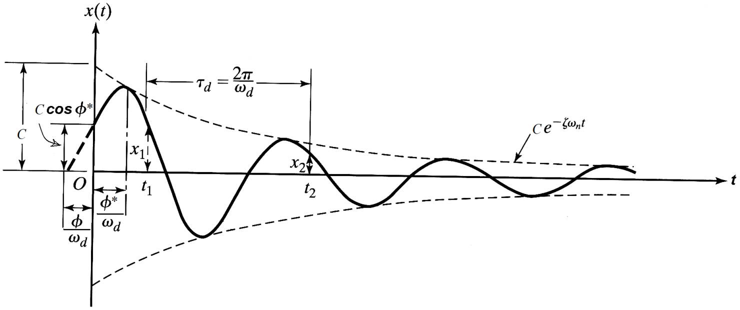


Underdamped system response

- The underdamped case is the only case having oscillatory motion
- In fact $x = Ce^{-\zeta\omega_n t} cos[\omega_d t - \phi^*]$ represents a harmonic motion where

  ω_d = angular frequency, ϕ^* = the phase angle, and

  C = amplitude, which exponentially decreases as $e^{-\zeta\omega_n t}$



S.S. Rao. *Mechanical Vibrations*. Pearson Education Inc., New Jersey, United States, 4th edition, 2004.

- Logarithmic Decrement $\equiv \delta$

The logarithmic decrement, δ, can be used to determine damping factor, ζ, of a mechanical system experimentally by recording a record of the system showing how its amplitude varies with time during damped free vibration of the system

Consider the rate of decrease of $x(t)$ of an underdamped system. For two successive peaks where t_1 is the time of the first peak and

$$t_2 = t_1 + \tau_d = t_1 + \frac{2\pi}{\omega_d}$$

is the time of the second peak, the ratio of response amplitudes is

$$\frac{x_1}{x_2} = \frac{Ce^{-\zeta\omega_n t_1} cos(\omega_d t_1 - \phi^*)}{Ce^{-\zeta\omega_n (t_1+\tau_d)} cos(\omega_d (t_1 + \tau_d) - \phi^*)} = e^{\zeta\omega_n\tau_d} \qquad (20)$$

The logarithmic decrement is defined as the natural logarithm of the ratio of two successive amplitudes x_j and x_{j+1}:

$$\delta = ln\frac{x_j}{x_{j+1}} = \zeta\omega_n\tau_d = 2\pi\zeta\frac{\omega_n}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\frac{c}{2m} \qquad (21)$$

For low damping, i.e., $0 < \zeta < 0.3$, we get:

$$\delta \approx 2\pi\zeta \tag{22}$$

δ can be used to determine the damping ratio ζ of the system. We determine experimentally x_j and some x_{j+k} at t_j and $t_j + t_{j+k}$

$$\frac{x_j}{x_{j+k}} = \frac{x_j}{x_{j+1}} \frac{x_{j+1}}{x_{j+2}} \frac{x_{j+2}}{x_{j+3}} \cdots \frac{x_{j+k-1}}{x_{j+k}} \tag{23}$$

Therefore:

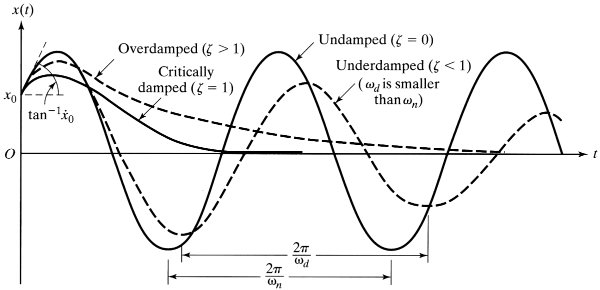
$$\frac{x_j}{x_{j+k}} = e^{k\zeta\omega_n\tau_d} = e^{k\delta} \tag{24}$$

And:

$$\delta = \frac{1}{k} ln\left(\frac{x_j}{x_{j+k}}\right) \tag{25}$$

Using δ, the damping ratio can then be evaluated.

Comparison of system responses



S.S. Rao. *Mechanical Vibrations*. Pearson Education Inc., New Jersey, United States, 4th edition, 2004.

Example 1.7

Design an underdamped shock absorber for a vehicle, such that: $x_{1.5} = \frac{1}{4}x_1$, m = 500 kg, $\tau_d = 1.0$ s and the clearance distance is 250 mm. It is required to find c, k and the minimum initial velocity $\dot{x}_0$ resulting in bottoming of the shock absorber.

$$x_{1.5} = \frac{1}{4}x_1 \text{ and } x_2 = \frac{1}{4}x_{1.5} \;\Rightarrow\; x_2 = \frac{1}{16}x_1$$

$$\delta = ln\left(\frac{x_1}{x_2}\right) = ln(16) = 2.7726 =$$

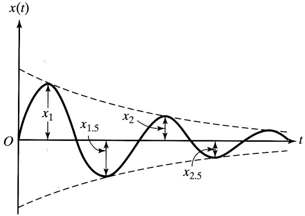
$$\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$
$$\Rightarrow \zeta = 0.4037$$

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 1$$
$$\Rightarrow \omega_n = 6.868 \text{ rad/sec}$$

$$c_c = 2m\omega_n = 2(500)(6.868) = 6868 \text{ Ns/m}$$

$$c = \zeta c_c = (0.4037)(6868) = 2772.5 \text{ Ns/m}$$

$$k = m{\omega_n}^2 = 500(6.868)^2 = 23582.6 \text{ N/m}$$



S.S. Rao. *Mechanical Vibrations*. Pearson Education Inc., New Jersey, United States, 4th edition, 2004.

It is known that for an underdamped system

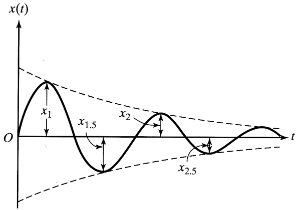
$x(t) = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$ where $\phi = \tan^{-1}\left[\frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}\right]$

From graph at t = 0

$x(0) = x_0 = 0 \Rightarrow \qquad \phi = 0$

$\dot{x}(t) = Ce^{-\zeta\omega_n t}\left[-\zeta\omega_n\sin(\omega_d t) + \omega_d\cos(\omega_d t)\right]$

Therefore at $t = 0 \Rightarrow \qquad \dot{x}(0) \equiv \dot{x}_0 = C\omega_d$



The equation of the envelope passing through the maximum value points is:

$$x = \sqrt{1-\zeta^2}\,Ce^{-\zeta\omega_n t}$$

Since the maximum displacement x_1 will occur at t_1, it can be shown that this happens when $sin(\omega_d t_1) = \sqrt{1-\zeta^2}$ which means that:

$$t_1 = \frac{sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_d} = 0.1839\text{s}$$

Then at $x = 250$ mm $= 0.25$ m and $t = 0.1839$ s:

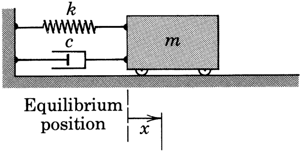
$$0.25 \text{ m} = \sqrt{1-(0.4037)^2}\,Ce^{-(0.4037)(6.868 \text{ rad/s})(0.1839 \text{ s})}$$

$$C = 0.455 \text{ m}$$

$$\dot{x}_0 = 0.455\omega_d = 0.455\omega_n\sqrt{1-\zeta^2} = 2.86 \text{ m/s}$$

Example 1.8

Consider the spring, mass, damper system shown. The body is moved 0.2 m to the right and released from rest. If m = 8 kg, c = 20 Ns/m and k = 32 N/m, what is the displacement at t = 2 s?



J.L. Meriam.*Engineering Mechanics Dynamics*. John Wiley and Sons Inc.,United States, 5th edition, 2002.

Solution:
1) Determine the amount of damping:

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{32\ N/m}{8\ kg}} = 2 \text{ rad/s}$$

Therefore

$$\zeta = \frac{20\ Ns/m}{2(8\ kg)(2\ rad/s)} = 0.625 \Rightarrow \zeta < 1 \text{ therefore the system is underdamped}$$

2) The general solution for underdamped systems is:

$$x(t) = Ce^{-\zeta\omega_n t}\sin[\omega_d t + \phi] = e^{-\zeta\omega_n t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]$$

The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.56$ rad/s

IC's:

$A = x_0 = 0.2$ m

$$B = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}$$

$$= \frac{(0.625)(2)(0.2)}{1.56} = 0.160 \text{ m}$$

$C = \sqrt{A^2 + B^2} = 0.256$ m

$\phi = \tan^{-1}(A/B) = 0.896$ rad

Hence $x(t) = 0.256e^{-1.25t}\sin(1.56t + 0.896)$

at t = 2 s $\Rightarrow x(2) = -0.01616 \text{ m} = -16.16 \text{ mm}$