MAAE 3004
Dynamics of Machinery

Lecture Slide Set 2

Harmonically-Excited Vibration

Department of Mechanical and Aerospace Engineering
Carleton University

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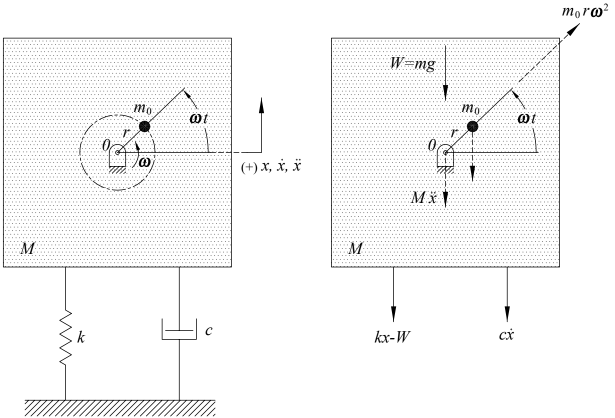
Outline

Introduction

- While all real mechanical systems have some damping, it is often neglected to simplify the analysis and to gain some insight into a complicated system.
- When a linear, constant-coefficient dynamic system is disturbed by certain types of exciting forces, the response consists of the sum of two distinct components
  1. The *forced response*, which resembles the excitation force in mathematical form.
  2. The *free (or natural) response*, which does not depend on the characteristics of the excitation function, but only on the physical parameters of the system. This is the response that is induced by initial conditions, regardless of the excitation function.
- These two components are obtained separately and consist of
  1. The *particular solution*, which is due to the excitation force, and is typically transient.
  2. The *homogeneous solution*, which is due to the system physical parameters and initial conditions, and is what typically remains when the transient particular response decays.
- The sum of the two solutions is referred to as the *total response of the system*.

Excitation Due to an Unbalanced Rotating Mass

- Unbalanced rotating masses in machines and mechanisms that subject them to periodic harmonic forces are very common sources of excitation.
- Consider a machine that has a total mass m consisting of a translating mass M to which is attached, by a slender rod of negligible mass, a mass m_0 which is rotating about an axis through 0.

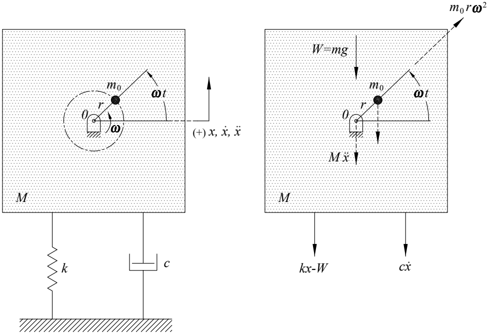


Excitation Due to an Unbalanced Rotating Mass

- The total mass of the mechanical system is

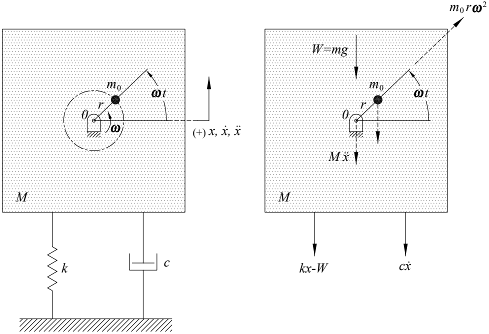
$$m = M + m_0$$

- Since the centre of gravity of m_0 is offset by radial distance r from its centre of rotation 0, it acts as a unbalanced rotating mass.
- The spring and dashpot have coefficients k and c and represent the stiffness and damping of the machine's mounting system.



Excitation Due to an Unbalanced Rotating Mass

- The positive sense for x, $\dot{x}$, and $\ddot{x}$ is up, and x is the displacement of M from its static equilibrium position (SEP).
- The total acceleration of m_0 is the vector sum of its normal acceleration $r\omega^2$, directed towards its centre of rotation 0 and the acceleration $\ddot{x}$ of mass M.
- The spring force is $kx - W$ because in the SEP the spring is compressed under the action of W.



Excitation Due to an Unbalanced Rotating Mass

- The positive sense for x, $\dot{x}$, and $\ddot{x}$ is up, and x is the displacement of M from its static equilibrium position (SEP).

- The total acceleration of m_0 is the vector sum of its normal acceleration $r\omega^2$, directed towards its centre of rotation 0 and the acceleration $\ddot{x}$ of mass M.

- Using d'Alembert's principle, sum the forces and inertia effects to zero:

$$\sum F_x = 0 = -W - (kx - W) - c\dot{x} - (M + m_0) + m_0 r\omega^2 \sin(\omega t) \quad (1)$$

- Letting $m = M + m_0$ and $F_0 = m_0 r\omega^2$, we can rewrite Equation (1) as

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin(\omega t) \quad (2)$$

where the compressive spring force $W = mg$ cancels out.

Excitation Due to an Unbalanced Rotating Mass

- We can rewrite Equation (2) in terms of the damping ratio ζ and undamped natural circular frequency ω_n as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin(\omega t) \qquad (3)$$

where $2\zeta\omega_n = \dfrac{c}{m}$ and $\omega_n^2 = \dfrac{k}{m}$.

- Equation (3) is the general form of the mathematical model of any single degree of freedom (DOF) system subject to any sinusoidal excitation force $\dfrac{F_0}{m}\sin(\omega t)$.

- The total response of the system is the sum of a *homogeneous* solution x_h and a *particular* solution x_p

$$x = x_h + x_p$$

Excitation Due to an Unbalanced Rotating Mass

- The homogeneous solution (the *free*, or *transient* solution of Equation (3) if $\zeta < 1$) is, as we have seen

$$x_h = e^{-\zeta \omega_n t} \left(A \cos(\omega_d t) + B \sin(\omega_d t) \right) \tag{4}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.

- The particular solution (the *free*, or *steady-state response*) is usually determined using complex numbers

- Using Euler's equation

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

where $e^{i\omega t}$ is a complex unit vector, we can write

$$\cos(\omega t) \stackrel{R}{=} e^{i\omega t} \quad \text{and} \quad \sin(\omega t) \stackrel{I}{=} e^{i\omega t}$$

where R means "the real part of" and I means "the imaginary part of".

Excitation Due to an Unbalanced Rotating Mass

- With this in mind, we can write the right hand side of Equation (3) as

$$\frac{F_0}{m}\sin(\omega t) = \frac{F_0}{m}e^{i\omega t} \tag{5}$$

  with the provision that only the imaginary part will be used in the solution process

- Then we assume a particular solution having the form

$$x_p = Xe^{i\omega t} \tag{6}$$

  where X is an imaginary constant to be determined so that it satisfies Equation (3)

- Successive time derivatives of Equation (6) give

$$\dot{x}_p = i\omega Xe^{i\omega t} \text{ and } \ddot{x}_p = -\omega^2 Xe^{i\omega t} \tag{7}$$

Excitation Due to an Unbalanced Rotating Mass

- Substituting Equations (5), (6), and (7) into Equation (3) yelds

$$\left(-\omega^2 + i\omega 2\zeta\omega_n + \omega_n^2\right) X e^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

- Dividing this equation by ω_n^2 and noting that $m\omega_n^2 = k$ we can write

$$\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}\right] X = \frac{F_0}{k} \quad (8)$$

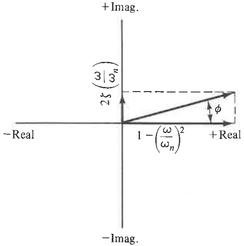
- We can rewrite the bracketed term in the complex plane as

$$\left[\underbrace{1 - \left(\frac{\omega}{\omega_n}\right)^2}_{\text{Real}} + \underbrace{i2\zeta\frac{\omega}{\omega_n}}_{\text{Imaginary}}\right] = \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2} e^{i\phi} \quad (9)$$

where, as can be seen in the following figure

$$\phi = \tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right] \quad (10)$$

Excitation Due to an Unbalanced Rotating Mass



- Substituting the right-hand side of Equation (9) into Equation (8) gives

$$X = \frac{(F_0/k)e^{-i\phi}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} = |X|e^{-i\phi} \qquad (11)$$

where the absolute value $|X|$ is the amplitude of the steady-state response

$$|X| = \frac{(F_0/k)}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \qquad (12)$$

Excitation Due to an Unbalanced Rotating Mass

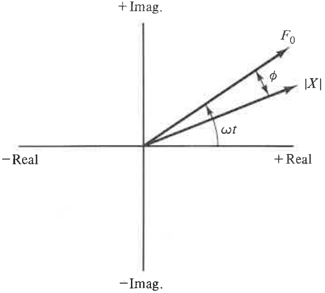
- Hence, the particular solution of Equation (3) is

$$x_p = |X|e^{-i\phi}e^{i\omega t} = |X|e^{i(\omega t-\phi)} \tag{13}$$

- Finally, using only the imaginary part of $e^{i(\omega t-\phi)}$ we obtain

$$x_p = |X|\sin(\omega t-\phi) \tag{14}$$

- The complex excitation force $F_0 e^{i\omega t}$ and the complex steady-state response $|X|e^{i(\omega t-\phi)}$ are rotating complex vectors, called *phasors*, in the complex plane
- Their magnitudes are F_0 and $|X|$, and their directions are given by $e^{i\omega t}$ and $e^{i(\omega t-\phi)}$, respectively
- They both rotate with angular velocity ω
- The *phase angle*, ϕ, between them is the angle by which the steady-state displacement lags the oscilating excitation force

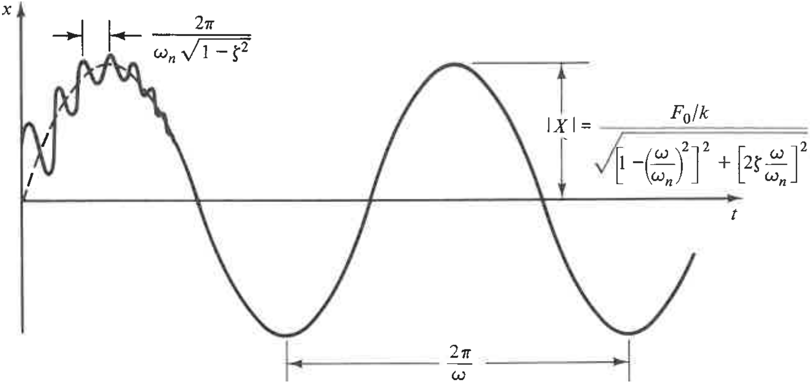


Excitation Due to an Unbalanced Rotating Mass

- Adding the transient homogeneous solution x_h to the steady-state solution x_p gives the *complete* solution of Equation (3)

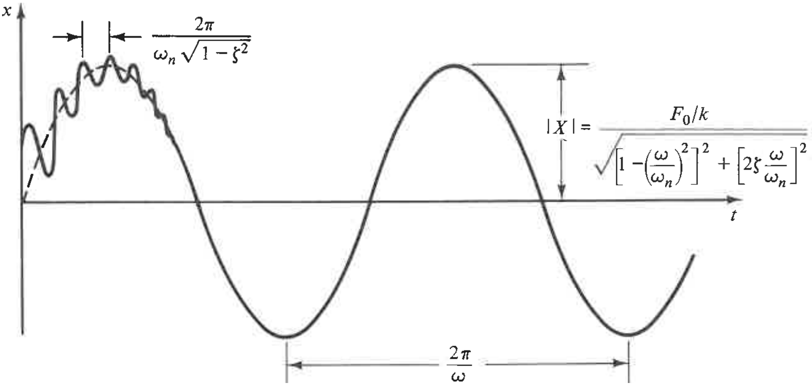
$$x = e^{-\zeta\omega_n t}\left(A\cos\left(\omega_d t\right) + B\sin\left(\omega_d t\right)\right) + |X|\sin\left(\omega t - \phi\right) \quad (15)$$

- The combined transient and steady-state response is shown for $\omega < \omega_n\sqrt{1-\zeta^2}$, and recall that $\omega_d = \omega_n\sqrt{1-\zeta^2}$



Excitation Due to an Unbalanced Rotating Mass

- The transient vibration portion of the complete response decays exponentially with time, until it disappears completely, leaving only the steady-state vibration
- The coefficients A and B of the transient vibration depend on initial conditions while the steady-state vibration depends only upon the forcing function and the physical parameters of the system

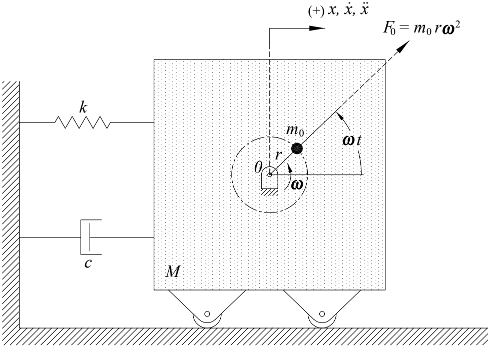


Excitation Due to an Unbalanced Rotating Mass

- To illustrate how to use the real part of $e^{i\omega t}$, consider the horizontal mass M rolling on a horizontal surface that is excited by a rotating unbalanced mass m_0
- The differential equation of motion of this system is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\cos(\omega t)$$

where $m = M + m_0$

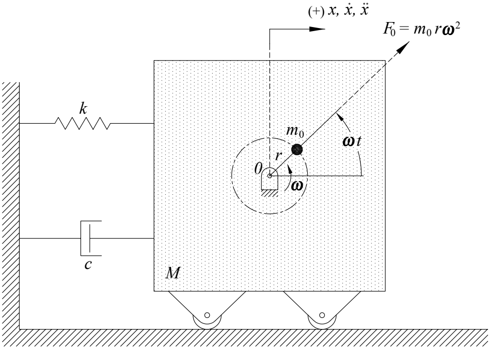


Excitation Due to an Unbalanced Rotating Mass

- The excitation term includes $\cos(\omega t)$, which is the real part of $e^{i\omega t}$
- The steady-state solution is

$$x_p = |X| \cos(\omega t - \phi)$$

The steady-state amplitude $|X|$ and phase angle ϕ are the same as Equations (12) and (10)



Steady-state Vibration Characteristics

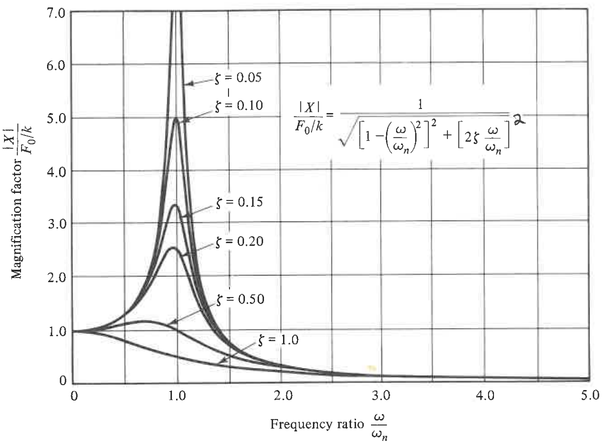
- Equations (12) and (10) describe the steady-state behaviour of a system subjected to a harmonic excitation force $F_0 \sin(\omega t)$ or $F_0 \cos(\omega t)$
- Excitation forces of this type arise from unbalanced rotating masses, or vortex shedding in systems subjected to a flow of gas
- It is obvious from Equation (12) that the amplitude of vibration increases with F_0
- To investigate how, we can rewrite Equation (12) in dimensionless form:

$$\frac{|X|}{F_0/k} = \frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2+\left[2\zeta(\omega/\omega_n)\right]^2}} \tag{16}$$

- The static displacement that the system would have if the force F_0 were applied very slowly is F_0/k, and $|X|$ is the amplitude of the steady-state vibration
- The ratio in Equation (16), known as the *magnification factor*, is a function of ζ and ω/ω_n

Steady-state Vibration Characteristics

- When $\omega/\omega_n \ll 1$, the magnification factor $|X|/(F_0/k)$ approaches 1
- When ω/ω_n approaches 1 and $\zeta \ll 1$, the magnification factor $|X|/(F_0/k)$ can become very large



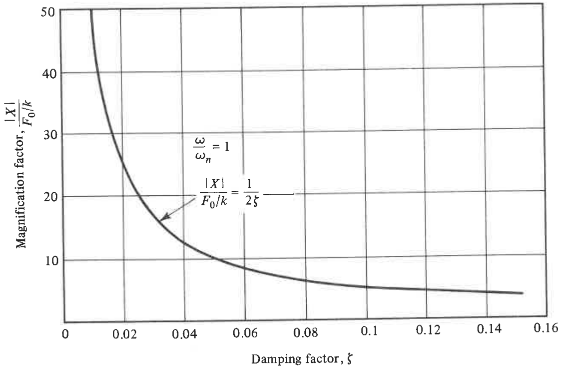
Steady-state Vibration Characteristics

- When $\omega/\omega_n = 1$, the magnification factor $|X|/(F_0/k)$ reduces to

$$\frac{|X|}{F_0/k} = \frac{1}{2\zeta} \tag{17}$$

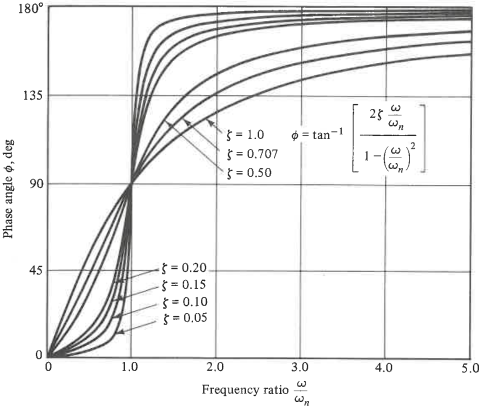
  which is the magnification factor at *resonance*

- This is one reason it is very important to be able to determine the approximate natural frequencies of systems



Steady-state Vibration Characteristics

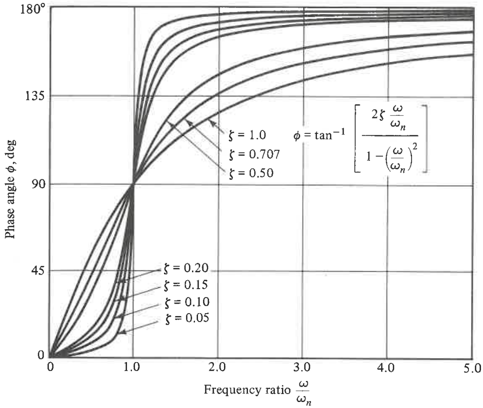
- Equation (10) reveals that the phase angle ϕ is also a function of ζ and ω/ω_n
- When $\omega/\omega_n \ll 1$ the phase angle is small, and the excitation force $F_0 \sin(\omega t)$ is nearly in phase with the response since $x = |X| \sin(\omega t - \phi)$



Steady-state Vibration Characteristics

- When $\omega/\omega_n = 1$ the phase angle is 90°, and the excitation force $F_0 \sin(\omega t)$ is in phase with the velocity since

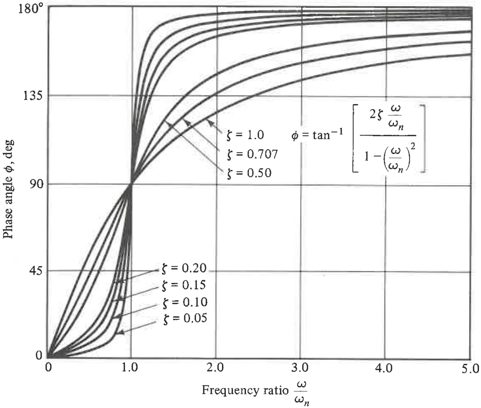
$$\dot{x} = |X| \cos\left(\omega t - \frac{\pi}{2}\right) = |X| \sin(\omega t)$$



Steady-state Vibration Characteristics

- When $\omega/\omega_n \gg 1$ the phase angle approaches 180°, and the sense of the excitation force $F_0 \sin(\omega t)$ is nearly completely out of phase (opposite) to that of the displacement since

$$x \approx |X| \sin(\omega t - \pi) = -|X| \sin(\omega t)$$



Example 2.1

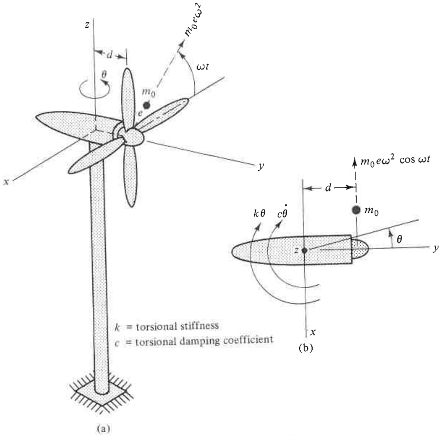
A small wind turbine is rigidly mounted to a cantilevered vertical steel pole having a torsional stiffness of k [Nm/rad] and a torsional damping coefficient of c [Nms/rad].

The centre of gravity of the rotor blade assembly having a total mass of m_0 is displaced by an eccentric distance e from the axis of rotation.

The mass moment of inertia about the z-axis of the complete turbine, rotor assembly, housing pod, and contents, is $\bar{I}_z$ [kgm^2].

The total mass of the mechanical system is m [kg].

The plane in which the blades rotate is located a distance d [m] from the z-axis.

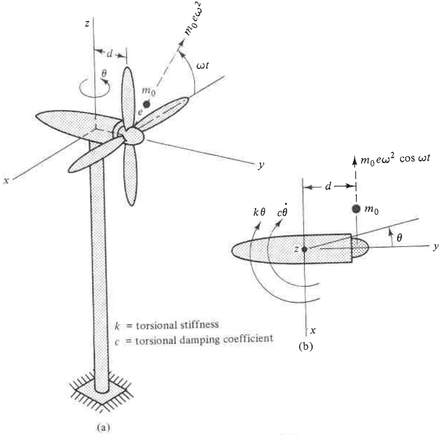


Determine the following. (a) The differential equation of motion of the torsional vibration system about the z-axis.

(b) The steady-state torsional response (the particular solution) of the system using complex algebra.

Neglect the following.

1. The effects of the mass and bending of the pole on the torsional response.
2. The gyroscopic effects of the rotating blades.



Solution

(a) The moment of the unbalanced rotating mass m_0 of the rotor blades about the z-axis is

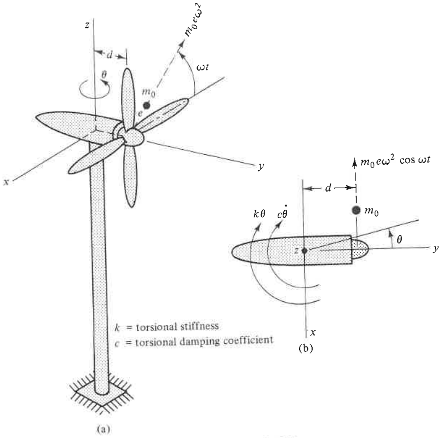
$$M(t) = \left(m_0 e\omega^2 \cos(\omega t)\right) d$$

or

$$M(t) = M_0 \cos(\omega t)$$

where $M_0 = m_0 e\omega^2 d$.

The vertical pole also exerts a restoring torque of $k\theta$, and a torsional damping moment of $c\dot{\theta}$.



Considering the moments about the z-axis, we can write

$$\sum M_z = \bar{I}_z \ddot{\theta} = -k\theta - c\dot{\theta} + M_0 \cos(\omega t)$$

Dividing by $\bar{I}_z$ and collecting terms containing the dependent variable θ yields

$$\ddot{\theta} + \frac{c\dot{\theta}}{\bar{I}_z} + \frac{k\theta}{\bar{I}_z} = \frac{M_0}{\bar{I}_z}\cos(\omega t)$$

Finally, let $c/\bar{I}_z = 2\zeta\omega_n$ and $k/\bar{I}_z = \omega_n^2$, leading to the differential equation of torsional vibration of the system about the z-axis

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{M_0}{\bar{I}_z}\cos(\omega t) \tag{18}$$

(b)

Using complex algebra to obtain the steady-state solution of this differential equation of motion, $\cos(\omega t)$ is replaced by the real part of $e^{i\omega t}$, and we assume a solution of the form

$$\theta_p = \Theta e^{i\omega t} \tag{19}$$

in which Θ is the complex constant to be determined such that it satisfies Equation (18), understanding that only the real part of $e^{i\omega t}$ will be used in determining this complex constant Θ

Taking successive time derivatives of Equation (19) we obtain

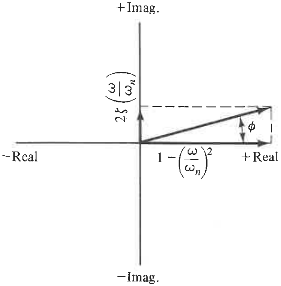
$$\dot{\theta}_p = i\omega\Theta e^{i\omega t} \text{ and } \ddot{\theta}_p = -\omega^2\Theta e^{i\omega t} \quad (20)$$

Substitute Equations (19) and (20) into Equation (18), and replace $\cos(\omega t)$ with $e^{i\omega t}$ to get

$$\left(-\omega^2 + i\omega 2\zeta\omega_n + \omega_n^2\right)\Theta e^{i\omega t} = \frac{M_0}{\bar{I}_z} e^{i\omega t}$$

Dividing both sides of this equation by ω_n^2, and recalling that $\bar{I}_z\omega_n^2 = k$ one obtains

$$\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}\right]\Theta = \frac{M_0}{\bar{I}_z\omega_n^2} = \frac{M_0}{k} \quad (21)$$



In the complex plane we can rewrite the bracketed term as

$$\left[1-\left(\frac{\omega}{\omega_n}\right)^2+i2\zeta\frac{\omega}{\omega_n}\right] = \sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}e^{i\phi}$$

where

$$\phi = \tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right] \tag{22}$$

Making this substitution in Equation (21), we can isolate Θ to obtain

$$\Theta = \frac{(M_0/k)e^{-i\phi}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} = |\Theta|e^{-i\phi} \quad (23)$$

where the absolute value $|\Theta|$ is the amplitude of the steady-state torsional response

$$|\Theta| = \frac{(M_0/k)}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad (24)$$

- The phase angle ϕ is the constant angle by which the response angular amplitude $|\Theta|$ lags the excitation moment amplitude M_0 and is given by Equation (22), which, as you can see is the same as Equation (10)
- Thus, the complex form of the steady-state solution is

$$\theta_p = |\Theta|e^{i(\omega t-\phi)}$$

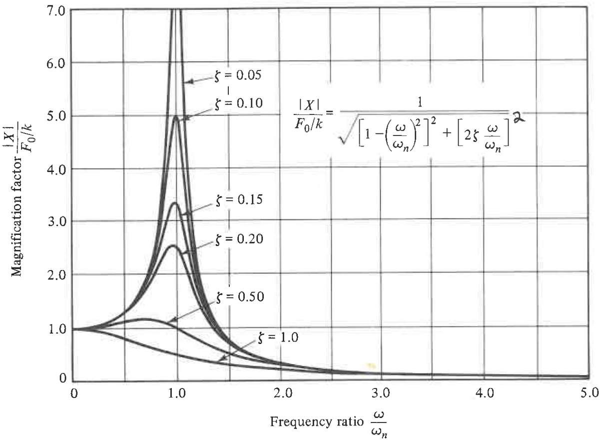
- Taking only the real part of $e^{i(\omega t-\phi)}$ we obtain the steady-state solution:

$$\theta_p = |\Theta|\cos(\omega t-\phi)$$

Steady-state Vibration Characteristics

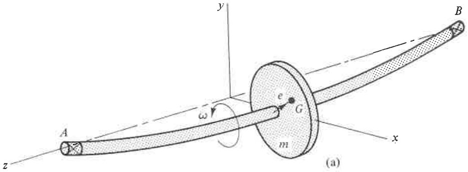
- It's important to note that the torsional and linear magnification factors are identical in form, and yield the same curves.

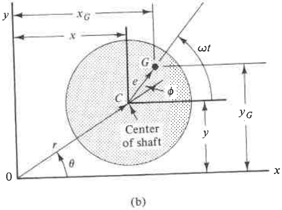
$$\frac{|X|}{F_0/k} = \frac{|\Theta|}{M_0/k} = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$$



Critical Speed of Rotating Shafts

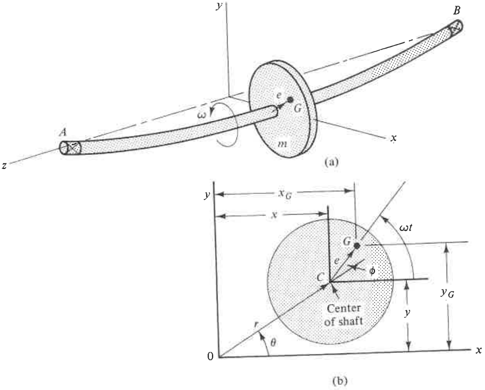
- When a shaft that is rotating about its longitudinal axis deforms about that axis, line *AB* in the figure, the deformed shaft will *whirl* about its original axis of rotation as well as continuing to rotate about its longitudinal axis
- Unbalanced disks, loose or worn bearings, and gyroscopic effects can, among other things, cause rotating shafts to whirl
- Consider the steady-state motion of the shaft in the figure, which is whirling because the disk attached to it is not balanced.
- Because of machining error, the mass centre G of the disk is eccentric by distance e from the geometric centre C.





Critical Speed of Rotating Shafts

- Let the disk centre C be located at the origin of the stationary xyz coordinate system when the shaft is aligned with the line AB passing through the bearing centres
- When the shaft deforms laterally, the shaft deflection at the disk centre is $r = OC$
- The deflected shaft and line AB form a rotating plane that makes an angle θ with the x-axis
- The angular velocity of the shaft-and-disk system with respect to the longitudinal axis of the shaft is ω
- The angular velocity of the plane formed by the shaft and line AB is $\dot{\theta}$

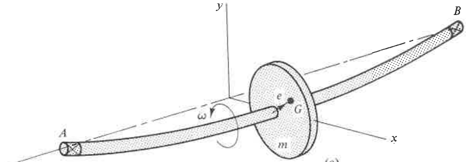


Critical Speed of Rotating Shafts

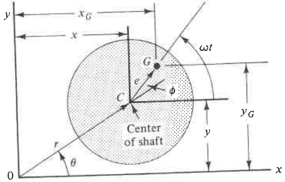
- The phase angle ϕ is the angle between the position vector $\mathbf{r}$ and the eccentricity vector $\mathbf{e}$ (whose magnitude is e)
- The shaft deflection r can become appreciable when the angular velocity ω approaches the natural circular frequency ω_n
- The *critical speed* ω occurs when

$$\frac{\omega}{\omega_n} = 1 \qquad (25)$$

- *Synchronous whirl* occurs when $\dot{\theta} = \omega$, implying that the magnitude of $\mathbf{r}$ is constant



(a)



(b)

Critical Speed of Rotating Shafts

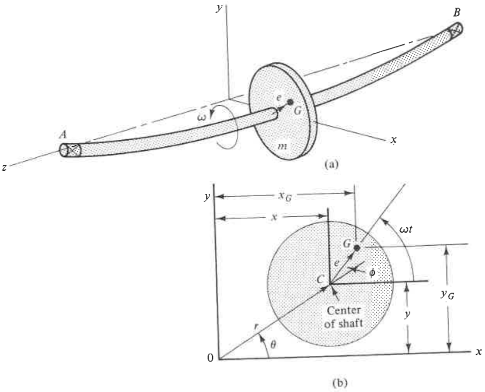
- Assume the weight of the shaft is small compared to that of the disk
- The coordinates of the mass centre G are

$$\left.\begin{aligned} x_G &= x + e\cos\omega t \\ y_G &= y + e\sin\omega t \end{aligned}\right\} \qquad (26)$$

where x and y are the coordinates of the disk centre C with respect to the stationary xyz coordinate system

- Taking successive time derivatives of Equation (26) we obtain the acceleration components of the disk centre of mass

$$\left.\begin{aligned} \ddot{x}_G &= \ddot{x} - e\omega^2\cos\omega t \\ \ddot{y}_G &= \ddot{y} - e\omega^2\sin\omega t \end{aligned}\right\} \qquad (27)$$



Critical Speed of Rotating Shafts

- For a circular-cylindrical shaft of uniform density, it is safe to assume that the stiffness and damping of the shaft are uniform regardless of the shaft orientation, which means

$$k = k_x = k_y$$

and

$$c = c_x = c_y$$

- With this assumption, we can use the acceleration components of Equation (27) and apply Newton's second law to obtain the differential equations of motion

$$-kx - c\dot{x} = m\left(\ddot{x} - e\omega^2 \cos(\omega t)\right)$$

and

$$-ky - c\dot{y} = m\left(\ddot{y} - e\omega^2 \sin(\omega t)\right)$$

Critical Speed of Rotating Shafts

- These two equations can be rewritten in the now familiar form of

$$\left.\begin{aligned} \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x &= e\omega^2 \cos(\omega t) \\ \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y &= e\omega^2 \sin(\omega t) \end{aligned}\right\} \quad (28)$$

where

$$2\zeta\omega_n = \frac{c}{m} \quad \text{and} \quad \omega_n = \sqrt{\frac{k}{m}}$$

- Again, we can assume complex steady-state solutions for Equations (28) as

$$x = Xe^{i\omega t} \quad \text{and} \quad y = Ye^{i\omega t}$$

- We proceed as before to obtain the complex form of the steady-state response

$$X = Y = \frac{e(\omega/\omega_n)^2 e^{i\phi}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad (29)$$

Critical Speed of Rotating Shafts

- Where the amplitudes are

$$|X| = |Y| = \frac{e(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad (30)$$

and the phase angle between position vectors **r** and **e** is again the same as Equation (10)

$$\phi = \tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right]$$

- The real displacement components of the disk centre C are then

$$\left.\begin{array}{l} x = |X|\cos(\omega t - \phi) \\ y = |Y|\sin(\omega t - \phi) \end{array}\right\} \quad (31)$$

- Squaring, and adding the two components gives

$$r = \sqrt{x^2 + y^2} = \sqrt{(|X|\cos(\omega t - \phi))^2 + (|Y|\sin(\omega t - \phi))^2}$$

Critical Speed of Rotating Shafts

- Since $|X| = |Y|$ we can conclude that the constant shaft deflection at the disk centre is

$$r = |X| = |Y|$$

- Using the expressions for x and y in Equation (31) we obtain

$$\tan\theta = \frac{y}{x} =$$

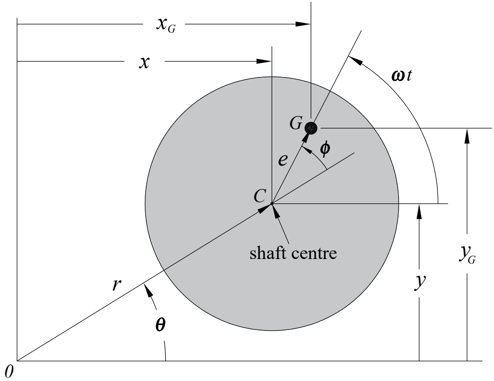
$$\frac{|Y|\sin(\omega t - \phi)}{|X|\cos(\omega t - \phi)} = tan(\omega t - \phi)$$

which shows that

$$\theta = \omega t - \phi$$

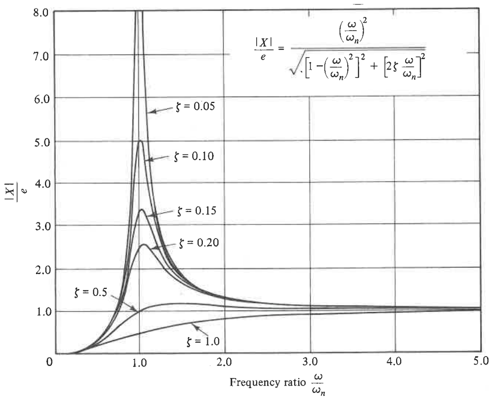
and

$$\dot{\theta} = \omega$$



Critical Speed of Rotating Shafts

- The plot of the dimensionless ratio $|X|/e$ versus the frequency ratio ω/ω_n reveals some familiar information

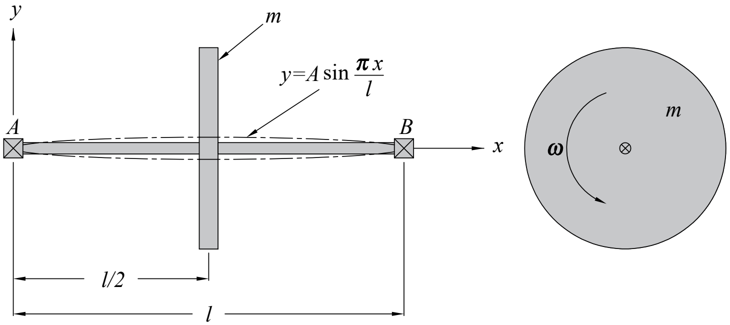


Critical Speed of Rotating Shafts

- The effective shaft spring constant k not only depends on the nature of the shaft, but also on the bending constraint imposed by the bearings
- For example, if the bearings are attached to a rigid support that prevents rotation of the bearings about any axis perpendicular to the line AB, the stiffness k would be that of a fixed-fixed beam
- If the bearings were self-aligning (free to rotate), the k used would be that of a pinned-pinned beam
- In starting and stopping rotating machines that may operate at speeds greater than their natural circular frequencies, such as gas turbines, large amplitudes of vibration can build up as the machine passes through the critical speed $\omega = \omega_n$
- These can be reduced by passing through the critical speed as quickly as possible since the amplitude buildup occurs over time
- However, the problem can be avoided if it is possible to design the system with a natural circular frequency ω_n that is much greater than the maximum operating circular frequency ω, meaning that $\omega/\omega_n \ll 1$

Example 2.2

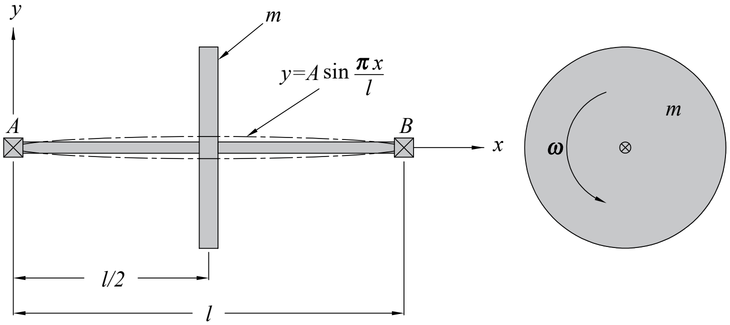
- Consider a shaft-and-disk system that is supported by self-aligning bearings; the steel shaft can be considered as a simply supported beam (pinned-pinned) for purposes of selecting a spring constant k for design purposes.
- The rotating disk is midway between the bearings A and B.



Example 2.2

For this system:

$$m = 12 \text{ kg (mass of disk)}$$
$$l = 0.5 \text{ m (length of shaft)}$$
$$d = 25.4 \text{ mm (diameter of shaft)}$$
$$\rho = 7843 \text{ kg/m}^3 \text{ (density of shaft)}$$
$$E = 206.8 \text{ GPa ((Young's) modulus of elasticity of shaft)}$$



Example 2.2

- The system to be designed is intended to operate over a range of speeds varying from 2400 to 3600 rpm and there is concern that the bearing reaction forces R_A and R_B could be dangerously large at the critical speed if the disk is not balanced.
- Manufacturing experience suggests that even with precise machining and homogeneous material, the eccentricity e of the disk can be kept to, at most, 0.05 mm
- Determine the following

  (a) The critical speed ω_n of the shaft-and-disk system including the distributed mass of the shaft

  (b) The maximum bearing reaction forces R_A and R_B that could be anticipated over the range of 2400 to 3600 rpm, with $e = 0.05$ mm and 2% damping ($\zeta = 0.02$)

Solution

(a) The critical speed ω_n

Since the distributed mass of the shaft is part of the system, the most straightforward approach is to use an energy method, known as *Rayleigh's energy method*

Considering the shaft as a simply supported (pinned-pinned) beam, the shape function

$$y = A\sin\left(\frac{\pi x}{l}\right)$$

satisfies the conditions of zero deflection and zero moment at its ends

With the disk located at the centre of the shaft, parameter A refers to the displacement of both the shaft and disk centres

It can be shown that the maximum kinetic energy of the shaft and disk is

$$T_{\max} = \frac{\gamma}{2}\int_0^{l} (y\omega_n)^2\,dx + \frac{m}{2}(A\omega_n)^2$$

where

$$\gamma = \text{mass per unit length of shaft [kg/m]}$$

$$\gamma l = \rho\left(\pi d^2/4\right) l = \text{mass of shaft [kg]}$$

$$m = \text{mass of disk [kg]}$$

Substituting the shape function gives

$$T_{\max} = \frac{\gamma A^2\omega_n^2}{2}\int_0^{l} \sin^2\left(\frac{\pi x}{l}\right)dx + \frac{mA^2\omega_n^2}{2}$$

Integrating and substituting the limits yields

$$T_{\max} = \frac{A^2\omega_n^2}{2}\left(\frac{\gamma l}{2} + m\right)$$

The mass of the shaft is

$$\gamma \mathrm{l} \;=\; \rho\left(\pi d^2/4\right)\mathrm{l} \;=\; 7843\ \mathrm{kg/m^3}\,\frac{\pi}{4}(0.0254\ \mathrm{m})^2(0.5\ \mathrm{m}) \;=\; 1.987\ \mathrm{kg}$$

Substituting this value and the mass of the disk into the maximum kinetic energy equation gives

$$T_{\max} \;=\; \frac{A^2\omega_n^2}{2}(12.99\ \mathrm{kg})$$

Since y is measured from the static equilibrium position of the system, the maximum *change* in potential energy from its potential energy in the static equilibrium position is given by the unfortunate equation

$$U_{\max} \;=\; \frac{EI}{2}\int_0^{\mathrm{l}}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

Fortunately, however, since the inertia effect due to the eccentricity of the mass centre of the disk has the same effect as a concentrated load at the shaft centre, the maximum *strain energy* can be determined more simply using

$$U_{\max} \;=\; \frac{1}{2}kA^2$$

The spring constant of a simply supported beam loaded at its centre is

$$k = \frac{48EI}{l^3}$$

where

$$I = \frac{\pi d^4}{64} = \frac{\pi (0.0254 \text{ m})^4}{64} = 2.043(10)^{-8} \text{ m}^4$$
$$E = 206.8(10)^9 \text{ Pa}$$
$$l = 0.5 \text{ m}$$

Substituting these values into the maximum strain energy equation gives

$$U_{\max} = \frac{A^2}{2}(1.622 \text{ N/m})(10)^6$$

According to Rayleigh's energy method, $T_{\max} = U_{\max} =$ constant, therefore

$$\frac{A^2\omega_n^2}{2}(12.99 \text{ kg}) = \frac{A^2}{2}(1.622 \text{ N/m})(10)^6$$

Solving for ω_n reveals that

$$\omega_n = 353.4\mathrm{rad/s}$$

so that the critical speed of the shaft-and-disk system is

$$N_{\mathrm{crit}} = \frac{353.4 \text{ rad/s}(60 \text{ s/min})}{2\pi \text{ rad/rev}} = 3374.7 \text{ rpm}$$

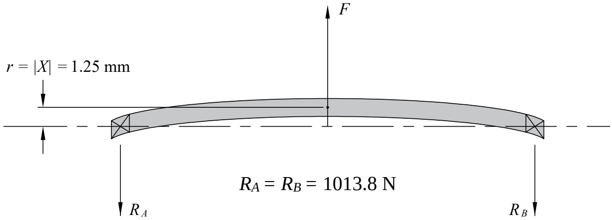
(b) Since the critical speed is within the operating range of 2400-3600 rpm, maximum displacement of the whirling shaft at the location of the disk will occur when $\omega/\omega_n = 1$, hence using Equation (30) we obtain

$$r = |X| = \frac{e}{2\zeta} = \frac{0.05 \text{ mm}}{2(0.02)} = 1.25 \text{ mm}$$

Considering the shaft as a spring of stiffness k and deflection r, the dynamic bearing reaction forces at resonance are

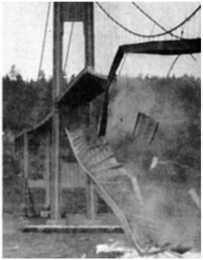
$$R_A = R_B = \frac{kr}{2} = \frac{(1.622 \text{ N/m})(10)^6(0.00125 \text{ m})}{2} = 1013.8 \text{ N}$$

Hence, it is clear that even a very small eccentricity, such as $e = 0.05$ mm in this example, can cause appreciable bearing reaction forces



Flow-induced Vibration

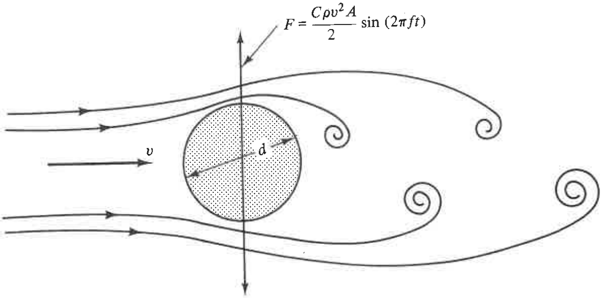
- Vibration caused by fluid flowing around or through objects is referred to as *flow-induced vibration*
- The catastrophic failure of the Tacoma Narrows bridge in 1940 is a well known example of the results of wind-induced oscillation
- Vibration caused by fluid flow through structures has been observed in pipelines, tubing in pumping systems, air compressors, et c.



Flow-induced Vibration

- There is a broad variety of flow-induced vibration
- Aircraft are designed to avoid the following aeroelastic problems:
  1. *Divergence* where the aerodynamic forces increase the angle of attack of a wing which further increases the force
  2. *Control reversal* where control activation produces an opposite aerodynamic moment that reduces, or in extreme cases, reverses the control effectiveness
  3. *Flutter* which is the uncontrollable vibration that can lead to the destruction of an aircraft
- Aeroelasticity problems can be prevented by adjusting the mass, stiffness, or aerodynamics of structures which can be determined and verified through the use of calculations, ground vibration tests and flight flutter trials
- Low frequency *galloping* in transmission lines occurs when lift and drag forces caused by air flowing over ice formations on the lines
- High frequency *singing* in transmission lines occurs due to vortex-shedding exciting the higher harmonics of the lines

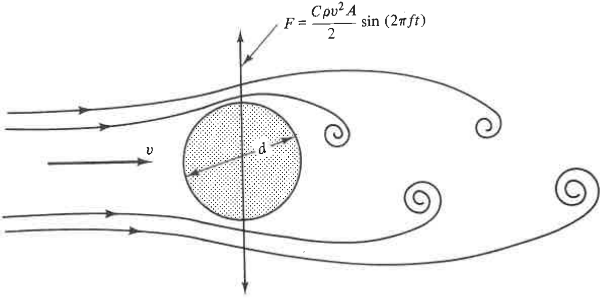
Vortex-shedding



- When fluid flows past a right-circular cylinder with sufficient velocity, vortices are formed in the fluid wake
- Such vortices are typically referred to as *Kármán* vortices, and shed in a regular pattern over a wide range of Reynolds numbers

$$R = \frac{vd\rho}{\mu}$$

Vortex-shedding



- The vortices shed alternately from opposite sides of the cylinder with a frequency f
- This causes an alternating pressure on each side of the cylinder, which acts as a sinusoidally varying force F which is perpendicular to the velocity of the fluid *before* the flow is disturbed
- The object will tend to move toward the low-pressure zone

Vortex-shedding

- The force is given by

$$F = \frac{C\rho v^2 A}{2} \sin(2\pi f t)$$

where

$$C = \text{drag coefficient (dimensionless)}, \ C \approx 1 \text{ for a cylinder}$$
$$v = \text{fluid velocity [m/s]}$$
$$A = \text{projected area of cylinder perpendicular to flow velocity } v \ [\text{m}^2]$$
$$\rho = \text{mass dendity of fluid } [\text{kg/m}^3]$$

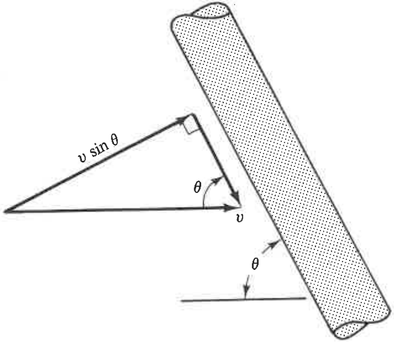
- Experiments confirm that the vortex shedding frequency is given by

$$f = \frac{Sv}{d}$$

where

$$S = \textit{Strouhal} \text{ number (dimensionless)}$$
$$d = \text{cylinder diameter [m]}$$
$$v = \text{fluid velocity [m/s]}$$

Vortex-shedding

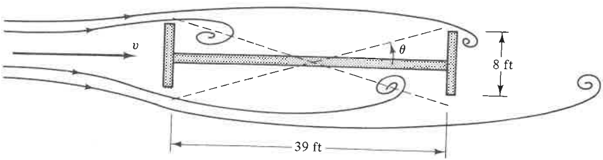


- If the fluid flow velocity is not perpendicular to the structure interrupting the flow has an angle of inclination θ, the vortex shedding frequency becomes

$$f = \frac{Sv\sin\theta}{d}$$

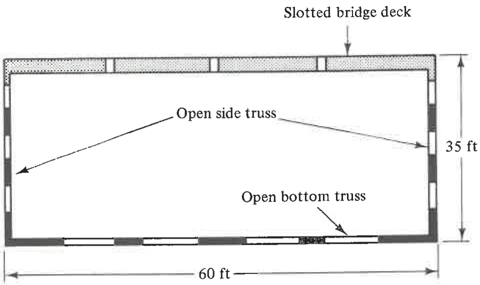
in which $v\sin\theta$ is the orthogonal component of the velocity

Vortex-shedding



- The destruction of the Tacoma Narrows bridge was caused by vortex shedding
- A steady wind of approximately 65 km/h resulted in a vortex shedding frequency at the natural *torsional* frequency of the bridge H-shaped deck
- This resonant condition caused torsional displacements (twist angle θ) along the bridge that reached amplitudes of more than 45°

Vortex-shedding



- The bridge replacing the destroyed one was built with open sided trusses between the bridge deck and the open bottom truss
- This rectangular section is many times stiffer in torsion than the H-shaped one
- Because it is stiffer, the natural frequency of the replacement bridge is considerably higher than any vortex shedding frequency that might be encountered