MAAE 3004
Dynamics of Machinery

Lecture Slide Set 3

Free and Forced Vibration of Multiple Degree of Freedom Systems

Part I

Department of Mechanical and Aerospace Engineering
Carleton University

September 28, 2023

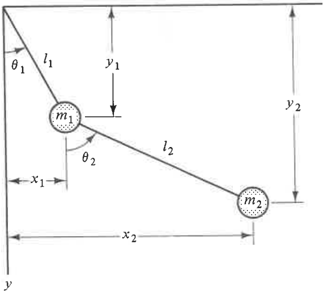
Outline

Differential Equations of Motion

- A mechanical system with n degrees of freedom (DOF) has n natural frequencies and n corresponding *normal modes* of vibration describing its configuration as it vibrates at the different natural frequencies
- When a system vibrates freely at a particular natural frequency, the normal mode corresponding to that frequency describes the configuration of the system for that particular mode
- There is a particular relationship between the amplitudes of the coordinates defining the motion of the system
- The differential equations of motion can be derived
  1. by direct application of Newton's second law
  2. by using the concept of dynamic equilibrium
  3. by using the definitions of *influence coefficients*
- We will examine all three

Generalised Coordinates

- Before we can derive the differential equations of motion for *n*-DOF systems, we must specify a sufficient number of *independent* coordinates to completely describe the configurations
- There is one coordinate associated with each DOF
- Any one of these coordinates can change without necessitating a change in the others, and is thus *independent* of the others
- Such coordinates are referred to as *generalised coordinates*



For the double pendulum we can write

$$x_1 = l_1 \sin\theta_1 \quad x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_1 = l_1 \cos\theta_1 \quad y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

The independent coordinates are θ_1 and θ_2, since x_1, x_2, y_1, and y_4 depend on θ_1 and θ_2

θ_1 and θ_2 can change independently, hence the system has 2-DOF

Using Newton's Second Law

- In deriving the differential equations of motion using Newton's second law, it is helpful to follow four steps

  1. Select a common positive sense for the displacements, velocities, and accelerations associated with the generalised coordinates used
  2. Measure linear and angular generalised coordinates from the static equilibrium position of the system
  3. Sketch a free-body diagram (FBD) of each mass element, elastic and damping forces acting on a mass element are shown on the FBD with senses consistent with the forces they exert on the mass element when it has a positive displacement and velocity
  4. Apply Newton's second law to the free body of each mass element, this will result in n differential equations of motion for an n-DOF mechanical system

- These equations are generally not independent of each other, since a particular generalised coordinate can be in more than one of them
- The equations are said to be *coupled*

Using Newton's Second Law

- Using the preceding steps, Newton's second law applied to the i^{th} mass element yields

$$\sum F_x = m_i \ddot{x}_i$$

where $\ddot{x}_i$ is the acceleration of the mass centre of the i^{th} mass element

- For a particular angular generalised coordinate θ_i, describing the angular displacement of the i^{th} mass

$$\sum M_0 = I_0 \ddot{\theta}_i$$

where $\sum M_0$ is the summation of moments of forces about an axis through 0, and I_0 is the mass moment of inertia about the same axis

- However, if the axis used coincides with the axis through the mass centre G of the element, then

$$\sum M_G = \bar{I} \ddot{\theta}_i$$

where $\bar{I}$ is the mass moment of inertia about an axis passing through the centre of mass.

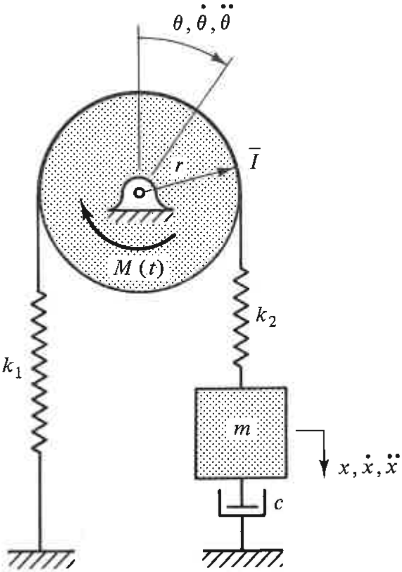
- For any other parallel axis, recall the parallel-axis theorem

$$I_0 = \bar{I} + md^2$$

where d is the distance between the two parallel axes

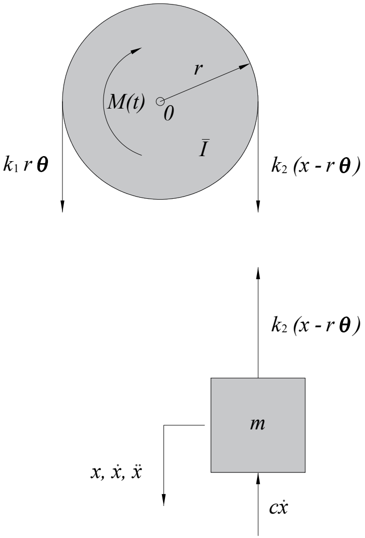
Example 3.1

- The 2-DOF system consists of a pulley, a mass m, two springs, a dashpot, and a cable connecting them
- The pulley's centroidal mass moment of inertia is $\bar{I}$, the dashpot has a damping coefficient of c, and the spring coefficients are k_1 and k_2
- There is sufficient friction between the cable and pulley to prevent the cable from slipping
- Determine the differential equations of motion when the pulley is subject to a time varying moment $M(t)$



Example 3.1 Solution

- The generalised coordinates are θ and x
- Clockwise rotations and linear motion downward are considered positive
- The forces exerted by the springs on the pulley and mass m accompany positive displacements θ and x
- The damping force $c\dot{x}$ acting on the mass m, has a sense opposing the positive downward velocity $\dot{x}$ of the mass m, since damping forces always oppose motion
- The weight force mg is not considered in the FBD since the spring forces cancel the effect of gravity in order to maintain static and dynamic equilibrium



Example 3.1 Solution

- Using the FBD of the pulley, we can write

$$\sum M_0 = \bar{I}\ddot{\theta}$$

- The sum of the moments is

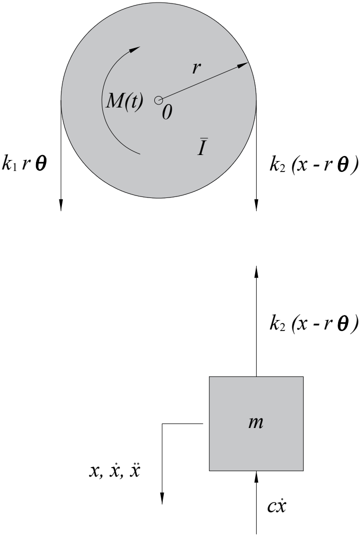
$$-k_1 r^2\theta + k_2(x - r\theta)r + M(t) = \bar{I}\ddot{\theta}$$

or

$$\bar{I}\ddot{\theta} + (k_1 + k_2)r^2\theta - k_2 r x = M(t) \quad (1)$$

- Applying Newton's second law to the FBD of the mass m gives

$$\sum F_x = m\ddot{x}$$



Example 3.1 Solution

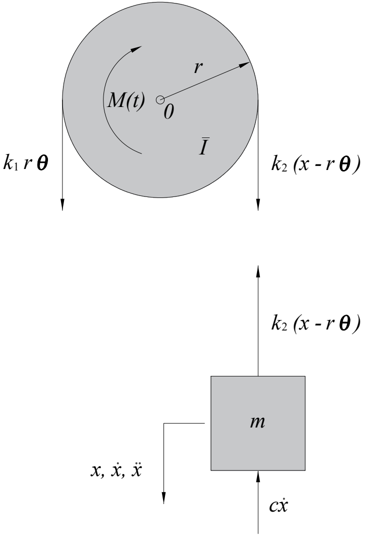
- The sum of the forces is

$$-k_2(x - r\theta) - c\dot{x} = m\ddot{x}$$

or

$$m\ddot{x} + c\dot{x} + k_2(x - r\theta) = 0 \quad (2)$$

- Equations (1) and (2) are the differential equations of motion of this system
- If the excitation moment $M(t)$ were zero, the system would have damped free vibration when either or both of the mass elements were disturbed from the static-equilibrium position



Mass, Damping, and Stiffness Matrices

- For n-DOF mechanical systems we need to express the n equations of motion in matrix form so that we can use tools from linear algebra to determine the n natural frequencies.
- For example, Equations (1) and (2) from the previous example can be expressed as

$$\begin{bmatrix} \bar{I} & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2)\, r^2 & -k_2 r \\ -k_2 r & k_2 \end{bmatrix} \begin{bmatrix} \theta \\ x \end{bmatrix} = \mathbf{I} \begin{bmatrix} M(t) \\ 0 \end{bmatrix}$$

where $\mathbf{I}$ is the 2×2 identity matrix

- Equation (1) can be recovered from the matrix equation by multiplying the first row of the square matrices by the corresponding column matrices (vectors), while Equation (2) can be recovered by multiplying the second row of the square matrices by the corresponding column matrices
- The matrix form of the n differential equations of motion for an n-DOF mechanical system will generally involve a mass matrix $\mathbf{M}$, a damping matrix $\mathbf{C}$, and a stiffness matrix $\mathbf{K}$

Mass, Damping, and Stiffness Matrices

- For Example 3.1, these matrices are

$$\mathbf{M} = \begin{bmatrix} \bar{I} & 0 \\ 0 & m \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} (k_1 + k_2)\, r^2 & -k_2 r \\ -k_2 r & k_2 \end{bmatrix},$$

Mass, Damping, and Stiffness Matrices

- The general form of the matrix differential equation of motion of an n-DOF system subjected to excitation forces and/or moments is

$$\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} \tag{3}$$

where x_1, x_2, $\cdots$, x_n are generalised coordinates that can be either linear or angular displacements, and F_1, F_2, $\cdots$, F_n are excitation forces or moments

Mass, Damping, and Stiffness Matrices

- Equation (3) can be written in the beautifully elegant and compact form as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F} \tag{4}$$

  where $\mathbf{F}$ is a column vector whose elements are excitation forces and/or moments and $\ddot{\mathbf{X}}$, $\dot{\mathbf{X}}$, and $\mathbf{X}$ are column vectors whose elements are generalised linear and/or angular accelerations, velocities, and displacements

- Equation (4) may represent any mechanical system where the number of DOF can be any integer between 1 and $\approx \infty$

- For undamped free vibration, Equation (4) reduces to

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0} \tag{5}$$

  where $\mathbf{0}$ is the *null* vector whose every column element is 0

- As we shall soon see, Equation (5) is used in obtaining the equations used to determine the natural frequencies and corresponding mode shapes of n-DOF mechanical systems

- But first, let's consider the properties of the mass, damping, and stiffness matrices

Mass, Damping, and Stiffness Matrices

- The following facts are true only for small displacements
- The curious engineer might ask "what are the bounding limits for small ... at what point does small become large?"
- The answer is, unfortunately, well beyond the scope of this course
- For small displacements, the mass, damping, and stiffness matrices, **M**, **C**, and **K**, are symmetric
- That is

$$m_{ij} = m_{ji} \qquad \text{(elements of mass matrix)}$$

$$c_{ij} = c_{ji} \qquad \text{(elements of damping matrix)}$$

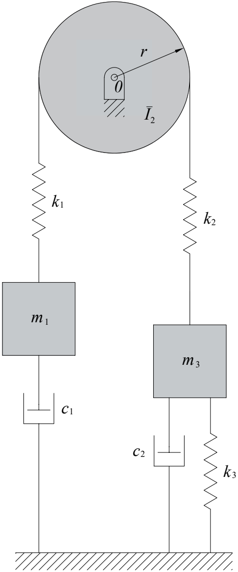
$$k_{ij} = k_{ji} \qquad \text{(elements of stiffness matrix)}$$

Mass, Damping, and Stiffness Matrices

- The mass matrix **M** is generally diagonal, where all elements not on the diagonal are zero, i.e.

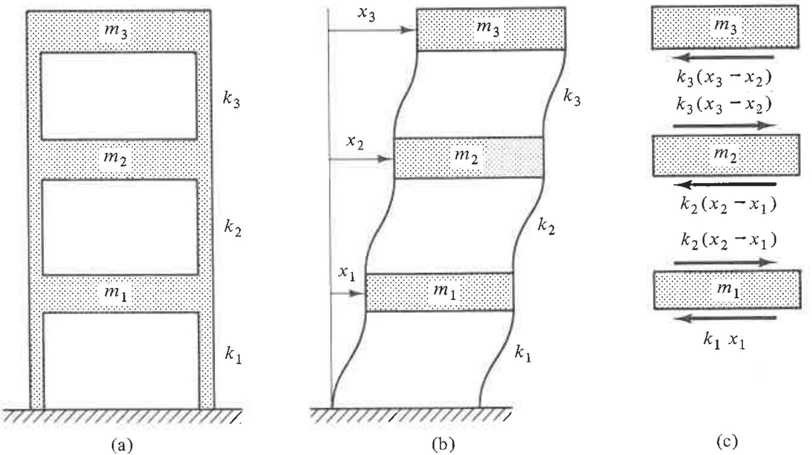
$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & \bar{I}_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

- This is the case for the system in the image
- However, this is not always true, e.g., when the equations are dynamically coupled, but **M** will still be symmetric when not diagonal
- In systems where the equations of motion are dynamically coupled, the mass matrix is not diagonal, but it is nonetheless diagonalisable, and diagonality is required to solve the equations
- The damping and stiffness matrices, **C** and **K**, are typically non-diagonal, but they are generally always symmetric



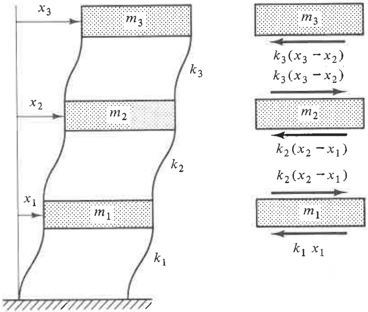
Example 3.2

- Determine the differential equations of motion for the undamped free vibration of the three-story building illustrated schematically in the image



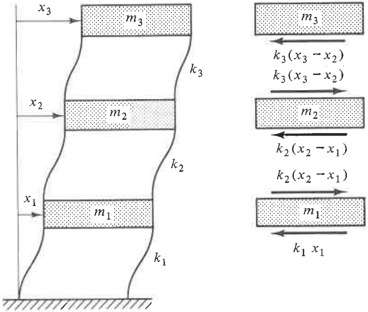
Example 3.2 Assumptions

- We assume that the mass distribution of the building can be represented by the lumped masses at the three different vertical levels, as illustrated
- We further assume that the structure of the lumped masses is non-deformable in comparison with the supporting columns, as illustrated in the left-hand side of the schematic diagram
- The spring constants shown in the figure are equivalent constants, representing the aggregate stiffness of all the columns supporting a given floor, which is done by considering the columns supporting the floor as springs in parallel



Example 3.2 Solution

- If the building is deformed by a force applied along a line of action of the mass centre m_3 of the third floor, then $x_3 > x_2 > x_1$, with the x_i, $\dot{x}_i$ and $\ddot{x}_i$ all positive to the right
- When the force is suddenly released the building will vibrate freely in the absence of damping
- The corresponding spring forces acting on each floor will be as illustrated in the FBD
- In applying Newton's second law, the spring forces have the same positive sense as the x_i, $\dot{x}_i$ and $\ddot{x}_i$



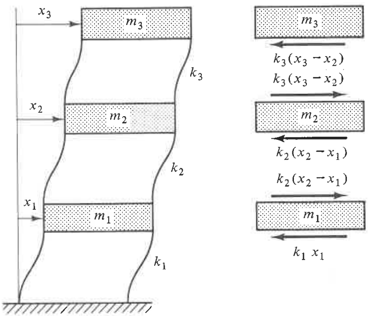
Example 3.2 Solution

- Applying Newton's second law to the FBDs of the three lumped masses yields

$$\begin{aligned} -k_1x_1 + k_2(x_2 - x_1) &= m_1\ddot{x}_1 \\ -k_2(x_2 - x_1) + k_3(x_3 - x_2) &= m_2\ddot{x}_2 \\ -k_3(x_3 - x_2) &= m_3\ddot{x}_3 \end{aligned}$$

- Collecting and rearranging the terms in each equation to put them into the appropriate form to allow us to determine the mass and stiffness matrices **M** and **K** leads to

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 &= 0 \\ m_3\ddot{x}_3 - k_3x_2 + k_3x_3 &= 0 \end{aligned}$$



Example 3.2 Solution

- By inspection, we can write these three differential equations of motion for the building in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- This matrix equation can be written in the compact symbolic form as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0}$$

Influence Coefficients: Stiffness; Flexibility; Damping

- As we have just seen, the differential equations of motion of a mechanical system can be written in the form of a mass matrix **M**, a damping matrix **C**, and a stiffness matrix **K**
- However, we can also write them in terms of a flexibility matrix **A**, which is the inverse of the stiffness matrix **K**
- That is

$$\mathbf{A} = \mathbf{K}^{-1}$$

or

$$\mathbf{K} = \mathbf{A}^{-1}$$

- The elements k_{ij}, a_{ij}, and c_{ij} of the stiffness, flexibility, and damping matrices are referred to as *influence coefficients*
- The subscript i is the row and subscript j is the column locating the element in the matrix
- The differential equations of motion of some n-DOF systems and the equations for determining their natural circular frequencies can be written by inspection and applying the definition of the influence coefficients used

Stiffness Coefficients

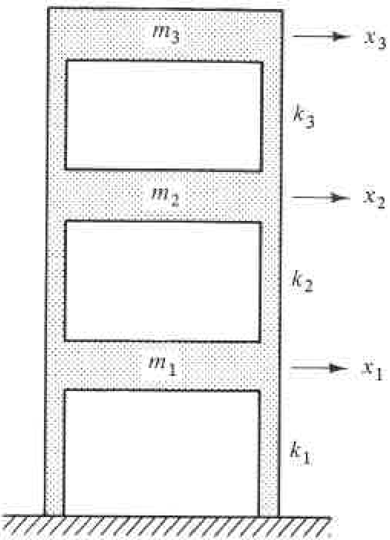
- The generalised coordinates of an n-DOF mechanical system that define the linear and/or angular displacements of the masses comprising the system are $q_1, q_2, \cdots, q_n$.
- We define the stiffness coefficient k_{ij} as the force or moment required to hold a particular coordinate q_i constant when a coordinate q_j is given a *unit* linear, or angular displacement with all other generalised coordinates held fixed
- The stiffness coefficient k_{jj} with $i = j$ then becomes the static force or moment required to give the coordinate q_j a unit linear, or angular displacement
- To determine the k_{ij}'s of a system using this definition, each mass in the system will have a generalised coordinate for each type of motion it can produce, and the subscripts i and j refer to these generalised coordinates
- For a mechanical system consisting of a positive unit linear, or angular, displacement q_j of one of the masses, the k_{ij}'s for that particular q_j are the forces and/or moments that must be applied to the masses associated with the coordinates q_i, $i = 1, 2, \cdots, n$ to maintain the system in that configuration, and are determined in total by applying the definition of k_{ij} for all q_j, $j = 1, 2, \cdots, n$

Stiffness Coefficients

- Hence, the stiffness coefficients k_{1j}, k_{2j}, $\cdots$, k_{nj} in the stiffness matrix $\mathbf{K}$ are the elements of the jth column and are the forces or moments that must be applied to the masses associated with the coordinates $q_1, q_2, \cdots, q_n$ to hold the system in a configuration consisting of a positive unit linear, or angular, displacement q_j of one of the masses, with all of the other mass displacements held at zero
- For example, if the mass associated with a translation coordinate q_1 and a rotation coordinate q_2 is given a positive unit linear displacement $j = 1$, and its angular displacement and all displacements of the other masses are held at zero, the resulting configuration yields the first column of the $\mathbf{K}$ matrix
- Similarly, if the same mass is given a unit angular displacement $j = 2$, while its linear displacement and all displacements of the other masses are held at zero, the resulting configuration yields the second column of $\mathbf{K}$
- For n-DOF, i.e. n generalised coordinates, this procedure is continued until $j = n$ to obtain all of the k_{ij} elements of the $\mathbf{K}$ matrix
- The stiffness coefficients are expressed in terms of the spring constants and can be positive or negative, relative to the specified positive sense, depending on how they act

Example 3.3

- Determine the stiffness matrix **K** of the three-story building from Example 3.2



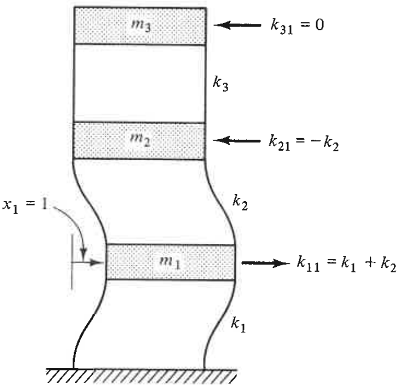
Solution

- To determine the first column of the stiffness matrix **K**, i.e. $j = 1$, the mass m_1 is given a positive unit displacement $x_1 = 1$, with m_2 and m_3 held fixed

Example 3.3 Solution

- Consider the constrained displacement where $x_1 = 1$, $x_2 = x_3 = 0$
- The image indicates that the required forces for $j = 1$ and $i = 1, 2, 3$ must be

$$\left.\begin{aligned} k_{11} &= k_1 + k_2 \\ k_{21} &= -k_2 \\ k_{31} &= 0 \end{aligned}\right\} \text{ first column of } \mathbf{K}$$

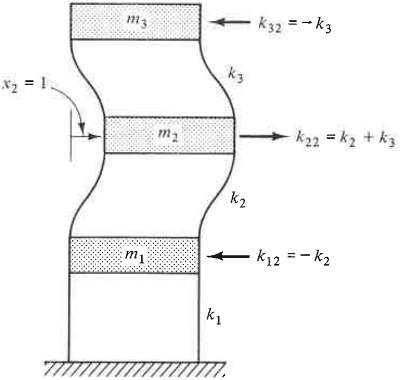


- k_{11} is the force required deform the elastic elements k_1 and k_2 by the amount $x_1 = 1$ giving the lumped mass m_1 a unit displacement
- To maintain $x_2 = 0$ requires force $x_1 k_2$ to act on m_2 in the opposite direction, hence $k_{21} = -k_2$, keeping m_2 stationary
- Since $x_2 = 0$, there is no tendency for m_3 to move so $k_{31} = 0$

Example 3.3 Solution

- To determine the second column of $\mathbf{K}$, i.e. k_{i2} where $i \in \{1, 2, 3\}$, m_2 is given the positive displacement $x_2 = 1$ while m_1 and m_3 remain stationary so that $x_1 = x_3 = 0$
- By inspection, it is to be seen that

$$\left.\begin{aligned} k_{12} &= -k_2 \\ k_{22} &= k_2 + k_3 \\ k_{32} &= -k_3 \end{aligned}\right\} \quad \text{second column of } \mathbf{K}$$



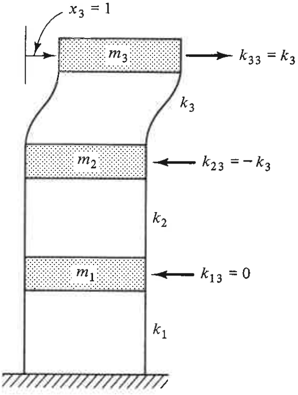
Example 3.3 Solution

- To determine the third column of **K**, i.e. k_{i3} where $i \in \{1,2,3\}$, m_3 is given the positive displacement $x_3 = 1$ while m_1 and m_2 remain stationary so that $x_1 = x_2 = 0$
- Again, by inspection, it is to be seen that

$$\left.\begin{aligned} k_{13} &= 0 \\ k_{23} &= -k_3 \\ k_{33} &= k_3 \end{aligned}\right\} \quad \text{third column of } \mathbf{K}$$

- Assembling the stiffness matrix **K**, we see that it is identical to the one obtained using Newton's second law in Example 3.2:

$$[k_{ij}] = \mathbf{K} = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \qquad (6)$$



Stiffness Coefficients

- It is **important to note** that the k_{ij}'s we have found using both the stiffness influence coefficient and Newton's second law yield the *reaction forces that the mass elements exert on the elastic elements*
- The spring forces that the elastic elements exert on the mass elements are equal in magnitude but opposite in sense
- For example, k_{11} is equal in magnitude and opposite in sense to the forces that the elastic elements exert on mass m_1 when $x_1 = 1$
- Since we are considering the steady-state response, i.e. $\sum F_i = 0$, without damping, Newton's second law states that

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0}, \qquad \text{then it must be that} \qquad \mathbf{M}\ddot{\mathbf{X}} = -\mathbf{K}\mathbf{X}$$

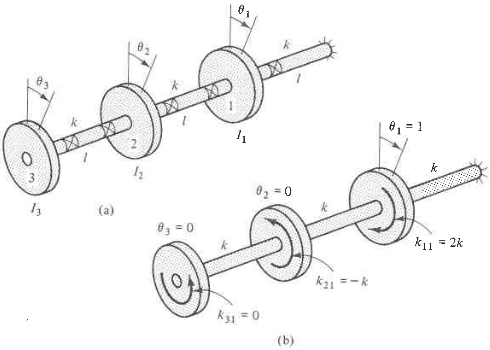
or

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = - \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- $-\mathbf{KX}$ are the forces the that the elastic elements exert on the masses and $\mathbf{KX}$ are the forces the masses exert on the elastic elements

Example 3.4

- A shaft with three evenly spaced discs rigidly attached to it is fixed at one end
- The mass moments of inertia of the discs are I_1, I_2, and I_3
- The torsional spring constant of each of the three shaft intervals of length l is $k = GJ/\mathrm{l}$ where G is the shear modulus and J is the polar mass moment of inertia of the shaft interval

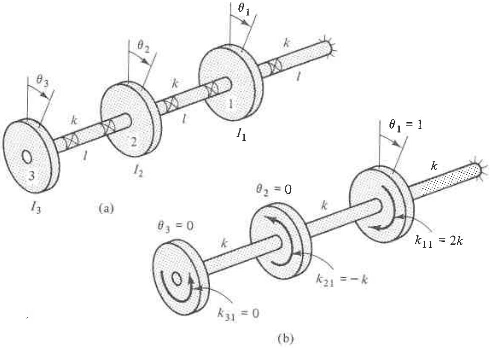


Example 3.4

- Using the angular displacements θ_1, θ_2, and θ_3 as generalised coordinates and the definition of k_{ij} determine

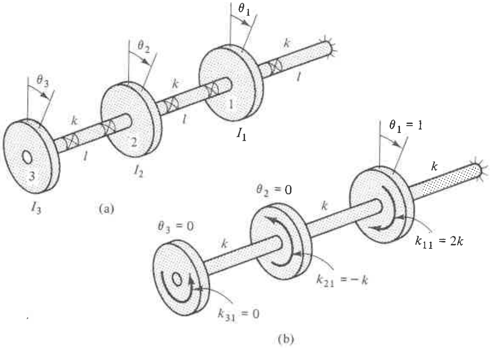
    a. The stiffness matrix $\mathbf{K}$

    b. The differential equations of motion of the undamped free vibration of the system in matrix form



Example 3.4 Solution

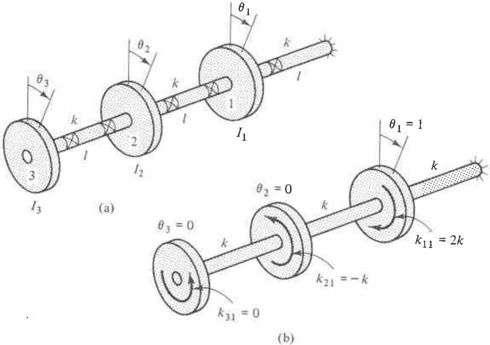
- a. To determine the first column of **K** disc 1 is given a unit angular displacement, $\theta_1 = 1$, with disks 2 and 3 held fixed so that $\theta_2 = \theta_3 = 0$
- The moment k_{11} required to rotate disc 1 through the angle $\theta_1 = 1$ is $2k$, and the moment k_{21} required to keep disc 2 fixed is $-k$
- There is no tendency for disc 3 to rotate, so $k_3 = 0$



Example 3.4 Solution

- Thus

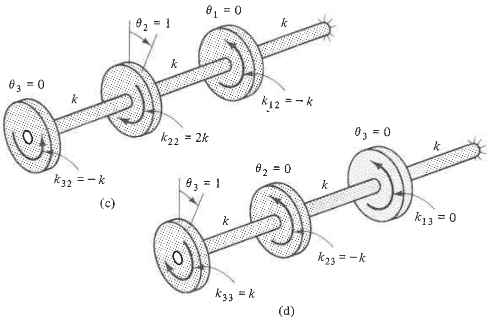
$$\left.\begin{aligned} k_{11} &= 2k \\ k_{21} &= -k \\ k_{31} &= 0 \end{aligned}\right\} \quad \text{first column of } \mathbf{K}$$



Example 3.4 Solution

- Similarly, giving disc 2 a unit angular displacement, $\theta_2 = 1$, with discs 1 and 3 held fixed so that $\theta_1 = \theta_3 = 0$, the moments required to maintain this configuration are

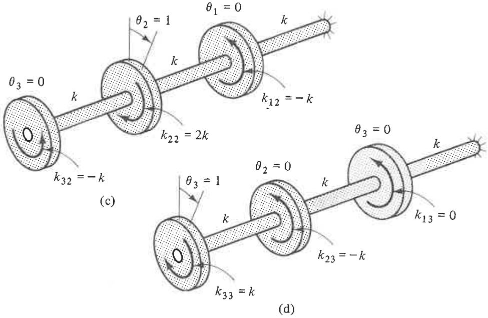
$$\left.\begin{aligned} k_{12} &= -k \\ k_{22} &= 2k \\ k_{32} &= -k \end{aligned}\right\} \quad \text{second column of } \mathbf{K}$$



Example 3.4 Solution

- Finally, with $\theta_3 = 1$ and discs 1 and 2 held fixed so that $\theta_1 = \theta_3 = 0$, the moments required to maintain this configuration are

$$\left.\begin{aligned} k_{13} &= 0 \\ k_{23} &= -k \\ k_{33} &= k \end{aligned}\right\} \quad \text{third column of } \mathbf{K}$$



Example 3.4 Solution

- b. The general form of the matrix equation expressing the equations of motion of the undamped free vibration is

$$\mathbf{M}\ddot{\mathbf{\Theta}} + \mathbf{K}\mathbf{\Theta} = \mathbf{0}$$

- Since the mass matrix $\mathbf{M}$ consists of the mass moments of inertia on the diagonal, the matrix equation expressing the differential equations of motion of the undamped free vibration of the disc-and-shaft mechanical system is

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \mathbf{0}$$

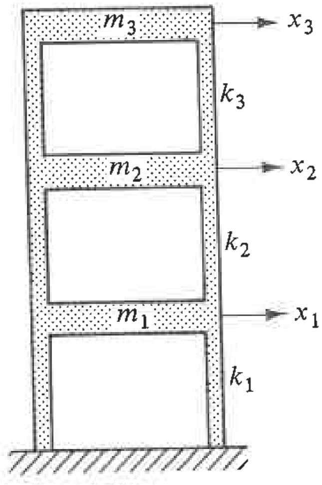
- The differential equations of motion for this disc-and-shaft mechanical system is analogous to the three story building from Example 3.3

Flexibility Coefficients

- The flexibility coefficient a_{ij} is defined as the linear or angular displacement that occurs for a particular generalised coordinate q_i when a static unit force or moment ($F = 1$, or $M = 1$) is applied at the location of a another generalised coordinate q_j with all other coordinates *free* to move
- To determine the a_{ij}'s of a mechanical system using this definition, note that each individual mass element in the system will have a generalised coordinate for each type of motion that it is capable of, and that the subscripts of the a_{ij}'s refer to these generalised coordinates
- Then, we can state that when a positive unit force F or moment M is applied to a system mass element having coordinate q_j, with all other coordinates free to move, the a_{ij}'s for that particular q_j are the linear and/or angular displacements of the masses associated with the coordinates q_i ($i = 1, 2, \cdots, n$), and are determined by applying the definition in turn for each q_j ($j = 1, 2, \cdots, n$,)

Flexibility Coefficients

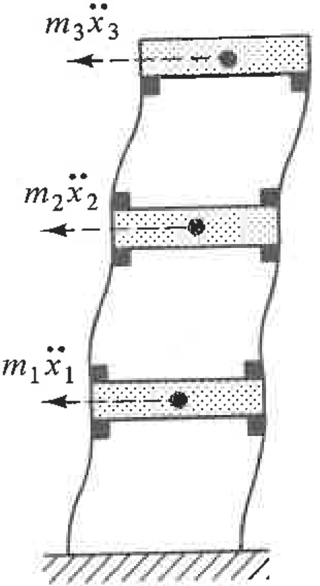
- As a demonstration, let us reconsider the three-story building, and ignore damping
- It is assumed that the distributed mass of the building can be represented effectively by the three lumped masses at the three different levels, as illustrated
- It is further assumed that the girders in the floor structures are infinitely rigid compared to the stiffness of the support columns
- Positive horizontal displacements, velocities, and accelerations of the three masses generalised coordinates x_i, $\dot{x}_i$, and $\ddot{x}_i$, $i \in \{1, 2, 3\}$, are to the right
- The use of flexibility coefficients to determine the differential equations of motion is facilitated with the concept of dynamic equilibrium



Flexibility Coefficients

- Sketch the inertia effects on the three floors with senses opposite to the assumed positive sense of the acceleration of each mass
- These inertia effects are equal in magnitude to the forces that the various *masses exert on the columns* and can be treated as if they were forces acting on the structure
- The total displacement of a particular mass is equal to the sum of the displacements caused by each inertia effect (force in this case) acting on the columns (the elastic elements in this model)
- Then, using the definition of flexibility coefficient, we can write

$$\left.\begin{aligned} x_1 &= -a_{11} m_1 \ddot{x}_1 - a_{12} m_2 \ddot{x}_2 - a_{13} m_3 \ddot{x}_3 \\ x_2 &= -a_{21} m_1 \ddot{x}_1 - a_{22} m_2 \ddot{x}_2 - a_{23} m_3 \ddot{x}_3 \\ x_3 &= -a_{31} m_1 \ddot{x}_1 - a_{32} m_2 \ddot{x}_2 - a_{33} m_3 \ddot{x}_3 \end{aligned}\right\} \quad (7)$$



Flexibility Coefficients

- Equation (7) is the set of 3 differential equations of motion for the undamped motion of the three story building in terms of the flexibility coefficients
- Recall that the actual acceleration of the structure is *always* toward its static equilibrium position for any configuration
- Although it was assumed that the accelerations were positive and to the right in the configuration illustrated, the actual accelerations for this configuration are negative and to the left
- It then follows that the negative values for $\ddot{x}_1$, $\ddot{x}_2$, and $\ddot{x}_3$, in Equation (7) will yield the positive displacements illustrated
- Assembling Equation (7) in matrix form leads to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

Flexibility Coefficients

- We can rewrite Equation (8) in the more compact form

$$\mathbf{X} + \mathbf{A}\mathbf{M}\ddot{\mathbf{X}} = \mathbf{0} \qquad (9)$$

where **A** and **M** are flexibility and mass matrices, respectively

- The mass matrix **M** is always diagonal, unless the generalised coordinates selected lead to dynamic coupling
- We will consider dynamic coupling, but later on
- We can multiply both sides of Equation (9) by $\mathbf{A}^{-1}$ leading to

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{A}^{-1}\mathbf{X} = \mathbf{0} \qquad (10)$$

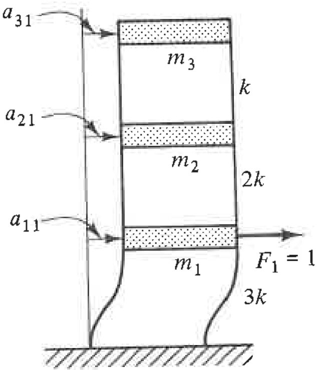
- Comparing this result to Equation (5), it is to be seen that

$$\mathbf{A}^{-1} = \mathbf{K}$$

- Thus, the inverse of the flexibility matrix, $\mathbf{A}^{-1}$, is the stiffness matrix **K**
- Since the inverse of a symmetric matrix is also symmetric, both $\mathbf{A}^{-1}$ and **K** are symmetric

Example 3.5

- Determine the flexibility matrix **A** and the differential equations of motion for the undamped free vibration of the three-story building illustrated schematically in the image using the definition of flexibility coefficients
- The equivalent spring constants for the first, second, and third floor masses m_1, m_2, and m_3 are $3k$, $2k$, and k, respectively



Solution

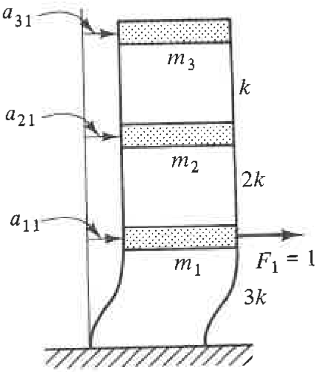
- To determine the first column of **A**, a static unit force $F_1 = 1$ is applied horizontally to m_1 with its line of action passing through the mass centre, which deforms the structure as illustrated
- From elementary statics we see that

$$F_1 = 1 = 3ka_{11} \Rightarrow a_{11} = \frac{1}{3k}$$

Example 3.5 Solution

- By inspection, it should be apparent that $a_{11} = a_{21} = a_{31}$
- Assembling the first column of **A** leads to

$$\left.\begin{aligned} a_{11} &= \frac{1}{3k} \\ a_{21} &= \frac{1}{3k} \\ a_{31} &= \frac{1}{3k} \end{aligned}\right\} \text{first column of } \mathbf{A}$$

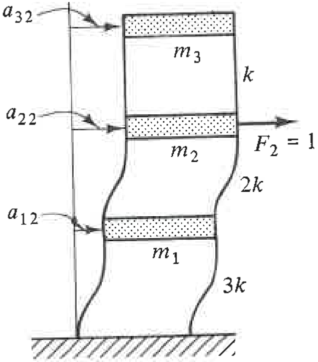


Example 3.5 Solution

- To determine the second column of **A**, a static unit force $F_2 = 1$ is applied horizontally to m_2 with its line of action passing through the mass centre, which deforms the structure as illustrated

- From statics and inspection, it can be seen that the columns in each of the first two stories are subjected to the force $F_2 = 1$, thus

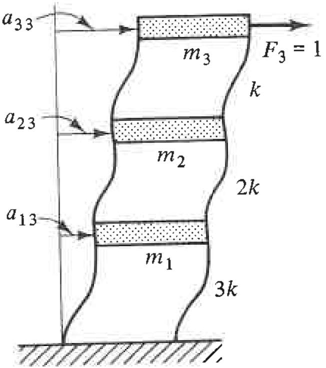
$$\left.\begin{aligned} a_{12} &= \frac{1}{3k} \\ a_{22} &= \frac{1}{3k} + \frac{1}{2k} = \frac{5}{6k} \\ a_{32} &= a_{22} = \frac{5}{6k} \end{aligned}\right\} \text{second column of } \mathbf{A}$$



Example 3.5 Solution

- To determine the third column of **A**, a static unit force $F_3 = 1$ is applied horizontally to m_3 with its line of action passing through the mass centre, which deforms the structure as illustrated
- From statics and inspection, it can be seen that the columns supporting all 3 masses are subjected to the force $F_3 = 1$, so that

$$\left.\begin{aligned} a_{13} &= \frac{1}{3k} \\ a_{23} &= \frac{1}{3k} + \frac{1}{2k} = \frac{5}{6k} \\ a_{33} &= \frac{1}{3k} + \frac{1}{2k} + \frac{1}{k} = \frac{11}{6k} \end{aligned}\right\} \text{third column of } \mathbf{A}$$



Example 3.5 Solution

- The differential equations of motion for the undamped free vibration of the three story building in terms of the flexibility matrix **A** are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3k} & \frac{1}{3k} & \frac{1}{3k} \\ \frac{1}{3k} & \frac{5}{6k} & \frac{5}{6k} \\ \frac{1}{3k} & \frac{5}{6k} & \frac{11}{6k} \end{bmatrix} \begin{bmatrix} m_1\ddot{x}_1 \\ m_2\ddot{x}_2 \\ m_3\ddot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Which can be simplified to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{6k} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix} \begin{bmatrix} m_1\ddot{x}_1 \\ m_2\ddot{x}_2 \\ m_3\ddot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

Example 3.5 Solution

- Is it true that the flexibility matrix **A** in Equation (11) is the inverse of the stiffness matrix **K** obtained in Example 3.3 as Equation (6)?
- Substituting $k_1 = 3k$, $k_2 = 2k$, and $k_3 = k$ into Equation (6) yields

$$\mathbf{K} = k \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{12}$$

- Computing $\mathbf{K}^{-1}$ we obtain

$$\mathbf{K}^{-1} = \begin{bmatrix} \frac{1}{3k} & \frac{1}{3k} & \frac{1}{3k} \\ \frac{1}{3k} & \frac{5}{6k} & \frac{5}{6k} \\ \frac{1}{3k} & \frac{5}{6k} & \frac{11}{6k} \end{bmatrix} = \mathbf{A} \tag{13}$$

Damping Coefficients

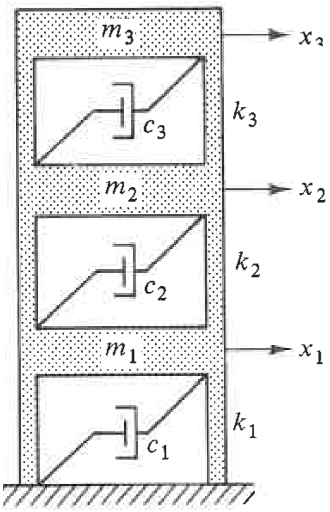
- When damping in the mechanical system is considered, the symmetric damping matrix **C**, comprised of elements c_{ij}, must be determined
- If a particular generalised coordinate q_j is given a positive unit linear or angular velocity $\dot{q}_j$ with all other coordinates held fixed, then for $i \in \{1, 2, \cdots, n\}$ the c_{ij}'s are the forces or moments required to maintain the specified velocity configuration
- The damping coefficient is similar to the stiffness coefficient in the sense that the c_{ij}'s are associated with forces and moments corresponding to unit *velocities*, while the k_{ij}'s are associated with forces and moments and corresponding unit *displacements*
- As a result, the procedure for determining the damping matrix **C** is similar to the procedure for determining the stiffness matrix **K**

Example 3.6

- Determine the damping matrix **C** and the differential equations of motion for the damped free vibration of the three-story building illustrated schematically in the image using the definition of damping coefficients
- The damping coefficients are c_1, c_2, and c_3, which are damping forces per unit velocity having units $[Ns/m]$

Solution

- Displacement and velocity of the building are assumed to be positive to the right
- To determine the first column of **C**, a unit velocity $\dot{x}_1 = 1$ is applied horizontally to m_1, with m_2 and m_3 held stationary so that $\dot{x}_2 = \dot{x}_3 = 0$



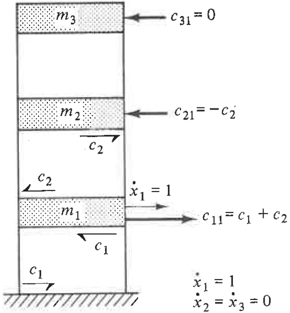
Example 3.6 Solution

- Recall that a damping force **always** opposes velocity, but the c_{ij}'s are equal in magnitude and opposite in sense to the forces the dampers exert on the masses
- The equivalent damping coefficient for series connection is the sum of the inverse of its reciprocal of individual damping coefficients

- The equivalent damping coefficient for parallel connection is the sum of individual damping coefficients
- The dampers acting on m_1 are c_1 and c_2 acting in parallel
- To maintain the velocity configuration, the first column of **C** must be

$$\left.\begin{aligned} c_{11} &= c_1 + c_2 \\ c_{21} &= -c_2 \\ c_{31} &= 0 \end{aligned}\right\} \quad \text{first column of } \mathbf{C}$$

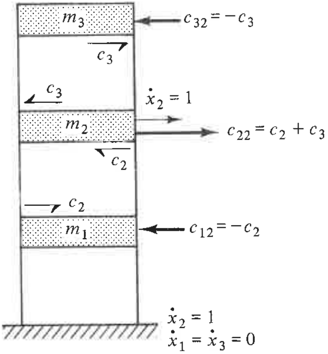
- The coefficient c_{21} is negative because it acts in the opposite direction assumed positive for x_2



Example 3.6 Solution

- To determine the second column of **C**, the second story mass m_2 is given a unit velocity $\dot{x}_2 = 1$ to the right while masses m_1 and m_3 remain stationary so that $\dot{x}_1 = \dot{x}_3 = 0$
- The second column of **C** as illustrated in the figure to maintain the dynamic equilibrium since the sum of the forces acting on the building must be zero given the constant velocity

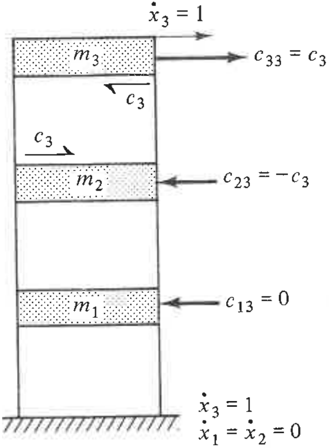
$$\left.\begin{aligned} c_{12} &= -c_2 \\ c_{22} &= c_2 + c_3 \\ c_{32} &= -c_3 \end{aligned}\right\} \text{ second column of } \mathbf{C}$$



Example 3.6 Solution

- To determine the third column of **C**, the third story mass m_3 is given a unit velocity $\dot{x}_3 = 1$ to the right while masses m_1 and m_2 remain stationary so that $\dot{x}_1 = \dot{x}_2 = 0$
- The third column of **C** as illustrated in the figure to maintain the dynamic equilibrium since the sum of the forces acting on the building must be zero given the constant velocity

$$\left.\begin{aligned} c_{12} &= 0 \\ c_{22} &= -c_3 \\ c_{32} &= c_3 \end{aligned}\right\} \text{third column of } \mathbf{C}$$



Example 3.6 Solution

- Combining the three columns of **C** reveals the symmetric damping coefficient matrix

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

  which has precisely the same form as **K** from Example 3.3

- The matrix form of the differential equations of motion for the damped free vibration of the 3 DOF system is

$$\mathbf{M}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \mathbf{C}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \mathbf{K}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

- Or, more compactly

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0}$$

Damping Coefficients

- In Example 3.6 it was fairly straightforward to determine the correct sense of the damping influence coefficients by inspection
- For damped systems where the correct senses of the damping coefficients is not readily apparent, a good way to proceed is to sketch FBDs of the mass elements and show the c_{ij}'s acting with the same sense as the positive sense prescribed for the generalised coordinates
- The correct signs for the c_{ij} values will reveal themselves when the sum of the forces or moments for the mass element are summed and equated to zero

Eigenvalues and Eigenvectors

- Differential equations of motion are **dynamically coupled** if the generalised coordinates used to describe the motion of the mechanical system result in a non-diagonal mass matrix $\mathbf{M}$
- Differential equations of motion are **statically coupled** if the generalised coordinates used to describe the motion of the mechanical system result in a non-diagonal stiffness matrix $\mathbf{K}$
- The advantage of a diagonal matrix $\mathbf{M}$ is that its inverse is simply another diagonal matrix $\mathbf{M}^{-1}$ whose diagonal elements are the reciprocals of the diagonal elements of $\mathbf{M}$
- An n-DOF mechanical system has n natural frequencies
- For each natural frequency there is a corresponding normal mode shape that defines a distinct relationship between the amplitudes of the generalised coordinates for that mode
- The squares of the natural circular frequencies ω_n^2 and corresponding sets of coordinate values describing the normal mode shapes are referred to as **eigenvalues** and **eigenvectors**, respectively, and they are of fundamental importance in the analysis of free and forced vibration of multiple DOF mechanical systems

Eigenvalues and Eigenvectors

- Recall the that the damped natural circular frequency ω_d is related to the undamped natural circular frequency ω_n by

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

- For many real systems having less than 20% damping, $\zeta < 0.20$, the damped natural circular frequency ω_d is approximately equal to the undamped natural circular frequency ω_n

- To obtain the algebraic equations needed to determine the eigenvalues and eigenvectors, we can pre-multiply Equation (5) by $\mathbf{M}^{-1}$, which yields

$$\ddot{\mathbf{X}} + \mathbf{M}^{-1}\mathbf{K}\mathbf{X} = \mathbf{0} \tag{14}$$

- Since we assume undamped free vibration, we have harmonic motion for each mass so that

$$\left.\begin{aligned} x_i &= X_i e^{i\omega t} \\ \ddot{x}_i &= -\omega^2 X_i e^{i\omega t} \end{aligned}\right\} \tag{15}$$

Eigenvalues and Eigenvectors

- Substituting Equation (15) into Equation (14) yields

$$-\omega^2\mathbf{X} + \mathbf{M}^{-1}\mathbf{K}\mathbf{X} = \mathbf{0}$$

which can be rewritten as

$$\left[\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}\right]\mathbf{X} = \mathbf{0} \tag{16}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, and the scalar product $\omega^2\mathbf{I}$ is a square $n \times n$ matrix with ω^2 on the diagonal, and all non-diagonal elements are 0

- For an n-DOF mechanical system, the matrix becomes

$$\omega^2\mathbf{I} = \omega^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{or,} \quad \begin{bmatrix} \omega^2 & 0 & \cdots & 0 \\ 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^2 \end{bmatrix}$$

Eigenvalues and Eigenvectors

- Setting $\omega^2 = \lambda$ in Equation (16), and if **M** is diagonal leads to

$$\begin{bmatrix} (k_{11}/m_1 - \lambda) & k_{12}/m_1 & \cdots & k_{1n}/m_1 \\ k_{21}/m_2 & (k_{22}/m_2 - \lambda) & \cdots & k_{2n}/m_2 \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1}/m_n & k_{n2}/m_n & \cdots & (k_{nn}/m_n - \lambda) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (17)$$

- Equation (16, 17) is a set of homogeneous algebraic equations.
- After setting $\omega^2 = \lambda$, the expansion of the determinant of the coefficient matrix neads to a polynomial of degree n

$$\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n \quad = 0 \qquad (18)$$

- Equation (18) is called the *frequency* or *characteristic* equation
- The roots of Equation (18) are the λ_i eigenvalues that make the determinant equal zero, and the b_i constants depend on the values of the stiffness coefficients and masses

Eigenvalues and Eigenvectors

- Since eigenvalues $\lambda_i = \omega_i^2$, an n-DOF system has n natural frequencies, ω_1, ω_2, ω_3, $\cdots$, ω_n, hence the reference to Equation (18) as the frequency equation
- After the eigenvalues are computed, they can be substituted, one value at a time, back into Equation (17), yielding the n eigenvectors, which are relationships between the unknown X_i's
- The eigenvectors describe the normal mode shapes, or mode configurations, corresponding to the natural frequencies

Example 3.7

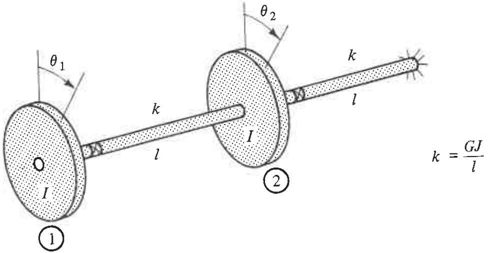
- Two identical discs each having centroidal mass moment of inertia I are rigidly attached to a steel shaft that is fixed at one end
- Each section of shaft has diameter d, segment length l, and a torsional spring constant k, where

$$\mathrm{l} = 0.6\ m \quad \text{(length of each segment of shaft)}$$

$$d = 30.0\ mm \quad \text{(shaft diameter)}$$

$$G = 800.0\ GPa \quad \text{(shaft shear modulus)}$$

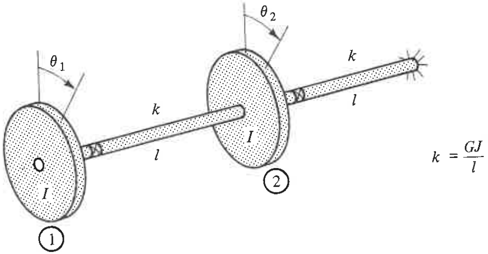
$$\mathrm{I} = 10.0\ kgm^2 \quad \text{(centroidal mass moment of inertia of each disc)}$$



Example 3.7 Solution

- Using the given system data, determine:
  a. the eigenvalues of the system;
  b. the natural frequencies of the system [Hz];
  c. the eigenvectors, i.e., the normal-mode shapes, of the system
- a. k_{11} is the moment required to give disc 1 a unit rotation, $\theta_1 = 1$
- k_{21} is the moment required to keep disc 2 stationary when k_{11} is applied, i.e., for $\theta_1 = 1$ and $\theta_2 = 0$

$$
\begin{aligned}
k_{11} &= k \\
k_{21} &= -k
\end{aligned}
$$



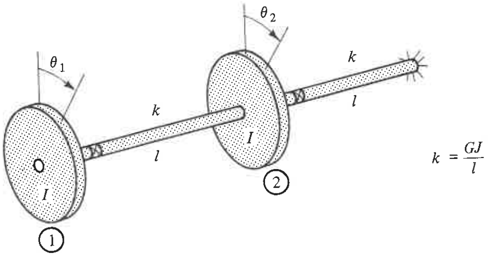
Example 3.7 Solution

- Similarly, for $\theta_1 = 0$ and $\theta_2 = 1$

$$\begin{aligned} k_{12} &= -k \\ k_{22} &= 2k \end{aligned}$$

- The stiffness and mass matrices are

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}$$



Example 3.7 Solution

- To determine the eigenvalues, we must first compute the characteristic (frequency) equation using the determinant of coefficient matrix from Equation (17)
- After setting $\omega^2 = \lambda$ we obtain

$$\det\left[\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right] = \det\begin{bmatrix} \left(\frac{k}{\mathrm{I}} - \lambda\right) & \frac{-k}{\mathrm{I}} \\ \frac{-k}{\mathrm{I}} & \left(\frac{2k}{\mathrm{I}} - \lambda\right) \end{bmatrix} = 0$$

- The determinant yields the characteristic equation

$$\left(\frac{k}{\mathrm{I}} - \lambda\right)\left(\frac{2k}{\mathrm{I}} - \lambda\right) - \left(\frac{k}{\mathrm{I}}\right)^2 = 0$$

in which $\lambda = \omega^2$

- Expanding this equation leads to the quadratic characteristic equation

$$\lambda^2 - \frac{3k}{\mathrm{I}}\lambda + \left(\frac{k}{\mathrm{I}}\right)^2 = 0$$

Example 3.7 Solution

- The roots of the quadratic characteristic equation yield the two eigenvalues for the 2-DOF mechanical system

$$\lambda_1 = \omega_1^2 = \frac{k}{\mathrm{I}}\left[\frac{3-\sqrt{5}}{2}\right] \quad \text{and} \quad \lambda_2 = \omega_2^2 = \frac{k}{\mathrm{I}}\left[\frac{3+\sqrt{5}}{2}\right]$$

- b. Using the given data and the tables for elastic elements as springs tables in Lecture Slide Set 1, the torsional spring constant is computed as

$$k = \frac{GJ}{\mathrm{l}}, \quad \text{where } J \text{ is the polar area moment of inertia,} \quad J = \frac{\pi d^4}{32}$$

which gives

$$k = \frac{800.0(10)^9 \ Pa}{0.6 \ m}\left[\frac{\pi(0.030 \ m)^4}{32}\right] = 106028.8 \ Nm/rad$$

and

$$\frac{k}{\mathrm{I}} = \frac{106028.8 \ Nm/rad}{10.0 \ kgm^2} = 10602.88 \ s^{-2}$$

Example 3.7 Solution

- substituting the value for k/I into the eigenvalue equations yields

$$\lambda_1 = \omega_1^2 = 4049.94\ s^{-2}$$
$$\lambda_2 = \omega_2^2 = 27758.70\ s^{-2}$$

- From which the two natural frequencies are

$$f_1 = \frac{\omega_1}{2\pi} = \frac{\sqrt{4049.94}}{2\pi} = 10.13\ Hz$$
$$f_2 = \frac{\omega_2}{2\pi} = \frac{\sqrt{27758.70}}{2\pi} = 26.52\ Hz$$

Example 3.7 Solution

- c. To determine the eigenvectors, which represent the normal mode shapes of this torsional vibration system, corresponding to each eigenvalue, we use the torsional version of Equation (17) to obtain

$$\begin{bmatrix} \left(\frac{k}{\mathrm{I}} - \lambda\right) & \frac{-k}{\mathrm{I}} \\ \frac{-k}{\mathrm{I}} & \left(\frac{2k}{\mathrm{I}} - \lambda\right) \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Performing the matrix-vector multiplication leads to

$$\left.\begin{aligned} \left(\frac{k}{\mathrm{I}} - \lambda\right)\Theta_1 - \frac{k}{\mathrm{I}}\Theta_2 = 0 \quad \text{(a)} \\ -\frac{k}{\mathrm{I}}\Theta_1 - \left(\frac{2k}{\mathrm{I}} - \lambda\right)\Theta_2 = 0 \quad \text{(b)} \end{aligned}\right\} \tag{19}$$

Example 3.7 Solution

- The relationships between the eigenvector components for the eigenvalues λ_1 and λ_2 can be determined from either Equation (19-a) or Equation (19-b)
- Without loss in generality, we select the first one, (a), to determine the ratio that depends on the eigenvalues

$$\frac{\Theta_2}{\Theta_1} = \frac{k/\mathrm{I} - \lambda}{k/\mathrm{I}} \tag{20}$$

- Substituting the values for λ_1 and λ_2 into Equation (20) gives

$$\left(\frac{\Theta_2}{\Theta_1}\right)_1 = \frac{10602.88 - 4049.94}{10602.88} = 0.62 \quad \text{(the first mode)}$$

and

$$\left(\frac{\Theta_2}{\Theta_1}\right)_2 = \frac{10602.88 - 27758.70}{10602.88} = -1.62 \quad \text{(the second mode)}$$

Example 3.7 Solution

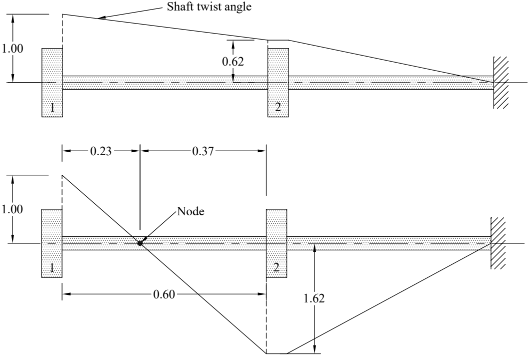
- Although there are an infinite number of values of Θ_1 and Θ_2 that will satisfy the ratios for the first and second mode, we can normalise the ratio by setting $\Theta_1 = 1$, meaning that for the first mode $\Theta_2 = 0.62$, and for the second mode $\Theta_2 = -1.62$, which gives the eigenvectors as

$$\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}_2 = \begin{bmatrix} 1 \\ -1.62 \end{bmatrix}$$

- The eigenvector for the first mode describes the undamped free vibration configuration of the system when it is vibrating at $f_1 = 10.13\ Hz$ where $\Theta_2 = 0.62\Theta_1$ at any instant in time, and since the ratio is positive, both discs are vibrating in phase with each other
- The eigenvector for the second mode describes the system configuration for the undamped free vibration mode when the system is vibrating at $f_2 = 26.52\ Hz$, with the discs vibrating at 180° out of phase with each other because the ratio is a negative number with $\Theta_2 = -1.62\Theta_1$

Example 3.7 Solution

- The figures illustrate the mode shapes, i.e., the twist configurations of the shaft where the vertical axis is the twist angle in radians and the horizontal axis is shaft length

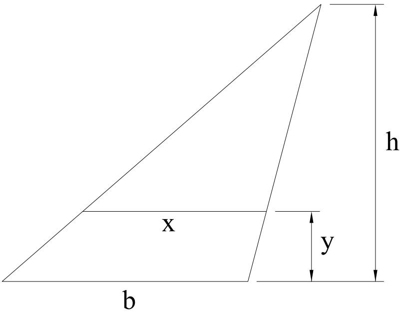


- The *node* for mode shape 2 is where the shaft has zero angular deflection, and the precise location along the length of the shaft segment between the two discs can be determined using similar triangles

Example 3.7 Solution

- In plane geometry, similar triangles are those with identical interior angles and proportional edge lengths
- The edge lengths are related according to the ratios

$$\frac{x}{b} = \frac{h-y}{h}$$



Orthogonality Properties of the Normal Modes

- There are two essential orthogonality relationships involving the normal modes, i.e., the eigenvectors, of an *n*-DOF system
- One involves the mass matrix **M**, the other involves the stiffness matrix **K**
- Familiarity with these orthogonality properties is the key to understanding the concepts of the modal analysis (eigenvector analysis) that will be discussed next
- Recall the matrix equation for undamped free vibration, Equation (5), repeated here for convenience:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0}$$

- Let's assume that the mass matrix is non-diagonal, but symmetric, and that the motion is harmonic so that the steady-state response is $x_i = X_i e^{i\omega t}$ so that from Equation (5) we obtain

$$\omega^2 \mathbf{M}\mathbf{X} = \mathbf{K}\mathbf{X} \tag{21}$$

Orthogonality Properties of the Normal Modes

- Equation (21) must be satisfied in general, so it must be true for any normal mode of vibration
- Let's designate $\mathbf{X}_r$ and $\mathbf{X}_s$ as the eigenvectors for the r^{th} and s^{th} modes, meaning that

$$\omega_r^2 \mathbf{M}\mathbf{X}_r = \mathbf{K}\mathbf{X}_r \quad (r^{th} \text{ mode}) \tag{22}$$

and

$$\omega_s^2 \mathbf{M}\mathbf{X}_s = \mathbf{K}\mathbf{X}_s \quad (s^{th} \text{ mode}) \tag{23}$$

in which ω_r^2 and ω_s^2 are the eigenvalues, the squares of the undamped natural circular frequencies of the r^{th} and s^{th} modes, respectively

- To obtain the desired result we must post-multiply the transpose of Equation (22) by $\mathbf{X}_s$, which gives

$$\omega_r^2 \left[\mathbf{M}\mathbf{X}_r\right]^T \mathbf{X}_s = \left[\mathbf{K}\mathbf{X}_r\right]^T \mathbf{X}_s \tag{24}$$

Orthogonality Properties of the Normal Modes

- From matrix algebra it is well known that

$$\left.\begin{aligned} [\mathbf{M}\mathbf{X}_r]^T &= \mathbf{X}_r^T\mathbf{M}^T \\ [\mathbf{K}\mathbf{X}_r]^T &= \mathbf{X}_r^T\mathbf{K}^T \end{aligned}\right\} \tag{25}$$

- Substituting these equalities into Equation (24) yields

$$\omega_r^2\mathbf{X}_r^T\mathbf{M}^T\mathbf{X}_s = \mathbf{X}_r^T\mathbf{K}^T\mathbf{X}_s \tag{26}$$

- However, because **M** and **K** are symmetric matrices $\mathbf{M}^T = \mathbf{M}$ and $\mathbf{K}^T = \mathbf{K}$ Equation (26) can be expressed as

$$\omega_r^2\mathbf{X}_r^T\mathbf{M}\mathbf{X}_s = \mathbf{X}_r^T\mathbf{K}\mathbf{X}_s \tag{27}$$

Orthogonality Properties of the Normal Modes

- Next, we can pre-multiply the s^{th} mode, Equation (23), by $\mathbf{X}_r^T$

$$\omega_s^2 \mathbf{X}_r^T \mathbf{M} \mathbf{X}_s = \mathbf{X}_r^T \mathbf{K} \mathbf{X}_s \tag{28}$$

- Subtracting Equation (28) from Equation (27) yields

$$\left(\omega_r^2 - \omega_s^2\right) \mathbf{X}_r^T \mathbf{M} \mathbf{X}_s = \mathbf{0} \tag{29}$$

- Since r and s are two distinct normal modes then $\omega_r^2 \neq \omega_s^2$ and it must be that

$$\mathbf{X}_r^T \mathbf{M} \mathbf{X}_s = \mathbf{0} \tag{30}$$

- Equation (30) expresses the orthogonality relationship between *any* two eigenvectors, $\mathbf{X}_r$ and $\mathbf{X}_s$, relative to the mass matrix $\mathbf{M}$
- Since the eigenvector $\mathbf{X}_r$ is a column matrix, $\mathbf{X}_r^T$ is a row matrix.
- Thus, for an n-DOF system Equation (30) can be written as

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}_r \mathbf{M} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_s = \mathbf{0} \tag{31}$$

Orthogonality Properties of the Normal Modes

- If the mass matrix is diagonal then Equation (31) becomes

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}_r \begin{bmatrix} m_1 X_1 \\ m_2 X_2 \\ \vdots \\ m_n X_n \end{bmatrix}_s = \mathbf{0} \qquad (32)$$

- Or, more compactly

$$\sum_{i=1}^{n} m_i (X_i)_r (X_i)_s = 0 \qquad (33)$$

- In the case of dynamic coupling in which $\mathbf{M}$ is symmetric, but not diagonal then the general form of Equation (31) is

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}_r \begin{bmatrix} m_{11} X_1 + m_{12} X_2 + \cdots + m_{1n} X_n \\ m_{21} X_1 + m_{22} X_2 + \cdots + m_{2n} X_n \\ \vdots \\ m_{n1} X_1 + m_{n2} X_2 + \cdots + m_{nn} X_n \end{bmatrix}_s = \mathbf{0} \qquad (34)$$

- Or, more compactly

$$\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} (X_i)_r (X_j)_s = 0 \qquad (35)$$

Orthogonality Properties of the Normal Modes

- Substituting Equation (30) into either Equation (27) or (28) for $r \neq s$ leads to

$$\mathbf{X}_r^T \mathbf{K} \mathbf{X}_s = \mathbf{0} \tag{36}$$

  which expresses the orthogonality relationship between any two eigenvectors, $\mathbf{X}_r$ and $\mathbf{X}_s$, relative to the stiffness matrix $\mathbf{K}$

- Recall that the term *orthogonal* means mutually perpendicular
- Two vectors are orthogonal if their scalar dot product is identically zero
- The orthogonality relationships expressed by Equations (30) and (36) are fundamental to decoupling equations of motion
- They are also useful in checking the accuracy of eigenvectors (normal modes) determined by numerical procedures required in the analysis of systems having a large number of degrees of freedom
- Measurement and truncation error will mean that Equations (30) and (36) will only approximately equal zero

Generalised Mass and Stiffness Matrices

- If we impose the condition that $r = s$ then Equations (30) and (36) will, in general, not equal zero.
- In general, when $\mathbf{M}$ is diagonal we will obtain

$$\mathbf{M}_r = \mathbf{X}_r^T \mathbf{M} \mathbf{X}_r = \sum_{i=1}^{n} m_i (X_i^2)_r, \quad r = 1, 2, \cdots, n \tag{37}$$

where $\mathbf{M}_r$ is called the *generalised mass element* for the r^{th} normal mode

- Note that $\mathbf{M}_r$ can be considered a 1×1 matrix, but it is really a scalar number
- In the case of dynamic coupling when $\mathbf{M}$ is symmetric but not diagonal, when $r = s$ the generalised mass element is the 1×1 matrix

$$\mathbf{M}_r = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} (X_i^2)_r (X_j^2)_r, \quad r = 1, 2, \cdots, n \tag{38}$$

Generalised Mass and Stiffness Matrices

- Similarly, when $r = s$ Equation (36) will yield the *generalised stiffness element* for the r^{th} normal mode:

$$\mathbf{K}_r = \mathbf{X}_r^T \mathbf{K} \mathbf{X}_r = \sum_{i=1}^{n} k_i (X_i^2)_r, \quad r = 1, 2, \cdots, n \tag{39}$$

- It now follows from Equations (27), (37), and (39) that

$$\omega_r^2 \mathbf{X}_r^T \mathbf{M} \mathbf{X}_r = \mathbf{X}_r^T \mathbf{K} \mathbf{X}_r$$

or

$$\omega_r^2 \mathbf{M}_r = \mathbf{K}_r \tag{40}$$

- Again, it is important to emphasise that $\mathbf{M}_r$ and $\mathbf{K}_r$ are considered to be 1×1 matrices and that Equations (37) and (39) are used to calculate the values of the single elements of these matrices

Example 3.8

- In Example 3.7, the eigenvalues were found to be

$$\lambda_1 = \omega_1^2 = 4049.94\ s^{-2}$$
$$\lambda_2 = \omega_2^2 = 27758.70\ s^{-2}$$

and the eigenvectors were found to be

$$\mathbf{X}_1 = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} \quad \text{and} \quad \mathbf{X}_2 = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}_2 = \begin{bmatrix} 1 \\ -1.62 \end{bmatrix}$$

- The mass moment of inertia of each disc was given as $10kgm^2$ meaning the mass matrix was

$$\mathbf{M} = \begin{bmatrix} 10\ kgm^2 & 0 \\ 0 & 10\ kgm^2 \end{bmatrix}$$

Determine

a. the orthogonality of the two modes;

b. the elements of the generalised mass matrix;

c. the elements of the generalised stiffness matrix

Example 3.8 Solution

a. The orthogonality relationship of the two modes can be checked with either Equation (32) or (36)

   Using the former, we can write that

$$\begin{aligned}\mathbf{X}_1^T\mathbf{M}\mathbf{X}_2 &= \begin{bmatrix} 1.0 & 0.62 \end{bmatrix}_1 \begin{bmatrix} 10\ kgm^2 & 0 \\ 0 & 10\ kgm^2 \end{bmatrix} \begin{bmatrix} 1.0 \\ -1.62 \end{bmatrix} \\ &= -0.044\ kgm^2\end{aligned}$$

   which is not identically zero because of the roundoff error because the eigenvalues and eigenvectors were determined with a scienticic calculator, then rounded to two decimal places

b. The generalised mass terms M_r are determined using Equation (37) yielding

$$M_1 = \sum_{i=1}^{2} m_i \left(\Theta_i^2\right)_1 = 10\ kgm^2(1.0)^2 + 10kgm^2(0.62)^2 = 13.84\ kgm^2$$

   and

$$M_2 = \sum_{i=1}^{2} m_i \left(\Theta_i^2\right)_2 = 10\ kgm^2(1.0)^2 + 10kgm^2(-1.62)^2 = 36.24\ kgm^2$$

Example 3.8 Solution

c. The generalised stiffness terms K_r can be determined using Equation (39), or more simply using Equation (40) since ω_1^2 and ω_2^2 are known

$$K_1 = \omega_1^2 M_1 = (4049.94)13.84\ kgm^2 = 56051.17\ kgm^2/rad$$

and

$$K_2 = \omega_2^2 M_2 = (27758.70)36.24\ kgm^2 = 1005975.29\ kgm^2/rad$$