MAAE 3004
Dynamics of Machinery

Lecture Slide Set 3

Free and Forced Vibration of Multiple Degree of Freedom Systems

Part II

Department of Mechanical and Aerospace Engineering
Carleton University

Outline

Modal Analysis

Forced Vibration

Support Excitation

Vibration Absorbtion

Principal Coordinates

- Modal analysis involves the decoupling of coupled differential equations of motion leading to a set of independent equations that can be used to determine the response of the *n*-DOF mechanical system
- The independent equations that result from the decoupling process are expressed in terms the coordinates of a new coordinate system that is different from the coordinate system that the coupled equations of motion were derived in
- These new coordinates are referred to as *principal coordinates*
- The new, independent equations are expressed in terms of the principal coordinates and in terms of the mormal-mode parameters that include the *n* natural circular frequencies and the modal damping properties of the mechanical system
- The decoupling process leads to one independent differential equation for each normal mode of vibration which can be solved as if it were the equation of a single-DOF mechanical system
- The total system response is then obtained by superposition of the responses of the individual normal modes

Modal Matrix U

- Although modal analysis is extremely useful in determining the response of forced n-DOF systems, it can also be used in obtaining the free vibration response of systems that comes from initial conditions
- The discussion that follows will examine both free and forced n-DOF vibration response
- The *modal matrix*, **U**, is required for the modal analysis and decoupling of the differential equations of motion
- The modal matrix is a square $n \times n$ matrix whose columns correspond to the n eigenvectors of the mechanical system where column 1 is mode 1, colum2 is mode 2, et c.
- For example, the 2×2 modal matrix for Example 3.7 where the two eigenvectors were found to be

$$\mathbf{X}_1 = \begin{bmatrix} 1.00 \\ 0.62 \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_2 = \begin{bmatrix} 1.00 \\ -1.62 \end{bmatrix},$$

so the corresponding modal matrix is

$$\mathbf{U} = \begin{bmatrix} 1.00 & 1.00 \\ 0.62 & -1.62 \end{bmatrix}$$

Decoupled Equations for Undamped Free Vibration

- The differential equations of motion of an *n*-DOF mechanical system are dynamically coupled if the generalised coordinates used lead to a non-diagonal mass matrix **M**, and statically coupled if the stiffness matrix **K** is non-diagonal
- It is often possible to select generalised coordinates that eliminate dynamic coupling, but it is generally not possible to select generalised coordinates that eliminate static coupling
- As a result, as we have seen in all of our examples involving stiffness and flexibility coefficients, static coupling is typically always present in the differential equations of motion and the stiffness matrix **K** is symmetric, but not diagonal

Decoupled Equations for Undamped Free Vibration

- To decouple these equations of motion we use the *linear coordinate transformation*

$$\mathbf{X} = \mathbf{U}\boldsymbol{\nu} \tag{41}$$

from which

$$\ddot{\mathbf{X}} = \mathbf{U}\ddot{\boldsymbol{\nu}} \tag{42}$$

- In these equations **X** is the vector of x_i generalised coordinates, **U** is the modal matrix, while $\boldsymbol{\nu}$ is the vector of principal coordinates ν_i
- The ν_i principal coordinates of points are described in the orthogonal principal coordinate system, while the modal matrix transforms the principal coordinate system into the generalised coordinate system, and the generalised coordinates x_i are the coordinates of the **same** points, but now described in the orthogonal generalised coordinate system
- The ν_i are obtained from

$$\boldsymbol{\nu} = \mathbf{U}^{-1}\mathbf{X} \tag{43}$$

Decoupled Equations for Undamped Free Vibration

- A well known theorem in linear algebra states that if $\mathbf{D}$ is a $n \times n$, i.e. square matrix, then the following two statements are always true:
  a. $\mathbf{D}$ is diagonalisable, and
  b. $\mathbf{D}$ has n linearly independent eigenvectors
- And hence, the $n \times n$ mass $\mathbf{M}$ and stiffness $\mathbf{K}$ matrices are always diagonalisable
- This is accomplished for the undamped free n-DOF system by pre-multiplying $\mathbf{M}$ and $\mathbf{K}$ by $\mathbf{U}^T$ and post-multiplying them by $\mathbf{U}$, respectively

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\boldsymbol{\nu}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\boldsymbol{\nu} = \mathbf{0} \tag{44}$$

- This results in the diagonalisation of the mass and stiffness matrices giving the $\mathbf{M}_r$ and $\mathbf{K}_r$ elements for $r = 1, 2, \cdots, n$

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \begin{bmatrix} M_{r=1} & 0 & \cdots & 0 \\ 0 & M_{r=2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{r=n} \end{bmatrix}, \tag{45}$$

Decoupled Equations for Undamped Free Vibration

and

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \begin{bmatrix} K_{r=1} & 0 & \cdots & 0 \\ 0 & K_{r=2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{r=n} \end{bmatrix} \tag{46}$$

- Recalling that

$$\omega_r^2 \mathbf{M}_r = \mathbf{K}_r, \quad r \in \{1, 2, \cdots, n\}$$

we can rewrite Equation (46) as

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \begin{bmatrix} \omega_1^2 M_1 & 0 & \cdots & 0 \\ 0 & \omega_2^2 M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 M_n \end{bmatrix} \tag{47}$$

where ω_r = undamped natural circular frequency of the r^{th} mode

$M_r = \sum_{i=1}^{n} m_i (X_i^2)_r = r^{th}$ mode generalised mass for diagonal $\mathbf{M}$

$M_r = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} (X_i^2)_r (X_j^2)_r = r^{th}$ mode generalised mass for non-diagonal $\mathbf{M}$

Decoupled Equations for Undamped Free Vibration

- Referring to Equations (45) and (47), the decoupled differential equations of motion for n-DOF free vibration in Equation (44) take on the pleasing form

$$\begin{bmatrix} \nwarrow & & \\ & M_r & \\ & & \searrow \end{bmatrix} \ddot{\boldsymbol{\nu}} + \begin{bmatrix} \nwarrow & & \\ & \omega_r^2 M_r & \\ & & \searrow \end{bmatrix} \boldsymbol{\nu} = \mathbf{0} \tag{48}$$

- Equation (48) shows that the decoupled differential equations of motion for n-DOF free vibration, in terms of the principal coordinates, are linearly independent and each has the form

$$\left.\begin{aligned} \ddot{\nu}_1 + \omega_1^2 \nu_1 &= 0 \\ \ddot{\nu}_2 + \omega_2^2 \nu_2 &= 0 \\ &\vdots \\ \ddot{\nu}_r + \omega_r^2 \nu_r &= 0 \\ &\vdots \\ \ddot{\nu}_n + \omega_n^2 \nu_n &= 0 \end{aligned}\right\} \tag{49}$$

Undamped Free Vibration Response

- The solution for any r^{th} mode of Equation (49) is simply

$$\nu_r = A_r \cos(\omega_r t) + B_r \sin(\omega_r t), \qquad r \in \{1, 2, \cdots, n\} \tag{50}$$

- Considering Equations (41) and (50), it follows that the response of the undamped free vibration of an n-DOF mechanical system due to initial conditions and system properties can be determined using

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{U} \begin{bmatrix} A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \\ A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \\ \vdots \\ A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \end{bmatrix} \tag{51}$$

  in which the x_i are the generalised coordinates used to model the motion of the system, and hence describe the vibratory motion.

- The constants A_r and B_r are determined from the specified initial conditions

$$(x_i)_{t=0}, \qquad \text{displacements at time } t = 0$$

$$(\dot{x}_i)_{t=0}, \qquad \text{velocities at time } t = 0$$

Undamped Free Vibration Response

- For computations involving some initial-condition problems when n is large, it is often more convenient to write Equation (51) in the following way

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \nu_1 \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_1 + \nu_2 \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_2 + \cdots + \nu_n \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_n \tag{52}$$

where $\nu_1, \nu_2, \cdots, \nu_n$ are computed according to Equation (50)

- And the $\mathbf{X}_i$ column vectors are the n individual eigenvectors contained in $\mathbf{U}$

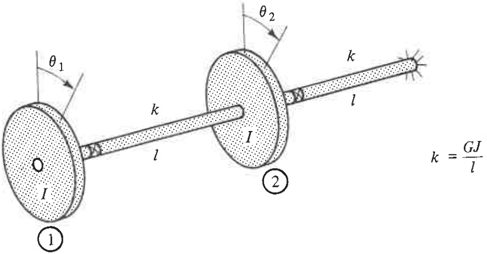
Example 3.9

- The undamped natural circular frequencies and modes, and hence modal matrix **U**, in Example 3.7 were found to be

$$\omega_1 = \sqrt{4049.94} = 63.64 \quad rad/s$$
$$\omega_2 = \sqrt{27758.70} = 166.61 \quad rad/s$$

and

$$\mathbf{U} = \begin{bmatrix} 1.00 & 1.00 \\ 0.62 & -1.62 \end{bmatrix}$$

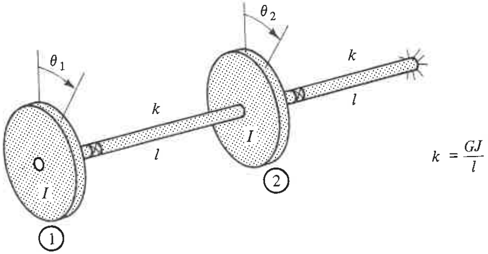


Example 3.9

- The desired initial conditions are

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 10.00^\circ \\ -12.00^\circ \end{bmatrix}_{t=0}, \qquad \text{and} \qquad \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0.00\ rad/s \\ 0.00\ rad/s \end{bmatrix}_{t=0}$$

- The system is carefully released from rest at time $t = 0$ with the stated initial conditions
- Determine the undamped free vibration response of the torsional system as a function of time



Example 3.9 Solution

- We will use four decimal place accuracy for the computations, and round to two in the computed expression of the response
- Using the initial angular displacements converted to radians, Equation (51) simplifies to

$$\begin{bmatrix} 0.1745 \\ -0.2094 \end{bmatrix} = \mathbf{U} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

- This matrix equation is a system of two linear equations in two unknowns, the coefficients A_1 and A_2
- Because the initial angular velocities are all identically zero, the time derivative of Equation (51) requires

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{U} \begin{bmatrix} B_1\omega_1 \cos 0 \\ B_2\omega_2 \cos 0 \end{bmatrix}$$

it must be that $B_1 = B_2 = 0$

Example 3.9 Solution

- Although there a number of methods suited to solving linear systems of equations, we will use linear algebra and matrix inversion to solve the system by multiplying both sides of the equation by the inverse of **U** giving

$$\mathbf{U}^{-1}\begin{bmatrix} 0.1745 \\ -0.2094 \end{bmatrix} = \mathbf{U}^{-1}\mathbf{U}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \mathbf{I}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

- The inverse of an $n \times n$ matrix is easily computable using a hand calculator when $n \leq 3$, otherwise it is very cumbersome
- For $n > 3$ numerical methods are more effective using Python, MatLAB, Maple, et c.

Example 3.9 Solution

- For reference, the inverse of a 2×2 matrix is

$$\text{if } \mathbf{A}_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } \mathbf{A}_{2\times 2}^{-1} = \frac{1}{\det\left(\mathbf{A}_{2\times 2}\right)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where $\det\left(\mathbf{A}_{2\times 2}\right) = a_{11}a_{22} - a_{21}a_{12}$

- The inverse of a 3×3 matrix is

$$\text{if } \mathbf{A}_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{then } \mathbf{A}_{3\times 3}^{-1} = \frac{1}{\det\left(\mathbf{A}_{3\times 3}\right)} \begin{bmatrix} (a_{22}a_{23} - a_{32}a_{23}) & -(a_{21}a_{33} - a_{31}a_{23}) & (a_{21}a_{32} - a_{31}a_{22}) \\ -(a_{12}a_{33} - a_{32}a_{13}) & (a_{11}a_{33} - a_{31}a_{13}) & -(a_{11}a_{32} - a_{31}a_{12}) \\ (a_{12}a_{23} - a_{22}a_{13}) & -(a_{11}a_{23} - a_{21}a_{13}) & (a_{11}a_{22} - a_{21}a_{12}) \end{bmatrix}$$

where

$$\det\left(\mathbf{A}_{3\times 3}\right) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Example 3.9 Solution

- For the problem at hand

$$\mathbf{U}^{-1}\begin{bmatrix} 0.1745 \\ -0.2094 \end{bmatrix} = \begin{bmatrix} 0.7232 & 0.4464 \\ 0.2768 & -0.4464 \end{bmatrix}\begin{bmatrix} 0.1745 \\ -0.2094 \end{bmatrix} = \begin{bmatrix} 0.0327 \\ 0.1408 \end{bmatrix}$$

- Therefore $A_1 = 0.0327\ rad$ and $A_2 = 0.1408\ rad$
- Meaning that $\nu_1 = 0.0327\cos(63.6391t)$ and $\nu_2 = 0.1408\cos(166.6094t)$
- The undamped free vibration response of the torsional system is given by Equation (51), $\mathbf{X} = \mathbf{U}\boldsymbol{\nu}$, giving

$$\begin{aligned} x_1 &= \theta_1 = 0.0327\cos(63.6391t) + 0.1408\cos(166.6094t) \\ x_2 &= \theta_2 = 0.0203\cos(63.6391t) - 0.2284\cos(166.6094t) \end{aligned}$$

- Rounding to two decimal places yields

$$\begin{aligned} x_1 &= \theta_1 = 0.03\cos(63.64t) + 0.14\cos(166.61t) \\ x_2 &= \theta_2 = 0.02\cos(63.64t) - 0.23\cos(166.61t) \end{aligned}$$

Decoupling Damped Free Vibration Equations

- The matrix form of the differential equations of motion for damped free vibration is

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0} \tag{53}$$

in which $\mathbf{C}$ is the damping matrix

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

- With inclusion of damping, the equations of motion can also be coupled by damping in addition to being statically and/or dynamically coupled
- Damping coupling corresponds to $\mathbf{C}$ containing non-zero off-diagonal elements

Decoupling Damped Free Vibration Equations

- Substituting the modal-matrix transformations into Equation (53) we obtain

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\boldsymbol{\nu}} + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\boldsymbol{\nu}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\boldsymbol{\nu} = \mathbf{0} \qquad (54)$$

- The first and last terms are transformed into diagonal matrices because of the orthogonality relationships between the eigenvectors relative to the mass and stiffness matrices, respectively
- Unfortunately, the damping term generally does not diagonalise because there is no such orthogonality relationship
- The way this has been dealt with is using the assumption of *proportional damping*
- In proportional damping, the damping matrix **C** is assumed to be proportional to either the mass matrix **M** or the stiffness matrix **K**

Decoupling Damped Free Vibration Equations

- If, for example, we assume that **C** is proportional to **M** by some constant of proportionality α then

$$\mathbf{C} = \alpha \mathbf{M}$$

- Then we have

$$\mathbf{U}^T\mathbf{C}\mathbf{U} = \alpha \mathbf{U}^T\mathbf{M}\mathbf{U} = \begin{bmatrix} \nwarrow & & \\ & \alpha M_r & \\ & & \searrow \end{bmatrix}$$

where M_r is the generalised mass of the r^{th} mode

- From this, it is common practise to assume that the proportional modal damping has the form

$$2\zeta_r \omega_r M_r = \alpha M_r \tag{55}$$

Decoupling Damped Free Vibration Equations

- Similarly, if we assume that damping is proportional to the stiffness matrix, then

$$\mathbf{C} = \beta \mathbf{K}$$

in which β is the proportionality constant, then we obtain

$$\mathbf{U}^T \mathbf{C} \mathbf{U} = \beta \mathbf{U}^T \mathbf{K} \mathbf{U} = \begin{bmatrix} \nwarrow & & \\ & \beta \omega_r^2 M_r & \\ & & \searrow \end{bmatrix}$$

And in this case, the modal damping can also be expressed as

$$2\zeta_r \omega_r M_r = \beta \omega_r^2 M_r \tag{56}$$

Decoupling Damped Free Vibration Equations

- With the assumption of proportional damping that leads either to Equation (55) or (56), the decoupled form of Equation (54) is

$$\begin{bmatrix} \nwarrow & & \\ & M_r & \\ & & \searrow \end{bmatrix} \ddot{\boldsymbol{\nu}} + \begin{bmatrix} \nwarrow & & \\ & 2\zeta_r\omega_r M_r & \\ & & \searrow \end{bmatrix} \dot{\boldsymbol{\nu}} + \begin{bmatrix} \nwarrow & & \\ & \omega_r^2 M_r & \\ & & \searrow \end{bmatrix} \boldsymbol{\nu} = \mathbf{0} \qquad (57)$$

- Equation (57) is a set of n uncoupled equations of the form

$$\ddot{\nu}_r + 2\zeta_r\omega_r\dot{\nu} + \omega_r^2\nu = 0, \qquad r \in \{1, 2, \cdots, n\} \qquad (58)$$

where ζ_r = modal damping factor of the r^{th} mode
ω_r = undamped natural circular frequency of the r^{th} mode

- Each of the decoupled r^{th} modes has the same form as the differential equation of motion of a single DOF mechanical system, which has the solution

$$\nu_r = e^{-\zeta_r\omega_r t}\left(A_r \cos\left(\omega_r\sqrt{1-\zeta_r^2}\,t\right) + B_r \sin\left(\omega_r\sqrt{1-\zeta_r^2}\,t\right)\right) \qquad (59)$$

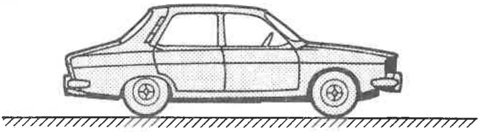
Equations of Motion

- Determining the vibration response of n-DOF mass systems that move is a critical consideration in their analysis, and design
- Time constraints will unfortunately limit our discussion to the equations of motion and some simple examples
- If you are interested in learning more please register in the 4^{th} year technical elective course **MAAE 4104, Vibration Analysis**
- We will now consider n-DOF mechanical systems that are subject to either excitation forces, or support excitation
- The matrix differential equation of an n-DOF system subjected to excitation forces and/or moments is

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{bmatrix} \tag{60}$$

Example 3.10

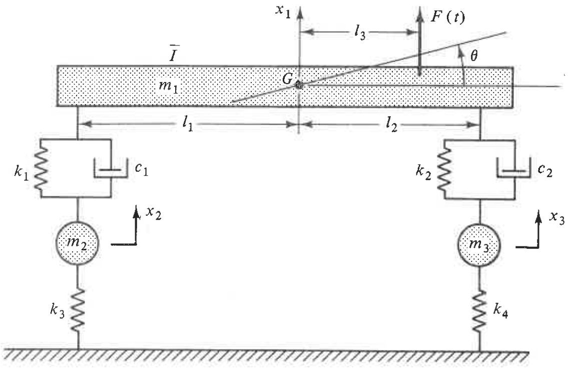
- One cylinder of the engine of a car is misfiring because of a fouled spark plug, which causes an excitation force $F(t)$ a distance l_3 from the mass centre G of the car



- Determine the elements of the column vector **F** which represents the forces and moments caused by the misfiring of the fouled plug

Example 3.10 Solution

- The standard half-car model is a four-lumped-mass system



c_1 and c_2 = damping coefficient of back and front shock absorbers, respectively

k_1 and k_2 = stiffness of back and front suspension springs, respectively

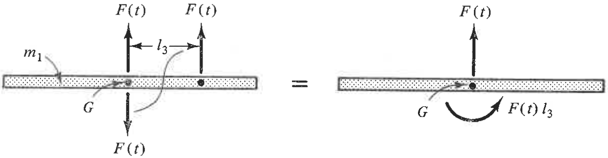
k_3 and k_4 = stiffness of back and front tires, respectively (assumed linear)

m_1 = main mass of car (everything but wheels)

m_2 and m_3 = mass of back and front wheels, respectively

$\bar{I}$ = centroidal mass moment of inertia of main mass

Example 3.10 Solution

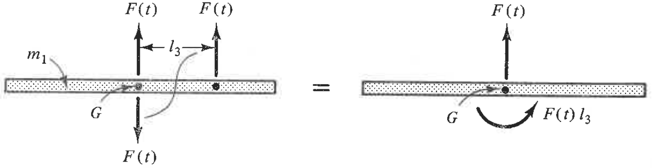


- Adding the two forces $F(t)$ that are equal in magnitude but oppositely directed, we generate an equivalent force system consisting of a force $F(t)$ acting through the mass centre G and a couple of magnitude $F(t)\mathrm{l}_3$
- The elements of **F** can be determined using the concept of *virtual work*
- The virtual work done by an excitation force $F_i(t)$ for an arbitrarily small displacement, δx_i is

$$\delta W_i \quad = \quad F_i(t)\delta x_i$$

- If we let all but one of the generalised coordinates of a system be held constant (fixed) and then consider a virtual displacement δx_i of the mass with the unfixed coordinate, the corresponding $F_i(t)$ in vector **F** will be the sum of all the excitation forces that do virtual work during that virtual displacement
- The elements of **F** that are moments are similarly determined using virtual angular displacements $\delta\theta_i$

Example 3.10 Solution

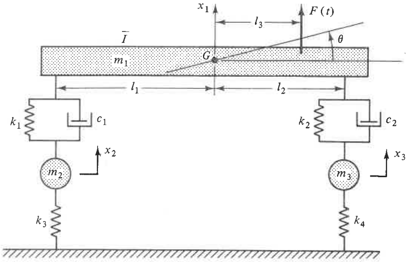


- The virtual work of the equivalent force $F(t)$ acting through the mass centre G due to a virtual displacement δx_1 of m_1 is

$$\delta W_1 = F(t)\delta x_1$$

which reveals that the first element of **F** is simply the force

$$\delta F_1(t) = F(t)$$



Example 3.10 Solution

- Since no external forces are acting on the masses of the wheels, it follows that

$$\delta W_2 = 0 \Rightarrow F_2(t) = 0$$
$$\delta W_3 = 0 \Rightarrow F_3(t) = 0$$

- The virtual work of the couple $F(t)\mathrm{l}_3$ due to a virtual angular displacement of $\delta\theta$ of the main mass m_1 is

$$\delta W_4 = F(t)\mathrm{l}_3\delta\theta$$

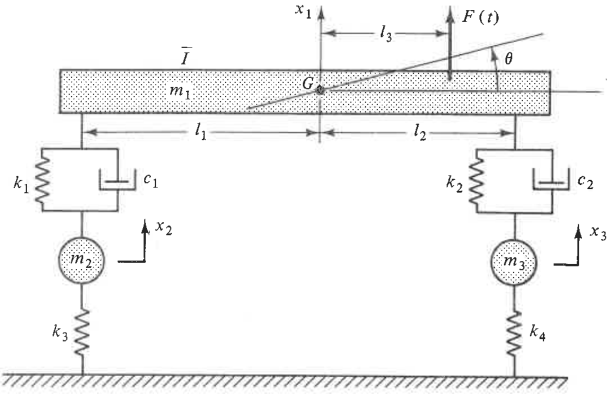
So the fourth element of **F** is

$$\delta W_4 = F(t)\mathrm{l}_3\delta\theta$$

- Thus, the column vector of forcing functions is

$$\mathbf{F} = \begin{bmatrix} F(t) \\ 0 \\ 0 \\ F(t)\mathrm{l}_3 \end{bmatrix}$$

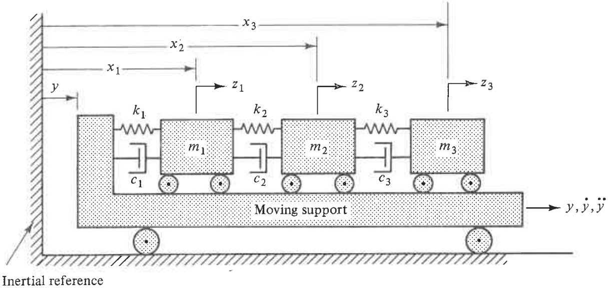
Example 3.10 Solution



- While the elements of **F** could easily have been determined by inspection of the figure, this simple example provides a good example of how to use the concept of virtual work for this type of application

Support Excitation

- Let us now consider an n-DOF system that is attached to a moving support frame



- The 3-DOF system in the image is excited by the motion of the support frame, and the generalised coordinates are assigned as

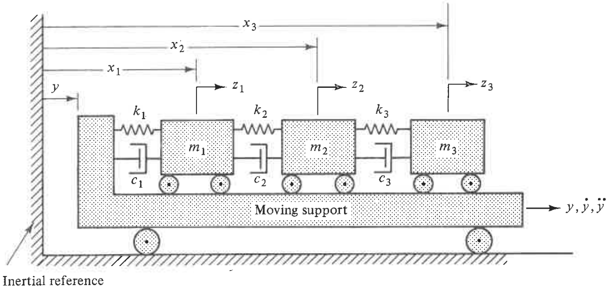
$$x_i = \text{absolute displacement of mass } m_i$$

$$y = \text{absolute displacement of moving support}$$

$$z_i = \text{displacement of } m_i \textit{ relative} \text{ to moving support}$$

Support Excitation

- The *absolute* displacements of m_1, m_2, and m_3 with respect to the relatively non-moving *inertial* coordinate reference frame are x_1, x_2, and x_3, respectively
- The displacements of the three masses *relative to the moving support* are z_1, z_2, and z_3
- The absolute displacement of the moving support with respect to the inertial frame is y

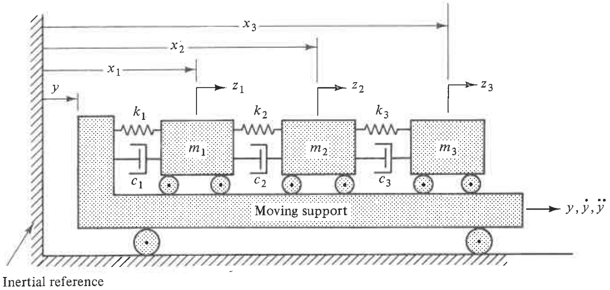


Support Excitation

- With these definitions in mind, for the i^{th} mass we can write that

$$\left.\begin{array}{rcl} x_i & = & y + z_i \\ \dot{x}_i & = & \dot{y} + \dot{z}_i \\ \ddot{x}_i & = & \ddot{y} + \ddot{z}_i \end{array}\right\} \quad (61)$$

- The only forces acting on the three m_i masses are the spring and damping forces
- The spring forces vary with the z_i relative displacements and the damping forces vary with the $\dot{z}_i$ relative velocities



Support Excitation

- Since Newton's second law applies to the *absolute* accelerations of the masses $\ddot{x}_1$, $\ddot{x}_2$, and $\ddot{x}_3$, we can write that

$$\mathbf{M}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix}+\mathbf{C}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix}+\mathbf{K}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\mathbf{0} \tag{62}$$

- As Equation (61) states, the x_i absolute displacements of the m_i masses are the sum of the m_i masses relative displacements **and** the y absolute displacement of the moving support, hence Equation (62) can be expressed as

$$\mathbf{M}\begin{bmatrix}\ddot{y}+\ddot{z}_1\\ \ddot{y}+\ddot{z}_2\\ \ddot{y}+\ddot{z}_3\end{bmatrix}+\mathbf{C}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix}+\mathbf{K}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\mathbf{0}$$

- We can rearrange this equation to interpret the forces $m_i\ddot{y}$ as excitation forces, giving

$$\mathbf{M}\begin{bmatrix}\ddot{z}_1\\ \ddot{z}_2\\ \ddot{z}_3\end{bmatrix}+\mathbf{C}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix}+\mathbf{K}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=-\mathbf{M}\begin{bmatrix}\ddot{y}\\ \ddot{y}\\ \ddot{y}\end{bmatrix} \tag{63}$$

Support Excitation

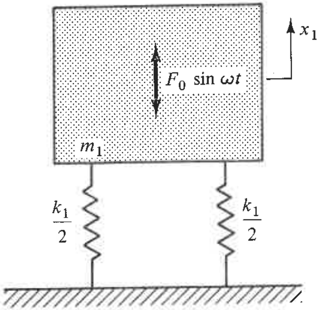
- Equation (63) expresses the differential equations of motion of the system in terms of its relative motion and an excitation force proportional to the $\ddot{y}$ acceleration of the support, and can be written in the compact form

$$\mathbf{M}\ddot{\mathbf{Z}} + \mathbf{C}\dot{\mathbf{Z}} + \mathbf{K}\mathbf{Z} = -\mathbf{M}\ddot{\mathbf{Y}} \tag{64}$$

- The elements of the damping matrix **C** and stiffness matrix **K** are determined as if the moving support were stationary and using the techniques we have seen earlier In the section on Influence Coefficients in Part I of Lecture Slide Set 3
- The modeling technique that can be used to obtain solutions to differential equations such as these will be illustrated with the concept of *vibration absorbtion*
- Vibration absorbers can be used to significantly attenuate vibration amplitudes to nearly undetectable levels in systems where vibration is undesirable

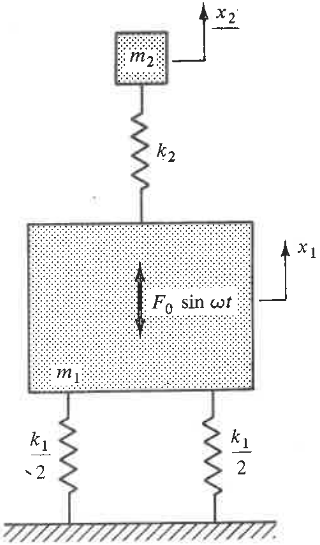
Vibration Absorber

- A common type of vibration absorber consists of a spring-and-mass system constructed such that its natural frequency is easily, and precisely varied
- This absorbtion system is rigidly attached to the principal system that is to have its vibration reduced, and the frequency of the absorber system is then adjusted until the desired result is achieved
- For example, if the circular frequency ω of the disturbing force $F_0 \sin(\omega t)$ acting on a system is close to the natural circular frequency $\omega_n = \sqrt{k/m}$ of the system, the amplitude of the response could become very large due to this resonance condition



Vibration Absorber

- Attaching an auxiliary spring-and-mass system consisting of k_2 and m_2, the vibration response amplitude can be reduced, essentially to zero, if the natural circular frequency of the absorber is adjusted until it equals that of the disturbing force, i.e., until $\sqrt{k_2/m_2} = \omega$
- This type of absorber is usually designed to have little damping and is “tuned” by varying either m_2, k_2, or both
- It is important to note that the original 1-DOF system becomes a 2-DOF system with this type of absorber added as shown



Vibration Absorber Design

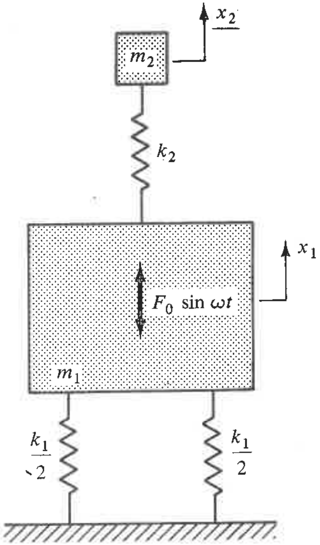
- The stiffness matrix of the 2-DOF system augmented with the absorber is determined by inspection, recalling the definitions of the stiffness coefficients in Part I of Lecture Slide Set 3

- The virtual work done by the excitation force $F_0 \sin(\omega t)$ due to the virtual displacement δx_1 is

$$\delta W_1 = F_0 \sin(\omega t) \delta x_1$$

and since there is no excitation force acting on m_2 we have

$$\delta W_2 = 0$$



Vibration Absorber

- The statically coupled differential equations of motion are therefore

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} F_0\sin(\omega t) \\ 0 \end{bmatrix} \quad (65)$$

- Next we premultiply Equation (65) by $\mathbf{M}^{-1}$, which gives

$$\left.\begin{aligned} \ddot{x}_1+\frac{k_1+k_2}{m_1}x_1-\frac{k_2}{m_1}x_2 &= \frac{F_0}{m_1}\sin(\omega t) \\ \ddot{x}_2-\frac{k_2}{m_2}x_1+\frac{k_2}{m_2}x_2 &= 0 \end{aligned}\right\} \quad (66)$$

- To determine the steady-state solution of these coupled equations, we let the imaginary part of $(F_0/m_1)e^{i\omega t}$ represent $(F_0/m_1)\sin(\omega t)$ and assume solutions of the form

$$x_1 = X_1 e^{i\omega t} \quad (67)$$

and

$$x_2 = X_2 e^{i\omega t} \quad (68)$$

Vibration Absorber Design

- Substituting Equations (67) and (68), and their appropriate time derivatives into Equation (66) leads to two algebraic equations

$$\left.\begin{aligned} \left(\frac{k_1+k_2}{m_1}-\omega^2\right)X_1-\frac{k_2}{m_1}X_2 &= \frac{F_0}{m_1} \\ -\frac{k_2}{m_2}X_1+\left(\frac{k_2}{m_2}-\omega^2\right)X_2 &= 0 \end{aligned}\right\} \tag{69}$$

- From this pair of algebraic equations in X_1 and X_2, we can express

$$X_2 = \left(\frac{\frac{k_2}{m_2}}{\left(\frac{k_2}{m_2}-\omega^2\right)}\right)X_1 \tag{70}$$

and

Vibration Absorber Design

$$X_1 = \frac{\frac{F_0}{m_1}\left(\frac{k_2}{m_2} - \omega^2\right)}{\left(\frac{k_1 + k_2}{m_1} - \omega^2\right)\left(\frac{k_2}{m_2} - \omega^2\right) - \frac{k_2^2}{m_1 m_2}} \tag{71}$$

- Equation (71) indicates that the amplitude $x_1 = X_1 = 0$ when k_2/m_2 of the vibration absorber is equal to the square of the circular frequency ω^2 of the excitation force

- If the purpose of the absorber is to perform this amplitude reduction when the principal system is in resonance with the excitation force $F_0 \sin(\omega t)$, that is, when $k_1/m_1 = \omega^2$, it then follows that when $X_1 = 0$

$$\frac{k_2}{m_2} = \frac{k_1}{m_1} = \omega^2 \Rightarrow X_1 = 0 \tag{72}$$

Vibration Absorber Design

- The two natural frequencies of the combined system depend upon the ratio of the absorber mass m_2 to the primary mass m_1
- Therefore, the *mass ratio* m_2/m_1 is an important parameter in the design of this type of vibration absorber
- To observe its effect on the total response of the system we first transform Equation (71) into non-dimensional form using the following notation

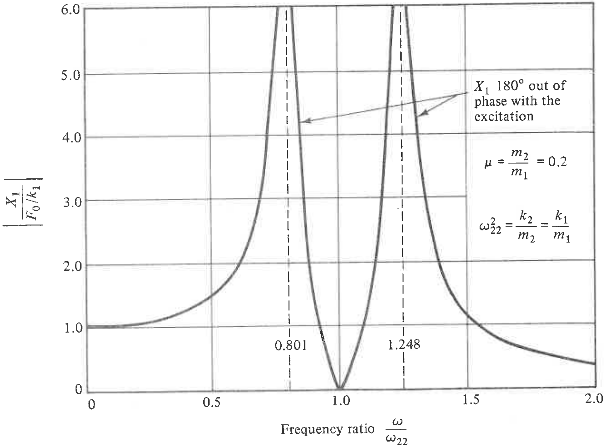
$$\left.\begin{aligned} \omega_{22}^2 &= \frac{k_2}{m_2} = \frac{k_1}{m_1} \\ \mu &= \frac{m_2}{m_1} = \frac{k_2}{k_1} \end{aligned}\right\} \tag{73}$$

- Using this notation we can rewrite Equation (71) in the following non-dimensional way

$$\frac{X_1}{F_0/k_1} = \frac{1-(\omega/\omega_{22})^2}{(\omega/\omega_{22})^4-(2+\mu)(\omega/\omega_{22})^2+1} \tag{74}$$

in which ω is the circular frequency of the disturbing force

Vibration Absorber Design



- This is a plot of the *absolute* values of Equation (74), $\left|\frac{X_1}{F_0/k_1}\right|$, as a function of $\frac{\omega}{\omega_{22}^2}$ for the mass ratio $\mu = 0.2$

Vibration Absorber Design

- The denominator of Equation (74) is an algebraic polynomial of degree 4 having four roots
- Equation (74) is infinite when the denominator vanishes, that is when

$$(\omega/\omega_{22})^4 - (2+\mu)(\omega/\omega_{22})^2 + 1 \quad = \quad 0$$

- This equation has four real roots that depend on the mass ratio μ, which are

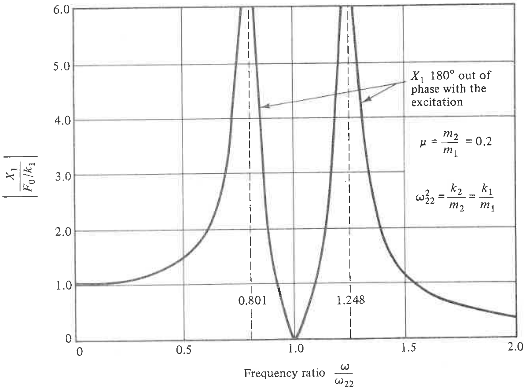
$$\pm\frac{1}{2}\left(\sqrt{4-2\sqrt{\mu^2+4\mu}+2\mu}\right) \quad \text{and} \quad \pm\frac{1}{2}\left(\sqrt{4+2\sqrt{\mu^2+4\mu}+2\mu}\right)$$

- For $\mu = 0.2$ the positive roots are

$$\frac{\omega}{\omega_{22}^2} \quad = \quad 0.801, \;\; 1.248,$$

as seen in the graph

Vibration Absorber Design



- The two natural circular frequencies of the composite system are

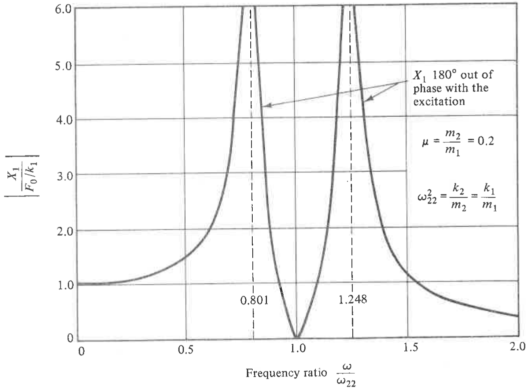
$$\omega_{n_1} = 0.801\omega_{22} = 0.801\sqrt{\frac{k_1}{m_1}}, \quad \text{and} \quad \omega_{n_2} = 1.248\omega_{22} = 1.248\sqrt{\frac{k_1}{m_1}}$$

in which $k_1/m_1 = k_2/m_2$

Vibration Absorber Design

- Hence, the two natural circular frequencies of the composite 2-DOF system are 0.801 and 1.248 times the natural circular frequency of the principal 1-DOF system

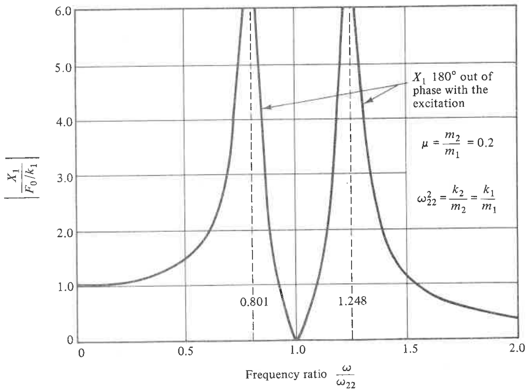
- We see that the vibration absorber has been *tuned* to eliminate vibration when the disturbing frequency is equal to the natural circular frequency of the principal system: $\omega/\omega_{22} = 1$



Vibration Absorber Design

- It can be shown using Equations (70), (73), and (74) that when $\omega_{22}^2 = k_2/m_2$ and $\omega/\omega_{22} = 1$ the amplitude X_2 of the of the absorber is 180° out phase with the disturbance force F_0:

$$X_2 = -\frac{F_0}{k_2}$$



Vibration Absorber Design

- In this case, the principal mass m_1 is subjected to both the disturbing excitation force $F_0 \sin(\omega t)$ and the absorber force $-k_2 X_2 \sin(\omega t)$
- The combination of these two forces corresponds to a condition of static equilibrium at any instant in time with $X_1 = 0$