MAAE 3004
Dynamics of Machinery

Lecture Slide Set 5

Displacement Analysis

Department of Mechanical and Aerospace Engineering
Carleton University

Outline

Introduction

Position specifies the location of a point relative to a selected reference coordinate system defined by:

1. an origin (labelled O)
2. coordinate axes
3. scale

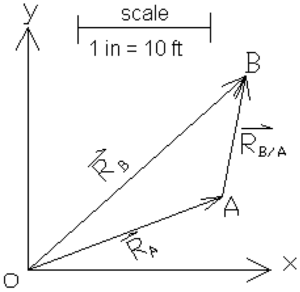
- $\vec{R}_A$ and $\vec{R}_B$: absolute position vectors
- $\vec{R}_{B/A}$: a relative position vector
- By applying vector addition

$$\vec{R}_B = \vec{R}_A + \vec{R}_{B/A}$$

or,

$$\vec{R}_A = \vec{R}_B + \vec{R}_{A/B}$$

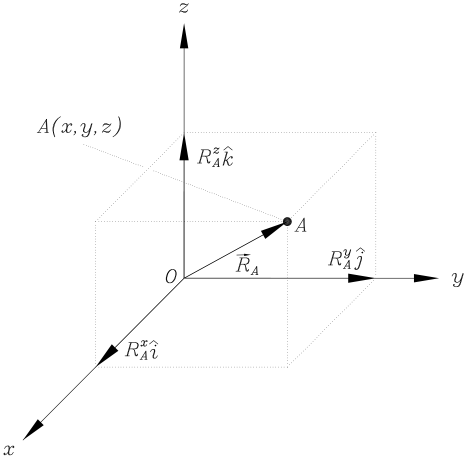
Note: $\vec{R}_{A/B} = -\vec{R}_{B/A}$



- The absolute coordinates of point A are conveniently expressed as the position vector starting at the origin of the coordinate system and terminating at point A in component form

$$\vec{R}_A = R_A^x \hat{i} + R_A^y \hat{j} + R_A^z \hat{k}$$

- $\vec{R}_A$ is an absolute position vector
- $R_A^x \hat{i}$, $R_A^y \hat{j}$, $R_A^z \hat{k}$ are the x, y, and z vector components (distances) in the directions of the $\hat{i}$, $\hat{j}$, $\hat{k}$ unit basis vectors.



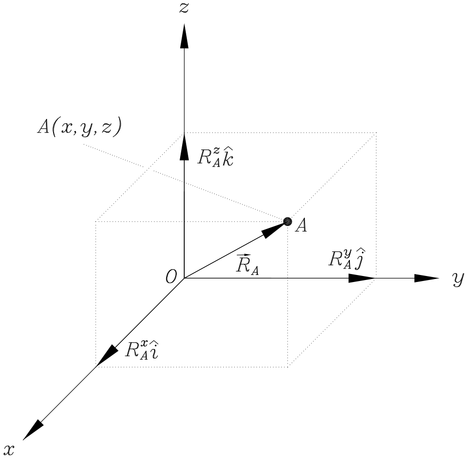
- The magnitude of position vector $\vec{R}_A$ (the relative distance of point A from the origin O) is

$$R_A = \left|\vec{R}_A\right| = \sqrt{\vec{R}_A \cdot \vec{R}_A}$$

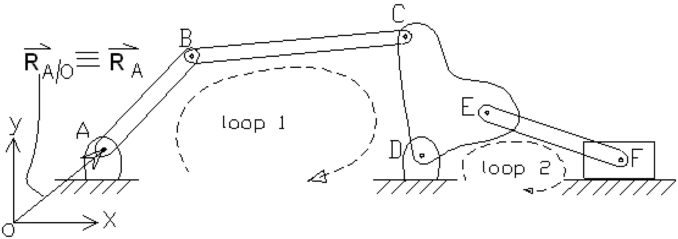
$$= \sqrt{(R_A^x)^2 + (R_A^y)^2 + (R_A^z)^2}$$

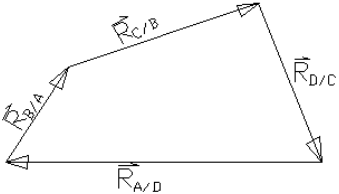
- The unit vector in the direction of $\vec{R}_A$ is

$$\hat{R}_A = \frac{\vec{R}_A}{R_A}$$



- Application of vector addition around closed loops results in the **Loop Closure Equation** that models a mechanism. e.g



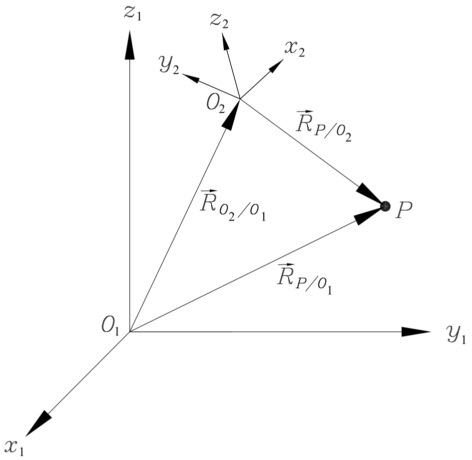


loop 1: $\vec{R}_{B/A} + \vec{R}_{C/B} + \vec{R}_{D/C} + \vec{R}_{A/D} = 0$

loop 2: $\vec{R}_{E/D} + \vec{R}_{F/E} + \vec{R}_{D/F} = 0$

- Note that various relative position equations can be written. e.g.
$\vec{R}_F = \vec{R}_D + \vec{R}_{E/D} + \vec{R}_{F/E} = \vec{R}_A + \vec{R}_{B/A} + \vec{R}_{C/B} + \vec{R}_{E/C} + \vec{R}_{F/E}$

- We will occasionally use more than one coordinate system to describe the relative position of a system of points.
- Consider the two coordinate systems with origins O_1 and O_2.
- We may find it convenient to describe point P relative to both coordinate systems where it may be that point P moves relative to O_1 but is stationary relative to O_2.
- That is coordinate system 2 moves with point P relative to coordinate system 1.



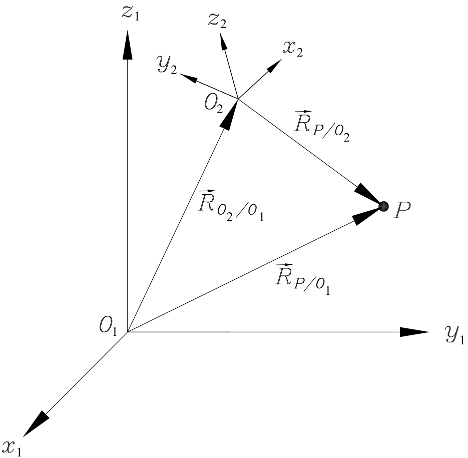
- Clearly

$$\vec{R}_{P/O_1} \neq \vec{R}_{P/O_2}$$

- Rather, we can express this situation with relative position vectors as

$$\vec{R}_{P/O_1} = \vec{R}_{O_2/O_1} + \vec{R}_{P/O_2}$$

- To express the position of point P relative to coordinate system 1 given the position and orientation of coordinate system 2 relative to 1 and the position of P in 2, we must first transform the coordinates of point P from coordinate system 2 to the corresponding coordinates in 1.



Graphical and Trigonometric Solution of Vector Equations

- Vector equations contain two types of information:
  1. magnitude
  2. direction
- A planar vector equation results in 2 scalar equations corresponding to the x and y coordinate directions. e.g.

$$\vec{R}_C = \vec{R}_A + \vec{R}_B$$
$$R_{C_X}\hat{i} + R_{C_Y}\hat{j} = R_{A_X}\hat{i} + R_{A_Y}\hat{j} + R_{B_X}\hat{i} + R_{B_Y}\hat{j}$$
$$R_{C_X} = R_{A_X} + R_{B_X}$$
$$R_{C_Y} = R_{A_Y} + R_{B_Y}$$

  $\therefore$ we can solve for <u>2</u> unknowns.
- Using graphical and trigonometric solution methods it is possible to solve equations containing up to 2 unknowns.

Symbology:	$\surd$	known quantity
	o	unknown quantity

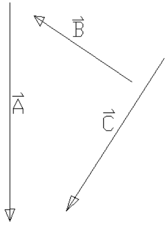
Magnitude status $\leftarrow\square\square\rightarrow$ direction status

$$\vec{R}_A$$

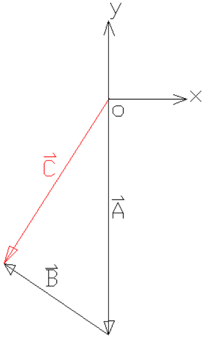
e.g. $\overset{o\surd}{\vec{R}_A} \Rightarrow$ magnitude unknown, direction known

Graphical Solution of Vector Equations

Consider the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$:



Four cases of two unknowns among three planar vectors can be identified



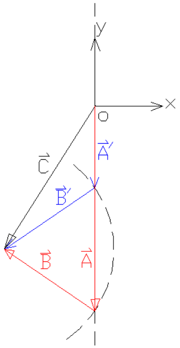
CASE 1: Magnitude and direction of the same vector unknown:

$$\overset{oo}{\vec{C}} = \overset{\surd\surd}{\vec{A}} + \overset{\surd\surd}{\vec{B}}$$

Solve by vector addition:

CASE 2: Magnitude of one & direction of other unknown: $\overset{\surd\surd}{\vec{C}} = \overset{o\surd}{\vec{A}} + \overset{\surd o}{\vec{B}}$

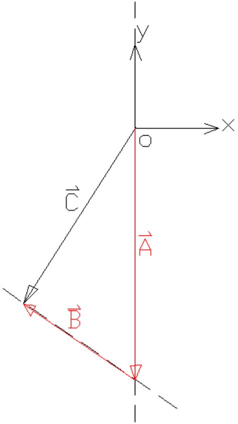
1. Draw $\vec{C}$
2. Draw direction of $\vec{A}$ passing through O
3. Draw arc with radius equal to the magnitude of $\vec{B}$ centred on the tip of $\vec{C}$
4. Identify $\vec{A}$ & $\vec{B}$ from intersection



Note: Two solutions are possible

CASE 3: Magnitude of two vectors unknown: $\overset{\surd\surd}{\vec{C}} = \overset{o\surd}{\vec{A}} + \overset{o\surd}{\vec{B}}$

1. Draw $\vec{C}$
2. Draw direction of $\vec{A}$ passing through O
3. Draw direction of $\vec{B}$ passing through the tip of $\vec{C}$
4. Intersection identifies $\vec{A}$ & $\vec{B}$

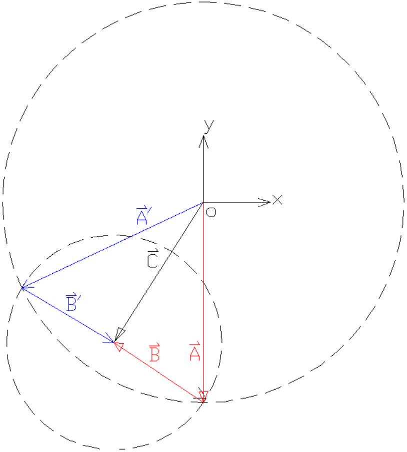


Note: The solution is distinct unless vectors are collinear when an infinite number of solutions are possible

<u>CASE 4:</u> Direction of the two vectors unknown: $\overset{\surd\surd}{\vec{C}} = \overset{\surd o}{\vec{A}} + \overset{\surd o}{\vec{B}}$

1. Draw $\vec{C}$
2. Draw arc with magnitude of $\vec{A}$ centered at O
3. Draw arc with magnitude of $\vec{B}$ centered on the tip of $\vec{C}$
4. Identify $\vec{A}$ & $\vec{B}$ from intersections

Note: Two solutions are possible



Example 5.1

Given: Slider crank mechanism where only link lengths and the position of the slider are known.

Determine: The mechanism configuration

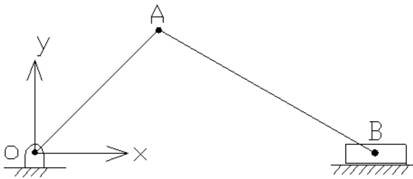
Solution: Lengths of $\vec{OA}$ & $\vec{AB}$ are known, i.e, we have $\overset{\surd o}{\vec{R}_A}$ and $\overset{\surd o}{\vec{R}_{B/A}}$, we also know position of B, i.e, $\overset{\surd\surd}{\vec{R}_B}$

The loop-closure equation may be written as:

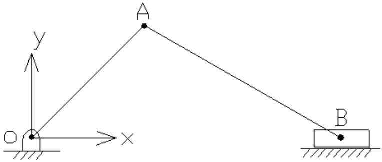
$$\vec{R}_{A/O} + \vec{R}_{B/A} + \vec{R}_{O/B} = 0$$

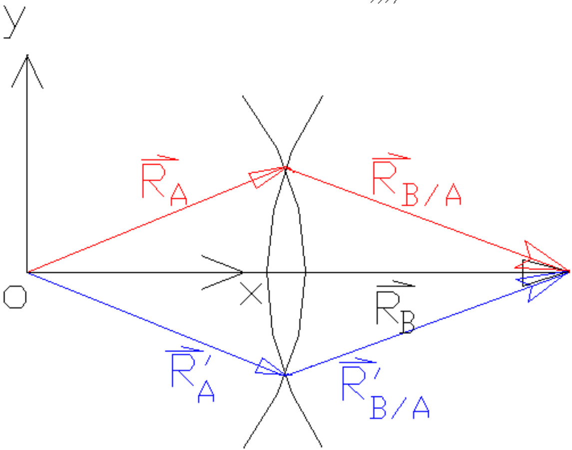
$$\vec{R}_{O/B} = -\vec{R}_{B/O}$$

$$\therefore \overset{\surd\surd}{\vec{R}_B} = \overset{\surd o}{\vec{R}_A} + \overset{\surd o}{\vec{R}_{B/A}}$$



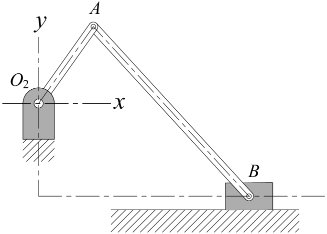
This is a problem of type 4, therefore solve as before

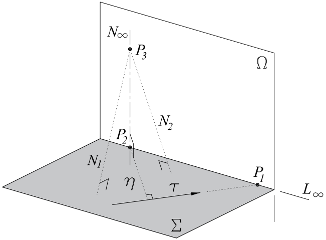




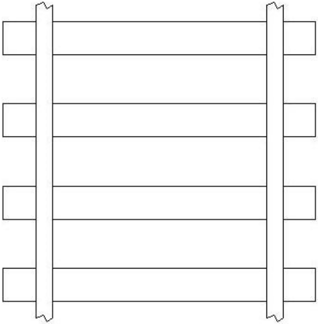
A Note on the Axes of R- and P-pairs (joints)

- The axis of an R-pair is a line through the invariant centre point of rotation perpendicular to the plane of motion.
- But it does not make sense to speak about the axis of a P-pair in the same way, as no real points in E_2 are invariant under a translation.
- The translation τ moves every point in the plane in the direction of the arrow by the amount equal to its length.
- The axis of a P-pair could be described as the line at infinity, N_∞, of all planes normal to the direction of τ.
- Σ is the plane containing the P-pair, τ is a particular translation effected by the P-pair, N_1 and N_2 are normal to Σ, and Ω represents the plane at infinity.



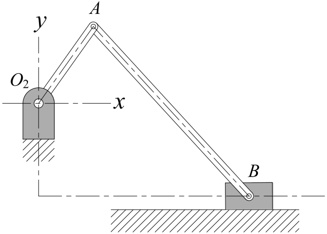


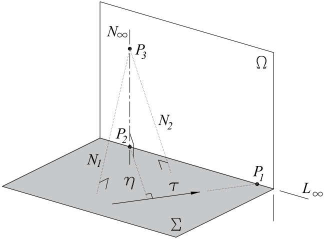
What we see.



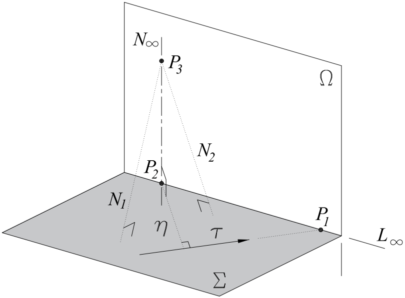
What is really there.

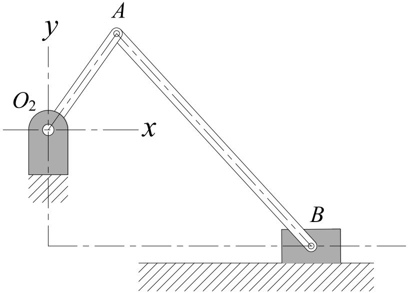
- The two planes Σ and Ω intersect in L_∞.
- Lines in the direction of τ intersect L_∞ in point P_1.
- Lines normal to τ in Σ, indicated by η, intersect L_∞ in P_2.
- The line N_∞ is the intersection of all planes normal to Σ and parallel to η.
- All normals to Σ, N_1, and N_2 being two of them, intersect N_∞ in the point P_3.
- The join of P_2 and P_3 is N_∞, which is the axis of the particular prismatic joint.



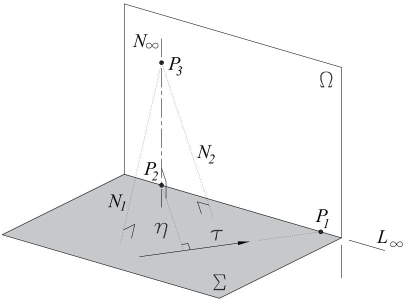


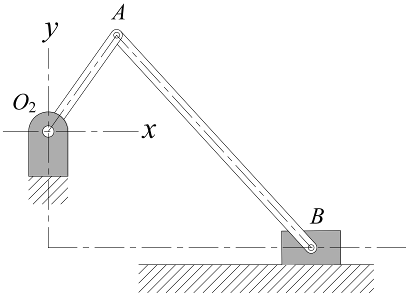
- In other words, the axis of a P-pair is the absolute polar line to the point at infinity of the direction of translation.
- What that means is (recall MAAE 2001, Engineering Graphical Design): the P-pair axis is a line at infinity perpendicular to the plane of translation, located where all the lines perpendicular to the direction of, and in the same plane as, the translation intersect the plane at infinity.
- A translation is therefore an arc length of an infinitely large circle where the change in angle generating the arc length is infinitely small.





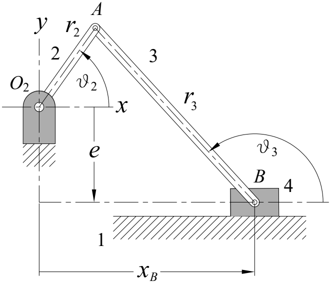
- The axis of a P-pair is **NOT** its longitudinal axis of symmetry.
- Regardless, P-pairs would be impossible to manufacture if they had no longitudinal axis of symmetry to establish the direction of translation, i.e. the longitudinal centreline.
- But one must not confuse this centreline with the joint axis, which is, for mechanical reasons, inaccessible.



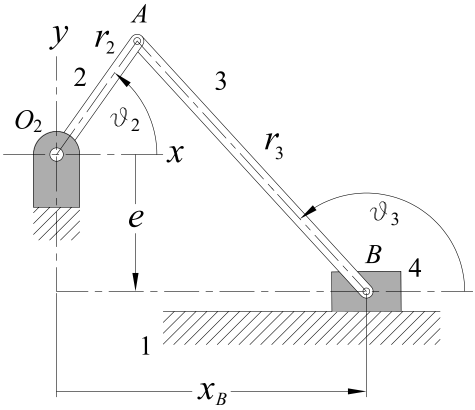


Trigonometric Solution of Vector Equations

- The graphical solution method is elegant and useful for visualisation, but unless using a CAD system, may not yield solutions that are sufficiently precise for analysis and synthesis.
- Most CAD systems transform the elements of the screen image into the underlaying trigonometric relationships imposed by the drawing elements.
- Hence, we will now examine the trigonometric approach.
- Consider the slider crank in the image.



- Note that by making the offset distance e equal to zero the same equations can be used for symmetric versions where the ground-fixed and slider R-pair centres are collinear on the x-axis.
- Two configuration analysis problems occur for planar slider cranks.
  1. Given ϑ_2 determine ϑ_3 and x_B.
     - This is known as the *forward kinematics* problem: given the input determine the output.
  2. Given x_B determine ϑ_2 and ϑ_3.
     - This is known as the *inverse kinematics* problem: given the output determine the input.



Given ϑ_2 determine ϑ_3 and x_B

- Identify the location of point A:

$$x_A = r_2 \cos\vartheta_2, \qquad y_A = r_2 \sin\vartheta_2.$$

- Next, note that

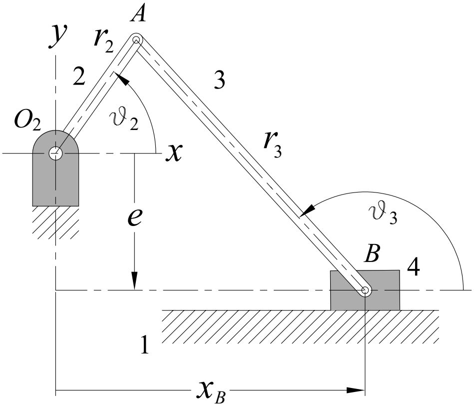
$$r_2 \sin\vartheta_2 = r_3 \sin\vartheta_3 - e,$$

- so that

$$\sin\vartheta_3 = \frac{1}{r_3}(e + r_2 \sin\vartheta_2). \quad (1)$$

- Furthermore we see that

$$x_B = r_2 \cos\vartheta_2 + r_3 \cos\vartheta_3. \quad (2)$$



- Now we use the trigonometric identity

$$\cos\vartheta_3 = \pm\sqrt{1-\sin^2\vartheta_3}$$

- which gives us from Equation (1)

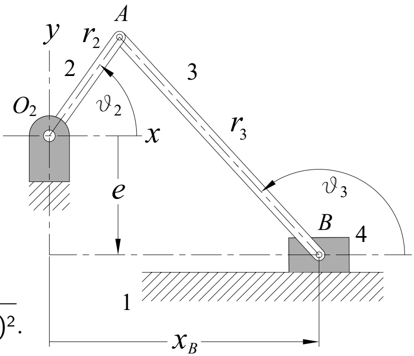
$$\cos\vartheta_3 = \pm\frac{1}{r_3}\sqrt{r_3^2-(e+r_2\sin\vartheta_2)^2}.$$

- Now we can compute the values of ϑ_3 and x_B using Equation (2) as

$$\vartheta_3 = \tan^{-1}\left(\frac{\sin\vartheta_3}{\cos\vartheta_3}\right),$$

$$x_B = r_2\cos\vartheta_2+\sqrt{r_3^2-(e+r_2\sin\vartheta_2)^2}.$$

- Note that we use positive value of $\cos\vartheta_3$ so that the slider will be on the right of the crank.



Given x_B determine ϑ_2 and ϑ_3

- Since we know the lengths of r_2 and r_3 we can write

$$x_B = r_2 \cos\vartheta_2 + r_3 \cos\vartheta_3.$$

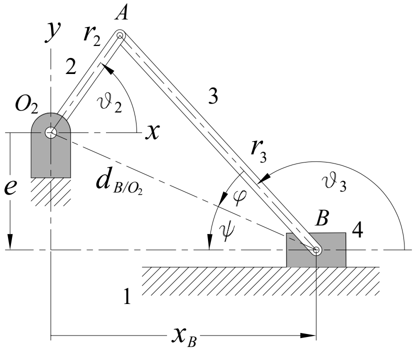
- Solving this for ϑ_2 yields

$$\vartheta_2 = \cos^{-1}\left(\frac{1}{r_2}(x_b - r_3\cos\vartheta_3)\right).$$

- To determine ϑ_3 we determine the quantities ψ, d_{B/O_2}, and ϕ.
- Clearly

$$\psi = \tan^{-1}\left(\frac{e}{x_B}\right), \text{ and}$$

$$d_{B/O_2} = (e^2 + x_B^2)^{1/2}.$$



- We can use the cosine law to determine ϕ

$$r_2^2 = d_{B/O_2}^2 + r_3^2 - 2r_3 d_{B/O_2} \cos\phi$$

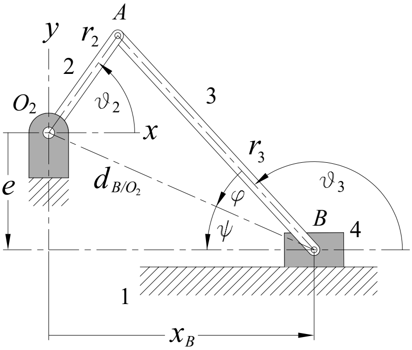
giving

$$\phi = \cos^{-1}\left(\frac{d_{B/O_2}^2 + r_3^2 - r_2^2}{2r_3 d_{B/O_2}}\right).$$

- Finally

$$\vartheta_3 = \pi - (\phi + \psi).$$

- These equations are also consistent when the offset distance $e = 0$.



Planar 4R Linkage

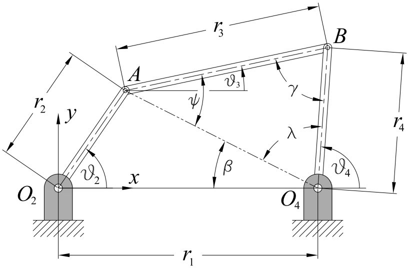
- The law of cosines can be used to determine the configuration of the planar 4R when the input angle, ϑ_2, and four link lengths r_i are known.

- The distance between points A and O_4 is given by

$$d_{O_4/A} = \left(r_1^2 + r_2^2 - 2 r_1 r_2 \cos\vartheta_2\right)^{1/2}$$

- Knowing $d_{O_4/A}$ we can now determine the three angles

$$\begin{aligned}
\beta &= \cos^{-1}\left(\frac{r_1^2 + d_{O_4/A}^2 - r_2^2}{2 r_1 d_{O_4/A}}\right), \\
\psi &= \cos^{-1}\left(\frac{r_3^2 + d_{O_4/A}^2 - r_4^2}{2 r_3 d_{O_4/A}}\right), \\
\lambda &= \cos^{-1}\left(\frac{r_4^2 + d_{O_4/A}^2 - r_3^2}{2 r_4 d_{O_4/A}}\right).
\end{aligned}$$



Planar $4R$ Linkage

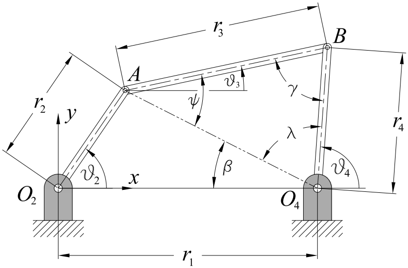
- If ϑ_2 is in the range $0 \leq \vartheta_2 \leq \pi$ then

$$\begin{aligned} \vartheta_3 &= -\beta \pm \psi, \\ \vartheta_4 &= \pi - \beta \mp \lambda. \end{aligned}$$

- If ϑ_2 is in the range $\pi \leq \vartheta_2 \leq 2\pi$ then

$$\begin{aligned} \vartheta_3 &= \beta \pm \psi, \\ \vartheta_4 &= \pi + \beta \mp \lambda. \end{aligned}$$

The upper signs correspond to the convex quadrangle configurations and the lower signs correspond to the concave quadrangle configurations.



Mechanical Advantage

- Mechanical advantage represents the ability of a of a particular mechanism to transmit force or torque from the actuated driver link to the output link via the coupler.
- Recall that the dot products of torque and angular velocity as well as force and linear velocity both represent power.
- In a conservative mechanical system where energy losses due to misalignment, friction, heat, etc. are negligible compared to the total energy transmitted by the system we can write

$$P_{\text{in}} = \vec{T}_{\text{in}} \cdot \vec{\omega}_{\text{in}} = \vec{T}_{\text{out}} \cdot \vec{\omega}_{\text{out}} = P_{\text{out}}$$

  or,

$$P_{\text{in}} = \vec{F}_{\text{in}} \cdot \vec{v}_{\text{in}} = \vec{F}_{\text{out}} \cdot \vec{v}_{\text{out}} = P_{\text{out}}$$

- This allows us to write the convenient ratios

$$\frac{T_{\text{out}}}{T_{\text{in}}} = \frac{\omega_{\text{in}}}{\omega_{\text{out}}} \text{ and } \frac{F_{\text{out}}}{F_{\text{in}}} = \frac{v_{\text{in}}}{v_{\text{out}}}$$

Mechanical Advantage

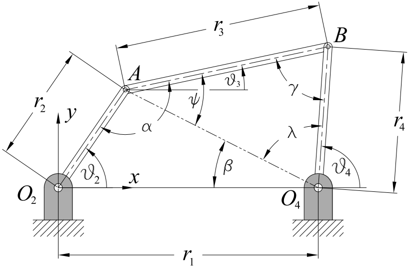
- Mechanical advantage is quantified as the ratio

$$\text{M.A.} = \frac{F_{\text{out}}}{F_{\text{in}}}.$$

- Since torque is the product of force and radius we can also write

$$\text{M.A.} = \frac{T_{\text{out}}}{r_{\text{out}}}\frac{r_{\text{in}}}{T_{\text{in}}} = \frac{r_{\text{in}}}{r_{\text{out}}}\frac{T_{\text{out}}}{T_{\text{in}}} = \frac{r_{\text{in}}}{r_{\text{out}}}\frac{\omega_{\text{in}}}{\omega_{\text{out}}}.$$

- Note the difference in the indices for the last product of ratios.
- The angle γ between the coupler and follower is the *transmission angle*.

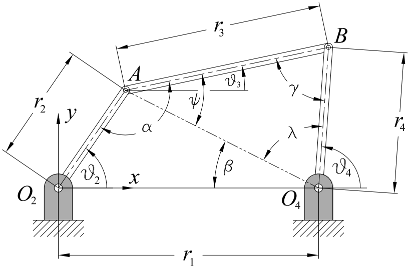


Mechanical Advantage

- The mechanical advantage can also be expressed as

$$\text{M.A.} \quad = \quad \frac{r_4 \sin\gamma}{r_2 \sin\alpha}$$

- As the transmission angle decreases the mechanical advantage decreases.
- At very small transmission angles even a very small amount of friction or misalignment of the joint axes might cause the mechanism to lock.
- A design rule of thumb, which goes back to antiquity and was used by the likes of Archimedes, is that a $4R$ linkage should not be used to overcome a load when $\gamma < 45^\circ$.



- The transmission angle varies continuously over the range of motion of the linkage, and is most favourable when $\gamma = 90^\circ$.
- If the purpose of the mechanism is to transmit force or torque, then by design, the transmission angle should stay in the range $90^\circ \pm 45^\circ$.

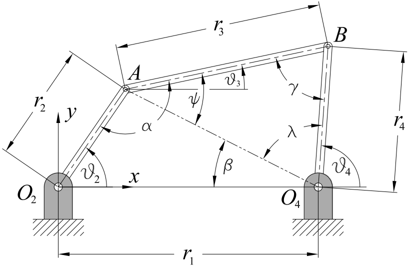
Mechanical Advantage

- The extreme values for γ occur when the input link aligns with the line joining the ground-fixed R-joints.
- Since the mechanical advantage is

$$\text{M.A.} \quad = \quad \frac{r_4 \sin \gamma}{r_2 \sin \alpha}$$

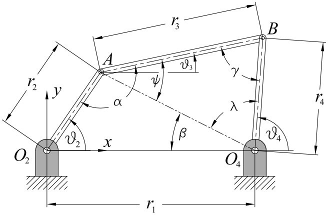
we can see that the magnitude of the angle α also has an effect.

- When $\sin \alpha = 0$ the mechanical advantage is infinite!
- In this case, an infinitesimally small input torque will produce a very large output torque.



Mechanical Advantage

- The extreme values for α occur when the input link aligns with the coupler, i.e. $\alpha = 0$ or 180°.
- This mode is called a *toggle* or *limit* configuration.
- Of course, both angles α and γ, and hence the mechanical advantage, continuously change as the linkage moves.
- A double rocker $4R$ has a configuration called *dead-centre* when links 3 and 4 lie on the same line.
- At the dead-centre $\gamma = 0^\circ$, or 180° and the mechanical advantage is 0.
- Since the linkage is locked, dead-centre should be avoided, or a spring should be added to make link 4 move through the dead-centre.



Complex Algebra Solution of Vector Equations

- For planar kinematics it is possible to solve problems using complex algebra.
- In component form a two dimensional vector is

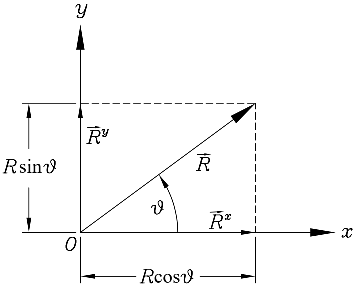
$$\vec{R} = R^x\hat{i} + R^y\hat{j}.$$

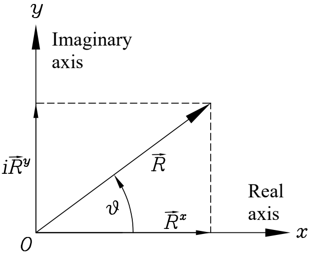
- It has two rectangular component magnitudes

$$R^x = R\cos\vartheta \quad \text{and} \quad R^y = R\sin\vartheta,$$

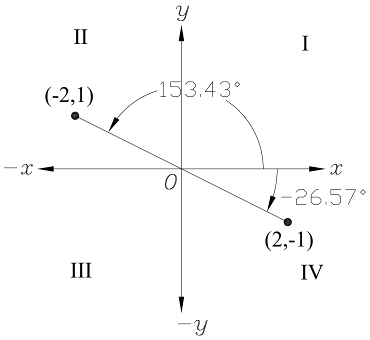
- where

$$R = \sqrt{(R^x)^2 + (R^y)^2} \quad \text{and} \quad \vartheta = \tan^{-1}\frac{R^y}{R^x}.$$





- Be careful to note the signs of R^x and R^y when evaluating the direction of $\vec{R}$.
- The angle ϑ is defined as positive in the counterclockwise sense from the positive x-axis.
- The signs, positive or negative, of R^x and R^y determine the quadrant in which the position vector $\vec{R}$ resides.



Sign R^x	Sign R^y	Quadrant
+	+	I
-	+	II
-	-	III
+	-	IV

Complex Polar Algebra Solution of Vector Equations

- A planar vector can be expressed in *polar notation* by specifying its magnitude and orientation

$$\vec{R} = R\angle\vartheta.$$

- In 2D kinematic problems one can use the complex plane by selecting an origin and by equating the x basis vector direction the *real* axis and the y basis vector direction the *imaginary* axis by scaling it with the *unit imaginary number* i, where

$$i = \sqrt{-1}.$$

- Hence, a planar vector can be represented by

$$\vec{R} = R\angle\vartheta = R\cos\vartheta + iR\sin\vartheta.$$

- Using the Euler trigonometric identity we can express the direction of $\vec{R}$ very efficiently as

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta.$$

- A planar vector can thus be represented in *complex polar form*

$$\vec{R} = Re^{i\vartheta}.$$

- Any planar vector equation involving sums such as

$$\vec{C} = \vec{A} + \vec{B}$$

can be compactly expressed as

$$Ce^{i\vartheta_C} = Ae^{i\vartheta_A} + Be^{i\vartheta_B}. \quad (3)$$

- From our graphical analysis we identified four pertinent cases for vector equations.

Case	Unknowns
1	C, ϑ_C
2	A, ϑ_B
3	A, B
4	ϑ_A, ϑ_B

Case 1: unknowns C and ϑ_C

- Substitute the Euler identity into Equation (3) to get

$$C(\cos\vartheta_C + i\sin\vartheta_C) \quad = \quad A(\cos\vartheta_A + i\sin\vartheta_A) + B(\cos\vartheta_B + i\sin\vartheta_B).$$

- Equate the real and imaginary terms to obtain a system of two independent equations

$$C\cos\vartheta_C \quad = \quad A\cos\vartheta_A + B\cos\vartheta_B, \tag{4}$$

$$C\sin\vartheta_C \quad = \quad A\sin\vartheta_A + B\sin\vartheta_B. \tag{5}$$

- The unknown angle ϑ_C can be eliminated from the two equations by squaring and adding them yielding

$$C \quad = \quad \pm\sqrt{A^2 + B^2 + 2AB\cos(\vartheta_B - \vartheta_A)}.$$

- Using the negative root would point C in the opposite direction and rotate ϑ_C by π.
- The unknown angle ϑ_C can be determined by dividing Eq (5) by (4)

$$\vartheta_C \quad = \quad \tan^{-1}\left(\frac{A\sin\vartheta_A + B\sin\vartheta_B}{A\cos\vartheta_A + B\cos\vartheta_B}\right),$$

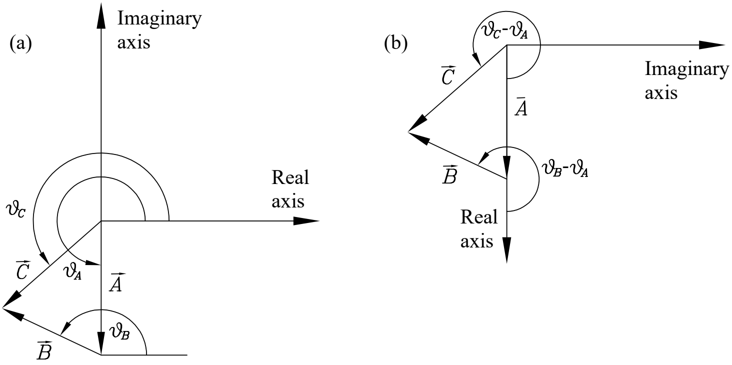
where the signs of the numerator and denominator must be considered to place the vector in the correct quadrant.

Case 2: unknowns A and ϑ_B

- This can be approached by dividing Equation (3) by $e^{i\vartheta_A}$ giving, after some algebra

$$Ce^{i(\vartheta_C - \vartheta_A)} = A + Be^{i(\vartheta_B - \vartheta_A)}.$$

- The effect of dividing Equation (3) by the complex unit vector represented as $e^{i\vartheta_A}$ is to rotate the real and imaginary axes clockwise by ϑ_A.



- Now we use the Euler identity to separate the real and imaginary componenets

$$C\cos(\vartheta_C - \vartheta_A) = A + B\cos(\vartheta_B - \vartheta_A), \quad (6)$$

$$C\sin(\vartheta_C - \vartheta_A) = B\sin(\vartheta_B - \vartheta_A). \quad (7)$$

- The solutions are then directly obtained from Equation (7) and (6), respectively, yielding:

$$\vartheta_B = \vartheta_A \pm \sin^{-1}\left(\frac{C\sin(\vartheta_C - \vartheta_A)}{B}\right),$$

$$A = C\cos(\vartheta_C - \vartheta_A) - B\cos(\vartheta_B - \vartheta_A).$$

- The solutions must be obtained in this order since the value for A is a function of ϑ_B.
- The first equation contains a $\sin^{-1}$ term which has positive and negative values, hence there are two distinct solutions.

Case 3: unknowns A and B

- We can begin to identify a solution by aligning the real axis with one of the vectors, $\vec{A}$ for instance, by dividing Equation (3) by $e^{i\vartheta_A}$.
- After using the Euler identity again, we separate the real and imaginary components and obtain the magnitude of B from Equation (7) and of A from Equation (6) respectively to obtain

$$\begin{aligned} B &= C\frac{\sin(\vartheta_C - \vartheta_A)}{\sin(\vartheta_B - \vartheta_A)} \\ A &= C\cos(\vartheta_C - \vartheta_A) - B\cos(\vartheta_B - \vartheta_A) \end{aligned}$$

- As expected, the solution is unique.
- It is important to remember that the angles are defined relative to the real axis which is aligned with vector $\vec{A}$ since we divided Equation (3) by $e^{i\vartheta_A}$.

Case 4: unknowns ϑ_A and ϑ_B

- We can identify a solution by aligning the real axis with vector $\vec{C}$ for this last case, giving

$$C = Ae^{i(\vartheta_A - \vartheta_C)} + Be^{i(\vartheta_B - \vartheta_C)}. \tag{8}$$

- We use the Euler identity again and separate the real and imaginary components and obtain

$$A\cos(\vartheta_A - \vartheta_C) = C - B\cos(\vartheta_B - \vartheta_C), \tag{9}$$
$$A\sin(\vartheta_A - \vartheta_C) = -B\sin(\vartheta_B - \vartheta_C). \tag{10}$$

- Squaring and adding these two equations gives

$$A^2 = C^2 + B^2 - 2BC\cos(\vartheta_B - \vartheta_C).$$

- Note that this equation is the law of cosines for the vector addition triangle, and we solve it for ϑ_B:

$$\vartheta_B = \vartheta_C \pm \cos^{-1}\left(\frac{C^2 + B^2 - A^2}{2BC}\right).$$

- We can rewrite Equation (9) as

$$A\cos(\vartheta_A - \vartheta_C) - C = -B\sin(\vartheta_B - \vartheta_C)$$

then square this equation and add to the square of Equation (10) to obtain another form of the law of cosines:

$$A^2 - B^2 + C^2 = 2AC\cos(\vartheta_A - \vartheta_C).$$

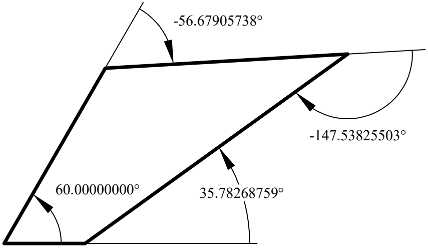
- Now, solve for ϑ_A yielding

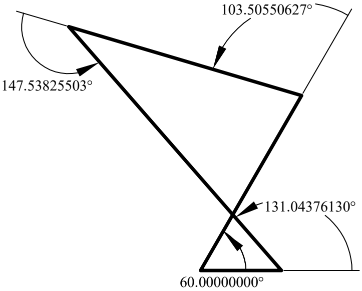
$$\vartheta_A = \vartheta_C \pm \cos^{-1}\left(\frac{A^2 - B^2 + C^2}{2AC}\right).$$

- The positive and negative values for the inverse cosine terms gives us two distinct solutions, as revealed by the graphical solution.

Displacement Analysis using the Six Algebraic v_i-v_j IO Equations

- The algebraic IO equations can also be used, very efficiently, to determine unknown angles given a set of link lengths and one specified variable parameter.
- The efficiency reveals itself when you realise that when the input or output variables are specified, the IO equations are quadratic in the other unknown value.
- If the linkage has two assembly modes, the two solutions of the quadratic equation represent the unknown value for both assembly modes of the linkage!





The Six v_i-v_j IO Equations

- Recall the algebraic IO equation for a planar 4R linkage:

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{11}$$

where

$$A = A_1A_2 = (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4),$$

$$B = B_1B_2 = (a_1 - a_2 + a_3 + a_4)(a_1 + a_2 + a_3 + a_4),$$

$$C = C_1C_2 = (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4),$$

$$D = D_1D_2 = (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4),$$

$$v_1 = \tan\frac{\theta_1}{2}, \quad v_4 = \tan\frac{\theta_4}{2}.$$

The Six v_i-v_j IO Equations

The remaining five 4R IO equations are:

$$A_1B_1v_1^2v_2^2 + A_2B_2v_1^2 + C_1D_2v_2^2 + 8a_2a_4v_1v_2 + C_2D_1 = 0, \tag{12}$$

$$A_2B_1v_1^2v_3^2 + A_1B_2v_1^2 + C_1D_1v_3^2 + C_2D_2 = 0, \tag{13}$$

$$B_1C_1v_2^2v_3^2 + A_1D_2v_2^2 + A_2D_1v_3^2 - 8a_1a_3v_2v_3 + B_2C_2 = 0, \tag{14}$$

$$A_1C_1v_2^2v_4^2 + B_1D_2v_2^2 + A_2C_2v_4^2 + B_2D_1 = 0, \tag{15}$$

$$A_2C_1v_3^2v_4^2 + B_1D_1v_3^2 + A_1C_2v_4^2 - 8a_2a_4v_3v_4 + B_2D_2 = 0. \tag{16}$$

Example 5.2

- Consider a planar 4R linkage with lengths $a_1 = 5\text{cm}$, $a_2 = 6\text{cm}$, $a_3 = 8\text{cm}$, and $a_4 = 2\text{cm}$, with an input angle of $\theta_1 = 60^\circ$. Determine θ_2, θ_3, and θ_4 using the appropriate v_i-v_j equation.
- We can start with determining θ_2 by substituting the lengths and $v_1 = \tan 60^\circ/2$ into Equation (12) and solve for v_2 revealing the values in both assembly modes, illustrated by the figure on the following page.

$$\theta_2 = \left(2\tan^{-1}(v_2)\right)\frac{180^\circ}{\pi} = -56.6791^\circ, \quad 103.5055^\circ$$

- Now determine both values for θ_3 using Equation (13), which yields

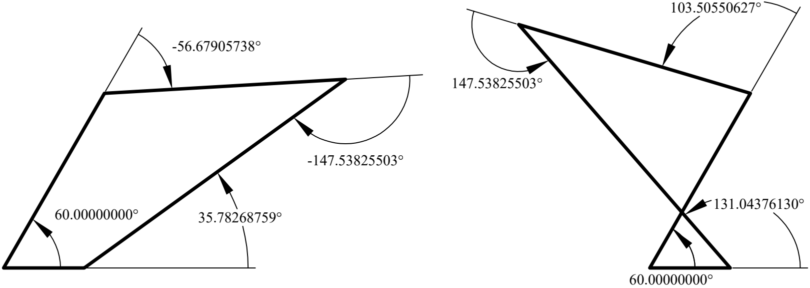
$$\theta_3 = \left(2\tan^{-1}(v_3)\right)\frac{180^\circ}{\pi} = -147.5383^\circ, \quad 147.5383^\circ$$

- Finally, determine both values for θ_4 using Equation (11), which yields

$$\theta_4 = \left(2\tan^{-1}(v_4)\right)\frac{180^\circ}{\pi} = 131.0438^\circ, \quad 35.7827^\circ$$

- It is similarly straightforward to perform displacement analysis for the planar RRRP linkage.

Example 5.2 Continued



Type, Number, and Dimensional Synthesis

The subject of linkage synthesis for motion generation contains three steps: type; number; and dimensional synthesis.

1. Type synthesis involves selecting the general topology of the mechanical system.
   - It might only contain a certain types of joints such as a slider crank.
   - It might contain gears, belts or chains, pulleys, or cams, etc.
2. Number synthesis concerns the numbers of links and/or joints that are required to obtain a desired mobility.
3. Dimensional synthesis involves determining the lengths of the links in the mechanical system, or the distance between adjacent joint axes.

Standard Synthesis Problems

There are several types of synthesis problems.

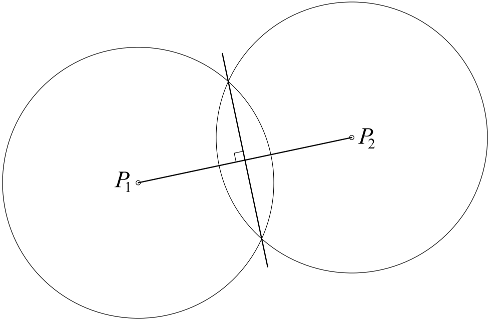
1. Guiding a point along a specific curve in the plane or in space.
2. The function generation problem, which we have already considered: designing a linkage to correlate the poses of the input and output links in a functional relationship that may also have some timing requirements.
3. The rigid body guidance problem: designing a linkage that can move a line on a rigid link through a finite number of poses in sequence.
4. Trajectory generation: position, velocity, and/or acceleration must be correlated along a specific curve.
5. We will only consider graphical methods to solve the rigid body guidance problem.

The Rigid Body Guidance Problem

- Guiding a rigid body through a finite set of positions and orientations.
- A good example is a landing gear mechanism which must retract and extend the wheels, having down and up locked poses with specific intermediate poses for collision avoidance.
- The rigid body guidance problem is also known as the Burmester problem.
- Many algebraic and trigonometric methods for the Burmester problem exist, but examining these methods is better suited to a graduate level course in kinematics or kinetics.
- We will restrict ourselves to graphical methods of synthesis to solve the rigid body guidance problem.

Ludwig Burmester (1840-1927), German mathematician and geometer whose research on kinematics greatly influenced the history of mechanism and machine science!

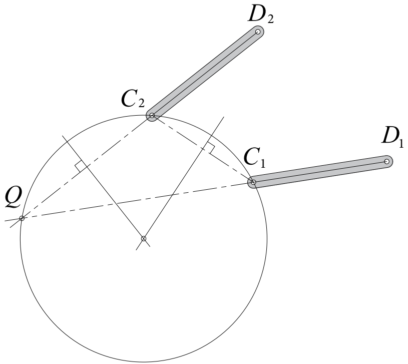
Right (Perpendicular) Bisector



- Given two points P_1 and P_2, construct the right (perpendicular) bisector of the line passing through the two points.
- Construct two circles with the same radii which is greater than one half of the distance between P_1 and P_2.
- Join the points of intersection of the two identical circles.
- This line right bisects the line joining P_1 and P_2.

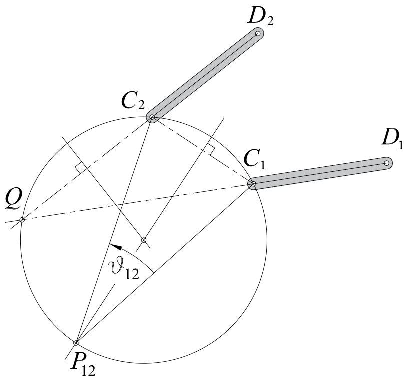
Pole of a Displacement

- A general planar displacement of the link *CD* consists of a translation of point C from pose 1 to 2 and a rotation about C_2
- There always exists a unique and instantaneously invariant point in the plane that is the centre of a single rotation that takes the link from pose 1 to pose 2, called the *pole* of the displacement.
- Extend the centre lines of link *CD* in each pose to find their point of intersection Q.
- Determine the circle on the points Q, C_1, and C_2.



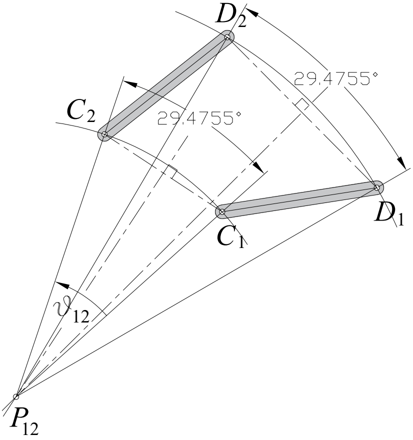
Pole of a Displacement

- Extend the right bisector of $C_1 C_2$ to intersect the circle.
- The point P_{12} is the *pole* of the displacement.
- The angle of rotation about point P_{12} is ϑ_{12}.



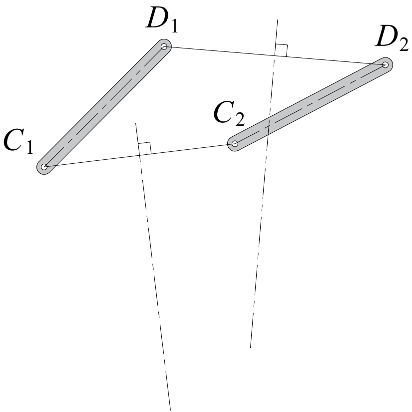
Alternate Pole Construction

- There is another way to graphically determine the pole of the general plane displacement of Link CD from pose 1 to 2.
- Construct the right bisectors of line segments C_1C_2 and D_1D_2 and locate their point of intersection: the pole point P_{12} as before.
- Note that all points on CD are rotated about the pole P_{12} through the same angle ϑ_{12}.

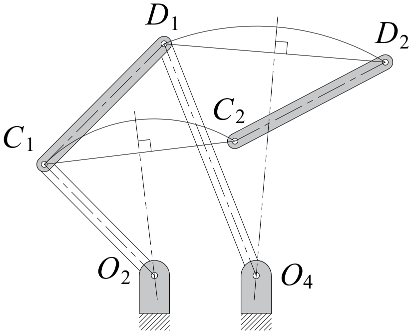


Exact Synthesis for Two Accuracy Poses

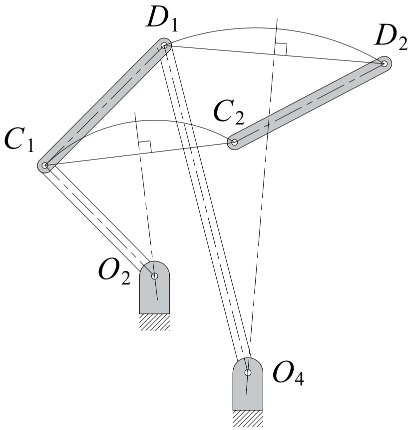
- If we are given only two distinct poses of a line segment in the plane it is a simple matter to determine a $4R$ linkage that will have the coupler, which contains the line segment, move exactly through the two poses.
- The moving revolute joint centres C and D must be connected to ground-fixed revolute joints via the input and output links, respectively.
- To identify the revolute joint centre locations fixed to ground, construct the right bisectors of line segments C_1C_2 and D_1D_2.



- The ground-fixed R-pair centres O_2 and O_4 may be placed anywhere on the right bisectors of C_1C_2 and D_1D_2, respectively.
- There are ∞^2 possible real solutions.

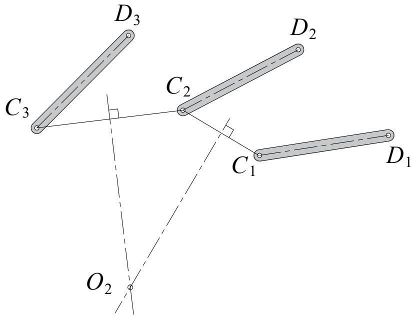


- The ground-fixed R-pair centres O_2 and O_4 may be placed anywhere on the right bisectors of C_1C_2 and D_1D_2, respectively.
- There are ∞^2 possible real solutions.

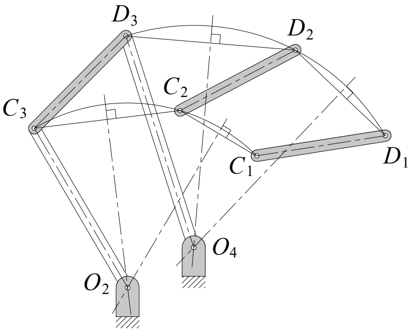


Exact Synthesis for Three Accuracy Poses

- If we are given three distinct poses of a line segment in the plane it is a simple matter to determine a $4R$ linkage that will have the coupler, which contains the line segment, move exactly through the three poses.
- The moving revolute joint centres C and D are connected to ground-fixed revolute joints via the input and output links, respectively.
- To identify one revolute joint centre fixed to ground, construct the right bisectors of line segments C_1C_2 and C_2C_3 and locate their point of intersection, which we will call O_2.

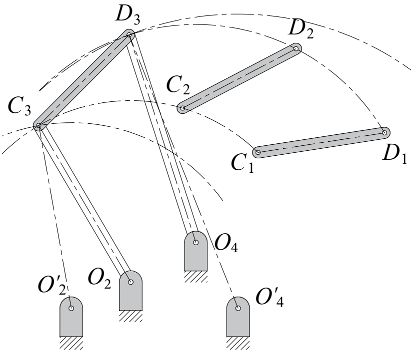


- Locate O_4 by finding the point of intersection of the right bisectors of line segments D_1D_2 and D_2D_3.
- The input and output link lengths and the distance between O_2 and O_4 are now determined.
- Points C and D now move on two circles centred at O_2 and O_4.



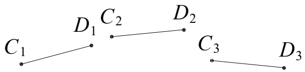
Synthesis for Three Constrained Accuracy Poses

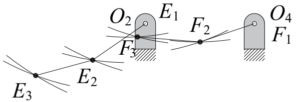
- Suppose now that the locations of the ground-fixed R pairs are required to be in different locations O_2' and O_4'.
- This means that points C and D will not move on circles centred at O_2' and O_4' and the three required poses are not possible to reach with these R pair locations.
- This additional constraint means that points C and D are no longer the centres of the moving R pairs, rather they are two points with coordinates in a coordinate system that moves with the coupler.



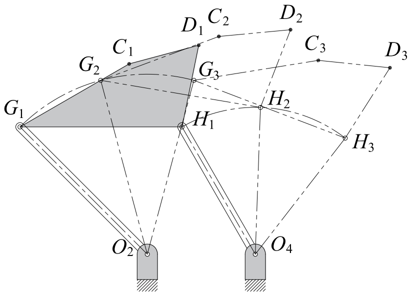
The following algorithm may be used to determine the linkage.

1. Consider the following example: the line segment CD must be moved through the three poses relative to O_2 and O_4.
2. Double-label points O_2 and O_4 as E_1 and F_1 then draw arcs of lengths C_2O_2 and D_2O_2 centred at C_1 and D_1, respectively and identify their point of intersection closest to O_2 and label it E_2.
3. Repeat this process drawing arcs of lengths C_2O_4 and D_2O_4 centred at C_1 and D_1, respectively and identify their point of intersection closest to O_4 and label it F_2.
4. Repeat steps 2 and 3 for points C_3 and D_3 and identify points E_3 and F_3, and construct line segments E_1E_2, E_2E_3, F_1F_2, and F_2F_3.



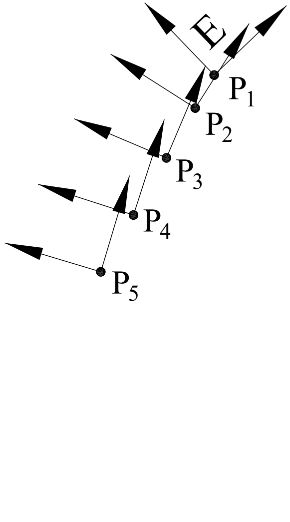


5. Bisect line segments E_1E_2, E_2E_3, F_1F_2, and F_2F_3 and label the points of intersection of the right bisectors of E_1E_2 and E_2E_3 as G and that of the right bisectors of F_1F_2 and F_2F_3 as H.
6. A particular line on the coupler of the desired linkage is the line joining points G and H.
7. The input link is represented by the line between G and O_2 while the output link is represented by the line joining H and O_4.
8. The actual shape of the coupler is arbitrary, as long as it contains line segments GH and C_1D_1.
9. Using the geometry of the coupler shape you choose, construct the linkage in poses 2 and 3.



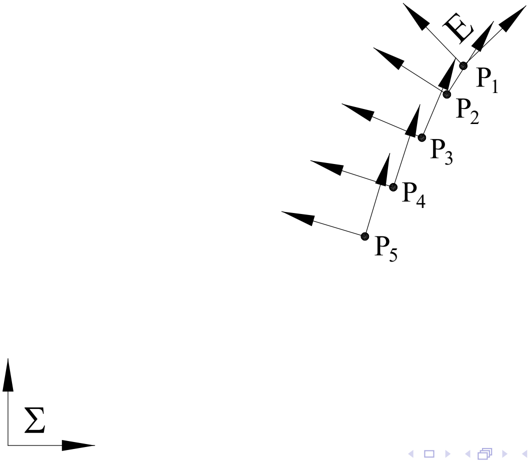
Note on Rigid Body Guidance

- Consider five specified poses of a coordinate system that moves with the coupler.



Note on Rigid Body Guidance

- Burmester theory states that as many as six different linkages can move a coordinate system that moves with the coupler through the five specified poses.



Note on Rigid Body Guidance

- Burmester theory states that as many as six different linkages can move a coordinate system that moves with the coupler through the five specified poses.
- These results were obtained using *kinematic mapping*, see https://carleton.ca/johnhayes/wp-content/uploads/CCToMM04.pdf for details