MAAE 3004
Dynamics of Machinery

Lecture Slide Set 6

Velocity Analysis

Department of Mechanical and Aerospace Engineering
Carleton University

Outline

Definitions

Velocity

The *particle* A travels from position $\vec{R}_A$ to position $\vec{R}'_A$ along an arbitrary curved path over small time increment Δt

$$\vec{R}'_A = \vec{R}_A + \Delta\vec{R}_A \qquad (1)$$

Total change in position

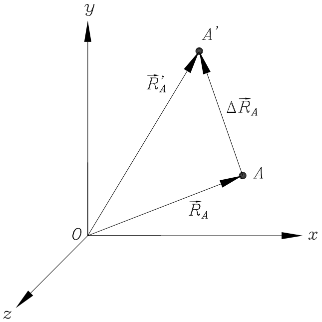
$$\Delta\vec{R}_A = \vec{R}'_A - \vec{R}_A \qquad (2)$$

Average velocity

$$\bar{\vec{V}}_A = \frac{\Delta\vec{R}_A}{\Delta t} \qquad (3)$$

Instantaneous velocity

$$\vec{V}_A = \lim_{\Delta t\to 0}\frac{\Delta\vec{R}_A}{\Delta t} = \frac{d\vec{R}_A}{dt} \qquad (4)$$



Notes:

- $\vec{V}_A$ applies to a specific point.
- $\vec{V}_A$ depends on the motion of the observer and the observer's reference coordinate system.
- $\vec{V}_A$ is an absolute velocity only if the observer's coordinate system is stationary.

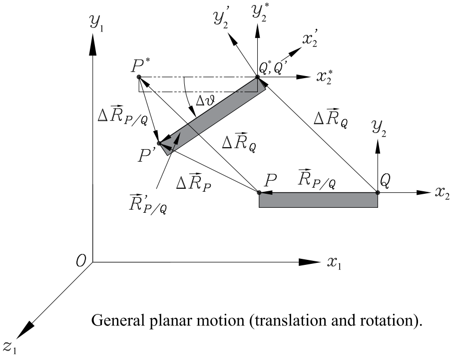
Angular Velocity

The *rigid body* changes position and orientation from PQ to $P'Q'$ over time increment Δt.

- The rigid body undergoes general planar motion.
  - Translation of $\Delta\vec{R}_Q$ from PQ to P^*Q^*
  - Rotation of $\Delta\theta$ from P^*Q^* to $P'Q'$
- Throughout Δt, an observer in the moving x_2y_2 coordinate system will observe no motion of Q.
- The magnitude of the angular velocity is

$$|\vec{\omega}| = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (5)$$

where $\vec{\omega}$ is directed along the axis of rotation and positive CCW.



General planar motion (translation and rotation).

Velocity Difference

Consider the following derivation:

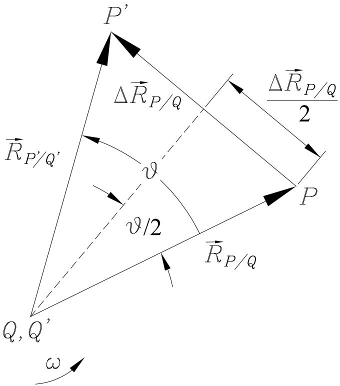
$$\sin\left(\frac{\Delta\theta}{2}\right) = \frac{\frac{\Delta R_{P/Q}}{2}}{R_{P/Q}} \qquad (6)$$

Therefore,

$$\Delta R_{P/Q} = 2R_{P/Q}\sin\left(\frac{\Delta\theta}{2}\right) \qquad (7)$$

For small angles (i.e., small Δt)

$$\sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2} \Rightarrow \Delta R_{P/Q} = R_{P/Q}\Delta\theta \qquad (8)$$



Then,

$$\left|\vec{V}_{P/Q}\right| \equiv \lim_{\Delta t\to 0}\frac{\Delta R_{P/Q}}{\Delta t} = \left(\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}\right)R_{P/Q} = \omega R_{P/Q} \qquad (9)$$

Noting that $\vec{\omega} \perp \vec{R}_{P/Q}$ and $\vec{V}_{P/Q} \perp \vec{R}_{P/Q}$, vector notation can be used to write the **velocity difference vector** as

$$\vec{V}_{P/Q} = \vec{\omega} \times \vec{R}_{P/Q} \qquad (10)$$

For two points, P and Q, on a *rigid body*, the velocity difference vector states

$$\vec{V}_{P/Q} = \vec{\omega} \times \vec{R}_{P/Q} \tag{11}$$

The position difference equation states

$$\Delta\vec{R}_P = \Delta\vec{R}_Q + \Delta\vec{R}_{P/Q} \tag{12}$$

For small Δt

$$\lim_{\Delta t \to 0} \frac{\Delta\vec{R}_P}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\vec{R}_Q}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta\vec{R}_{P/Q}}{\Delta t} \tag{13}$$

This is the **velocity difference equation**

$$\vec{V}_P = \vec{V}_Q + \vec{V}_{P/Q} \tag{14}$$

Notes:

- Equation 14 can be applied to any pair of points
- Equation 11 makes sense only when applied to a pair of points on the same rigid body.
- Equations 14 and 11 can be combined for points on the same rigid body

$$\vec{V}_P = \vec{V}_Q + \vec{\omega} \times \vec{R}_{P/Q} \tag{15}$$

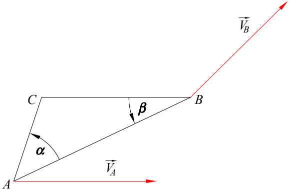
Velocity Polygons

Standard Velocity Polygon

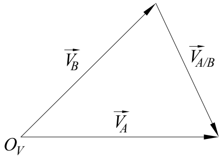
The velocities of points A and B on the unconstrained *rigid body ABC* are known to be $\vec{V}_A$ and $\vec{V}_B$ respectively. The objective is to determine the angular velocity $\vec{\omega}$ and $\vec{V}_C$.

- The velocity difference equation can be written as

$$\overset{\surd\surd}{\vec{V}}_A = \overset{\surd\surd}{\vec{V}}_B + \overset{oo}{\vec{V}}_{A/B} \quad (16)$$
$$\vec{V}_B = \vec{V}_A + \vec{V}_{B/A} \quad (17)$$



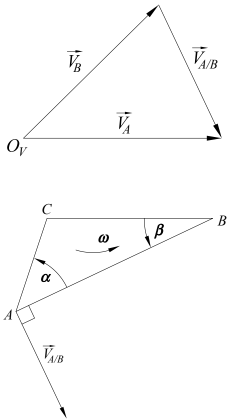
- Choose a scale.
- Draw $\vec{V}_A$ and $\vec{V}_B$ from a common origin.
- Obtain $\vec{V}_{A/B}$ (magnitude and direction) by solving Equation 16 graphically



- The angular velocity is determined from application of the velocity difference vector.

$$\vec{V}_{A/B} = \vec{\omega} \times \vec{R}_{A/B} \quad (18)$$
$$\mid V_{A/B} \mid = \omega R_{A/B} \quad (19)$$
$$\omega = \frac{\mid V_{A/B} \mid}{R_{A/B}} \quad (20)$$

- The direction of $\vec{\omega}$ is obtained by inspecting the direction of $\vec{V}_{A/B}$ and considering the RH rule.

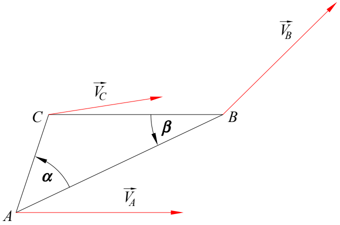


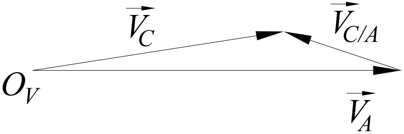
- To obtain $\vec{V}_C$, use the velocity difference equation

$$\overset{oo}{\vec{V}}_C = \overset{\surd\surd}{\vec{V}}_A + \overset{o\surd}{\vec{V}}_{C/A} = \overset{\surd\surd}{\vec{V}}_B + \overset{o\surd}{\vec{V}}_{C/B} \tag{21}$$

-Note that the directions of $\vec{V}_{C/A}$ and $\vec{V}_{C/B}$ are known to be perpendicular to $\vec{R}_{C/A}$ and $\vec{R}_{C/B}$ respectively.

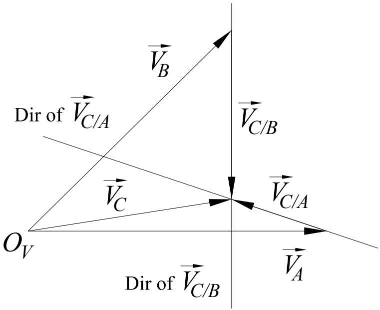
- To find $\vec{V}_C$, we can continue in one of two ways:
  1. By calculating the velocity difference magnitude
     - Calculate $|\vec{V}_{C/A}| = \omega R_{C/A}$ or $|\vec{V}_{C/B}| = \omega R_{C/B}$
     - Complete the polygon for $\overset{\surd\surd}{\vec{V}}_A$ and $\overset{\surd\surd}{\vec{V}}_{C/A}$ (or $\overset{\surd\surd}{\vec{V}}_B$ and $\overset{\surd\surd}{\vec{V}}_{C/B}$) to obtain $\overset{oo}{\vec{V}}_C$

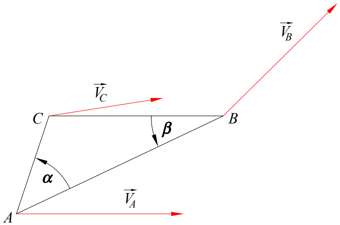




2. By using the directions of the velocity differences exclusively
   - From the graphical solution to Equation 21 which states

$$\vec{V}_C = \overset{o\surd}{\vec{V}}_{C/A}\,(\perp \text{ to} \vec{R}_{C/A}) + \overset{\surd\surd}{\vec{V}}_{A} = \overset{o\surd}{\vec{V}}_{C/B}\,(\perp \text{ to} \vec{R}_{C/B}) + \overset{\surd\surd}{\vec{V}}_{B} \qquad (22)$$

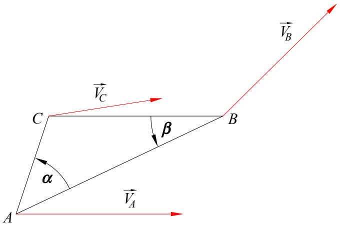




Enhanced Velocity Polygon

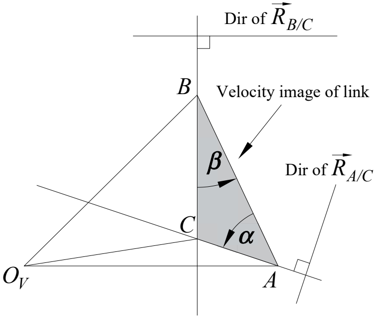
Consider the construction of an enhanced velocity polygon

- Choose an appropriate scale and point O_V for the origin.
- Draw $\vec{V}_A$ and $\vec{V}_B$ from O_V with termini labeled as A and B.
- Proceed as with the standard velocity polygon above.



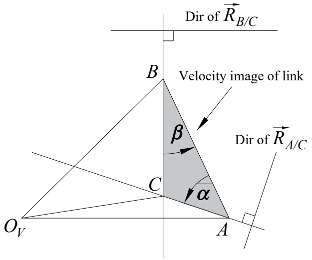
The *enhanced velocity polygon* can be interpreted as follows:

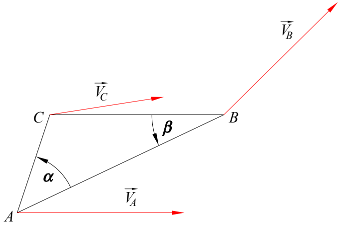
- O_V is the velocity image of the fixed link and all points with zero absolute velocity.
- The absolute velocity of any point on the velocity polygon is represented by a line from O_V to the image of the point.
- The velocity difference vector between any two points is represented by the line between the image of the two points.



Characteristics of a *velocity image* are:

- The velocity image of a link has a shape similar to the link with a scale factor equal to ω of that link (e.g., $V_{B/A} = \omega R_{B/A}; V_{C/A} = \omega R_{C/A}; V_{B/C} = \omega R_{B/C}$).
- The velocity image of a translating link is a single point in the velocity polygon.
- The velocity image of a link is rotated 90 deg in the direction of $\vec{\omega}$.
- The velocity image concept can be used to obtain $\vec{V}_C$ very quickly as follows:
  - Obtain $O_V A$ and $O_V B$.
  - Construct ABC on the velocity polygon by means of angles α and β.
  - Read $\vec{V}_C$ as $O_V C$ (after appropriate scaling).





Example 6.1: Graphical Solution

Four-bar linkage

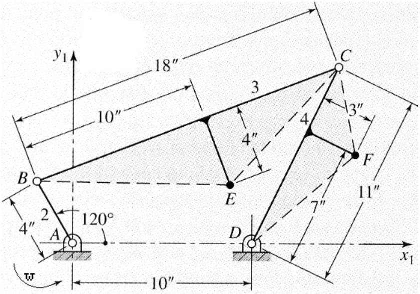
A four-bar linkage is shown schematically below. It is required to find $\vec{V}_E$, $\vec{V}_F$, $\vec{\omega}_3$, and $\vec{\omega}_4$ for the case where $\vec{\omega}_2 = 900$ rpm ccw.

Solution:

Loop closure equation

$$\underbrace{\vec{V}_{B/A}}_{\vec{V}_B} + \vec{V}_{C/B} + \underbrace{\underbrace{\vec{V}_{D/C}}_{-\vec{V}_{C/D}}}_{-\vec{V}_C} + \underbrace{\vec{V}_{A/D}}_{0} = 0$$

$$\overset{\surd\surd}{\vec{V}}_B + \overset{o\surd}{\vec{V}}_{C/B} = \overset{o\surd}{\vec{V}}_C$$

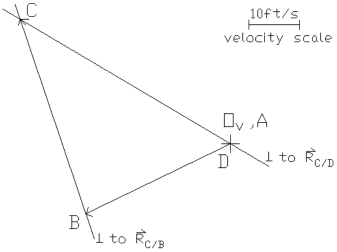


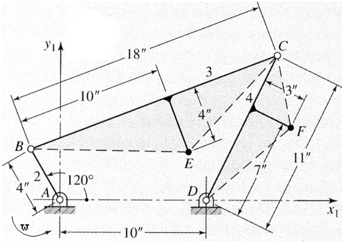
J J.Uicker. *Theory of Machines and Mechanisms*. Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.

- $\omega_2 = (900 \text{ rev/min})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ sec}) = 94.2 \text{ rad/s ccw}$
- $\vec{V}_B = \vec{V}_A + \vec{V}_{B/A} = \vec{\omega}_2 \times \vec{R}_{B/A} = (94.2 \text{ rad/s})(4/12 \text{ ft}) = 31.4 \text{ ft/s} \angle 210 \text{ deg}$
- $$\overset{o\checkmark}{\vec{V}_C} = \overset{\checkmark\checkmark}{\vec{V}_B} + \overset{o\checkmark}{\vec{V}_{C/B}} = \overset{\checkmark\checkmark}{\vec{V}_D} + \overset{o\checkmark}{\vec{V}_{C/D}}$$

$$\vec{V}_C = \cancelto{0}{\vec{V}_D} + \vec{V}_{C/D}$$

- $$V_{C/B} = 38.4 \text{ ft/s}; V_C = V_{C/D} = 45.6 \text{ ft/s}$$





J J.Uicker. *Theory of Machines and Mechanisms*. Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.

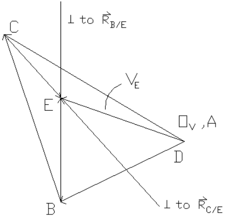
- Then,

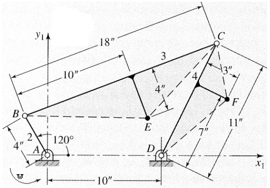
$$\vec{V_{C/B}} = \vec{\omega_3} \times \vec{R_{C/B}} \Rightarrow \omega_3 = \frac{V_{C/B}}{R_{C/B}} = \frac{38.4}{\frac{18}{12}} = 25.6 \text{ rad/s ccw}$$

$$\vec{V_{C/D}} = \vec{\omega_4} \times \vec{R_{C/D}} \Rightarrow \omega_4 = \frac{V_{C/D}}{R_{C/D}} = \frac{45.5}{\frac{11}{12}} = 49.6 \text{ rad/s ccw}$$

- $$\overset{oo}{\vec{V}_E} = \overset{\surd\surd}{\vec{V}_B} + \overset{o\surd}{\vec{V}_{E/B}} = \overset{\surd\surd}{\vec{V}_C} + \overset{o\surd}{\vec{V}_{E/C}}$$

$$\vec{V}_E = 27.6 \text{ ft/s}$$

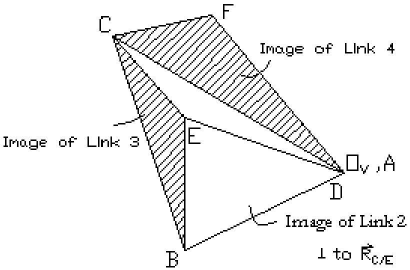


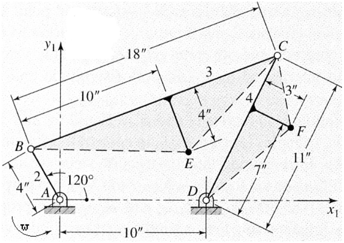


J J.Uicker. *Theory of Machines and Mechanisms*. Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.

Note that an alternative approach, in this case, would have been to construct $\triangle CEB$ similar to link BEC.

- Similarly, $\vec{V}_F = 31.8\ \mathrm{ft/s}$





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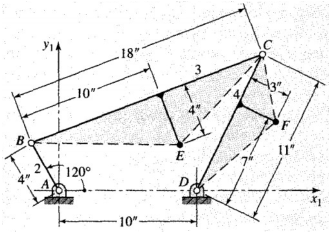
Example 6.1: Analytic Solution

- Using the position analysis techniques from Lecture Slide Set 5 and using the laws of cosines and sines we find that for the given link lengths and input angle of $\vartheta = 120^\circ$ that the orientations of lines BC and DC on links 3 and 4, relative to the x_1-axis are

$$\begin{aligned} \vartheta_3 &= 20.92^\circ \\ \vartheta_4 &= 64.05^\circ \end{aligned}$$

The relative position vectors are



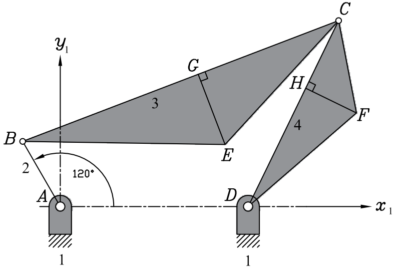
$$\begin{aligned}
\vec{R}_{B/A} &= 4\text{ in }\angle 120^\circ = -2\hat{i} + 3.46\hat{j}\text{ in} \\
\vec{R}_{C/B} &= 18\text{ in }\angle 20.92^\circ = 16.81\hat{i} + 6.43\hat{j}\text{ in} \\
\vec{R}_{C/D} &= 11\text{ in }\angle 64.05^\circ = 4.81\hat{i} + 9.89\hat{j}\text{ in} \\
\vec{R}_{D/A} &= 10\text{ in }\angle 0^\circ = 10\hat{i} + 0\hat{j}\text{ in} \\
\vec{R}_{E/B} &= 10.77\text{ in }\angle -0.88^\circ = 10.77\hat{i} - 0.17\hat{j}\text{ in} \\
\vec{R}_{F/D} &= 7.62\text{ in }\angle 40.85^\circ = 5.76\hat{i} + 4.98\hat{j}\text{ in}
\end{aligned}$$

- Calculate the angular velocity of link 2 in $\mathrm{rad/s}$

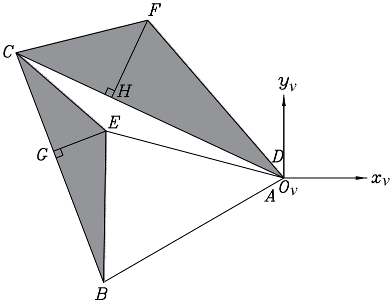
$$\omega_2 = \left(900\,\frac{\mathrm{rev}}{\mathrm{min}}\right)\left(2\pi\,\frac{\mathrm{rad}}{\mathrm{rev}}\right)\left(\frac{1\ \mathrm{min}}{60\ \mathrm{s}}\right) = 94.25\ \mathrm{rad/s\ ccw}$$

- Calculate $\vec{V}_B$ after converting units of inches to feet and noting that point A is relatively fixed

$$\begin{aligned}\vec{V}_B &= \cancelto{0}{\vec{V}_A} + \vec{V}_{B/A} = \vec{\omega}_2 \times \vec{R}_{B/A} \\ &= -27.21\hat{i} - 15.74\hat{j}\ \mathrm{ft/s} = 31.42\ \mathrm{ft/s}\ \angle 210^\circ\end{aligned}$$



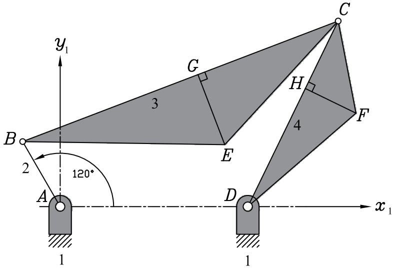
Pose diagram



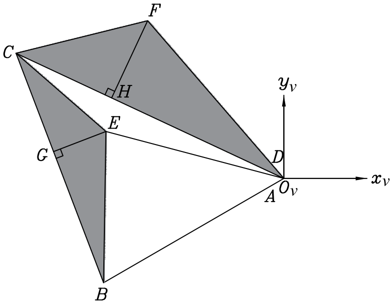
Velocity polygon

- The velocity of point C can be expressed as

$$\begin{aligned}
\overset{o\surd}{\vec{V}_C} &= \overset{\surd\surd}{\vec{V}_B} + \overset{o\surd}{\vec{V}_{C/B}} = \cancelto{0}{\vec{V}_D} + \overset{o\surd}{\vec{V}_{C/D}} \\
&= \vec{V}_B + \vec{\omega}_3 \times \vec{R}_{C/B} = \vec{\omega}_4 \times \vec{R}_{C/D}.
\end{aligned} \tag{23}$$



Pose diagram



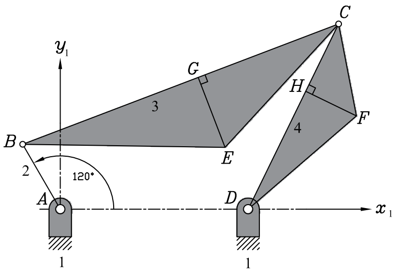
Velocity polygon

- Substituting the vector elements of $\vec{R}_{C/B}$, $\vec{R}_{C/D}$, and $\vec{V}_B$ into Equation (23) yields the following vector equation after converting units of inches to feet

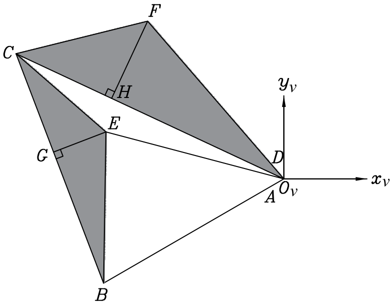
$$\begin{bmatrix} -27.21 \text{ ft/s} \\ -15.74 \text{ ft/s} \end{bmatrix} + \begin{bmatrix} -0.54 \text{ ft} \\ 1.40 \text{ ft} \end{bmatrix} \omega_3 = \begin{bmatrix} -0.82 \text{ ft} \\ 0.40 \text{ ft} \end{bmatrix} \omega_4.$$

- Solving the two vector element equations simultaneously yields the angular velocities of links 3 and 4

$$\omega_3 = 25.28 \text{ rad/s (ccw)}, \qquad \omega_4 = 49.51 \text{ rad/s (ccw)}.$$



Pose diagram



Velocity polygon

- The velocities of points E and F can be determined using the relative velocity equations choosing points B and D as the relative points:

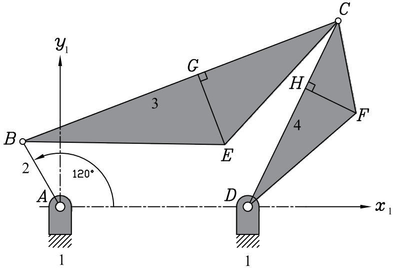
$$\vec{V}_E = \vec{V}_B + \vec{V}_{E/B} = \vec{V}_B + \vec{\omega}_3 \times \vec{R}_{E/B}, \text{ and}$$

$$\vec{V}_F = \vec{V}_D + \vec{V}_{F/D} = \cancelto{0}{\vec{V}_D} + \vec{\omega}_4 \times \vec{R}_{F/D}.$$

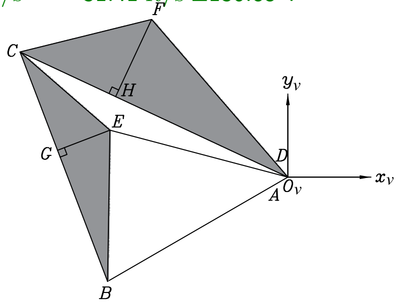
- This yields the desired velocities after units of inches are converted to feet:

$$\vec{V}_E = -26.86\hat{i} + 7.07\hat{j} \text{ ft/s} = 27.77 \text{ ft/s} \angle 165.25^\circ, \text{ and}$$

$$\vec{V}_F = -20.55\hat{i} + 23.76\hat{j} \text{ ft/s} = 31.41 \text{ ft/s} \angle 130.85^\circ.$$



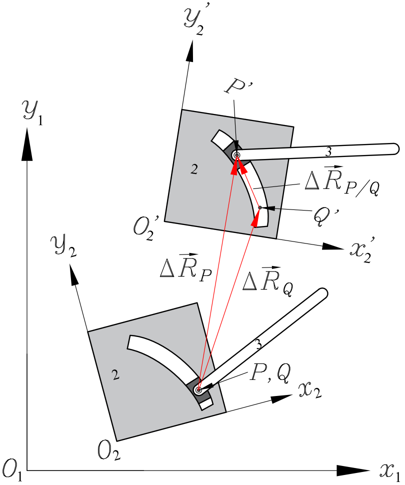
Pose diagram



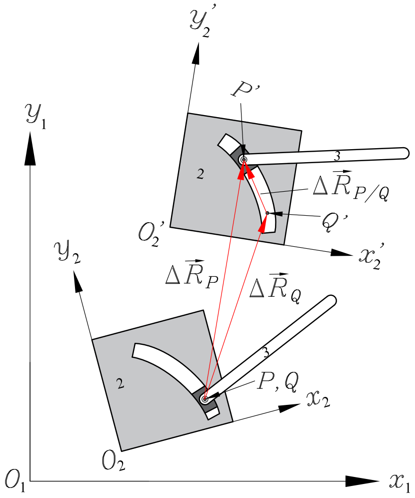
Velocity polygon

Relative (Apparent) Linear Velocity of a Point

- Often it is easier to work with the velocity of a point constrained to move on a path on a moving body relative to another point which remains stationary on the moving link rather than the absolute velocity of the point.
- Consider the schematic illustration.
- Point Q moves with the slotted plate (link 2) but does not move with respect to the plate.
- Points P and Q are initially coincident.
- Point P is on link 3, but is constrained to move in the slot on link 2.



- During a short time interval Δt
  - $x_2 y_2$ moves to $x_2' y_2'$
  - Pin P (attached to Link 3) moves relative to Link 2 along a known constrained path in Link 2 (the slot).
- Q is initially coincident with P, but attached to Link 2.
- $\Delta \vec{R}_P$ is the absolute displacement of Pin P as observed by a stationary observer (SO) attached to $x_1 y_1$.
- $\Delta \vec{R}_Q$ is the absolute displacement of the coincident point Q as observed by the SO.
- $\Delta \vec{R}_{P/Q}$ is the relative (apparent) displacement of Pin P as observed by a moving observer (MO) attached to $x_2 y_2$.



By the relative displacement equation

$$\Delta\vec{R}_P = \Delta\vec{R}_Q + \Delta\vec{R}_{P/Q}$$

Dividing by Δt and taking the limit as $\Delta t \to 0$, results in the relative velocity of P with respect to Link 2

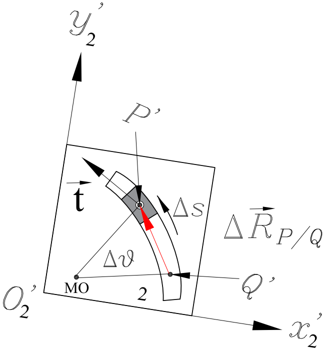
$$\vec{V}_P = \vec{V}_Q + \vec{V}_{P/Q}$$

Note that for the MO in x_2y_2, during the small Δt, i.e., small $\Delta\theta$:

$$\vec{V}_{P/Q} = \lim_{\Delta t\to 0}\frac{\Delta\vec{R}_{P/Q}}{\Delta t}$$

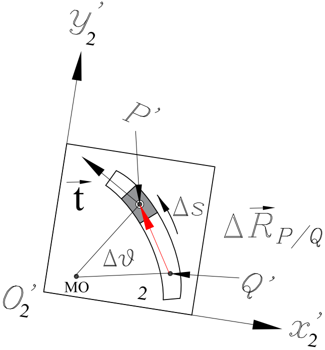
$$= \lim_{\Delta t\to 0}\frac{\Delta\vec{R}_{P/Q}}{\Delta s}\cdot\frac{\Delta s}{\Delta t}$$

$$= \frac{\mathrm{d}\vec{R}_{P/Q}}{\mathrm{d}s}\cdot\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\vec{t}$$



for $\vec{V}_{P/Q}$:

- The direction of the relative velocity is always known and it is tangent to the known path of P in Link 2.
- The magnitude of the relative velocity is equal to the relative speed with which Point P moves along the path.



Relative (Apparent) Angular Velocity

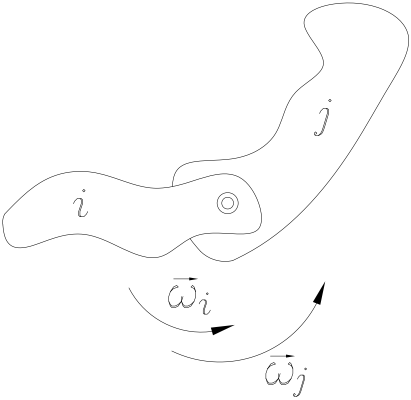
The two bodies, i and j, have angular displacement relative to each other:

- $\vec{\omega}_i$ is the angular velocity of Body i;
- $\vec{\omega}_j$ is the angular velocity of Body j; and
- $\vec{\omega}_{i/j}$ is the apparent angular velocity (the angular velocity of Body i with respect to Body j).

To an observer attached to Body j and rotating with it

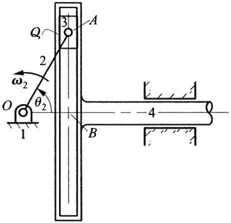
$$\vec{\omega}_{i/j} = \vec{\omega}_i - \vec{\omega}_j \tag{24}$$

$$\Rightarrow \vec{\omega}_i = \vec{\omega}_j + \vec{\omega}_{i/j} \tag{25}$$



Example 6.2

Given: The scotch yoke mechanism as shown where ω_2 is known
Determine: A complete velocity analysis of the mechanism



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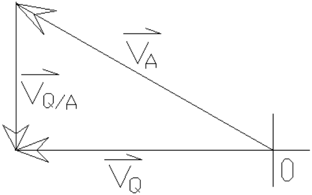
Solution:

$$\vec{V_A} = \underbrace{\vec{V_O}}_{0} + \vec{V_{A/O}} = \vec{\omega_2} \times \vec{R_{A/O}}$$

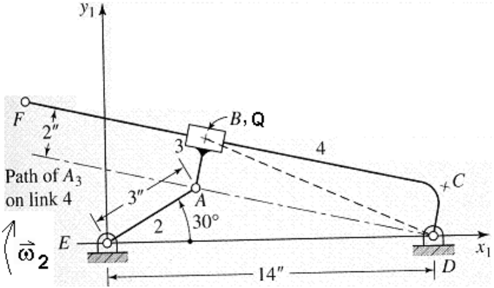
$$\therefore \quad |\vec{V_A}| = \omega_2 R_{A/O} \quad \perp \text{ to OA}$$

$$\overset{o\surd}{\vec{V_Q}} = \overset{\surd\surd}{\vec{V_A}} + \overset{o\surd}{\vec{V_{Q/A}}}$$



Example 6.3

An inversion of the slider-crank mechanism is shown. Link 2 is driven at an angular velocity of 36 rad/s cw. It is required to find $\vec{\omega_4}$. and the velocity of the slider along link 4.



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- *B* is defined to be a point attached to Link 3.
- *Q* is defined to be a point that is coincident with B but on Link 4.

- Velocity difference between A and E

$$\vec{V}_A = \vec{V}_E + \vec{V}_{A/E} = \vec{\omega}_2 \times \vec{R}_{A/E} = (36 \text{ rad/s})(3/12 \text{ ft}) = 9 \text{ ft/s} \perp \vec{R}_{A/E}$$

- Apparent velocity between B and Q

$$\vec{V}_B = \vec{V}_Q + \vec{V}_{B/Q} = \underbrace{\vec{V}_D}_{0} + \vec{V}_{Q/D} + \vec{V}_{B/Q}$$

- $\vec{V}_B$ can also be written using the velocity difference between B and A

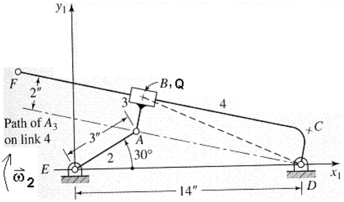
$$\vec{V}_B = \vec{V}_A + \vec{V}_{B/A} = \underbrace{\vec{V}_E}_{0} + \vec{V}_{A/E} + \vec{V}_{B/A}$$

- Combining the two expressions for $\vec{V}_B$:

$$\overset{\surd\surd}{\vec{V}_{A/E}} + \overset{o\surd}{\vec{V}_{B/A}} = \overset{o\surd}{\vec{V}_{Q/D}} + \overset{o\surd}{\vec{V}_{B/Q}}$$

- $\vec{V}_{B/A}$ and $\vec{V}_{B/Q}$ have the same direction, therefore

$$\overset{\surd\surd}{\vec{V}_{A/E}} + (\overset{o\surd}{\vec{V}_{B/A}} - \overset{o\surd}{\vec{V}_{B/Q}}) = \overset{o\surd}{\vec{V}_{Q/D}}$$



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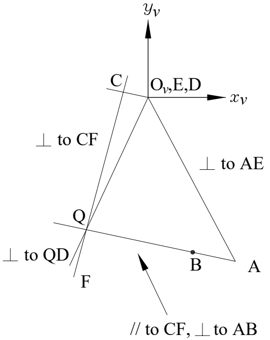
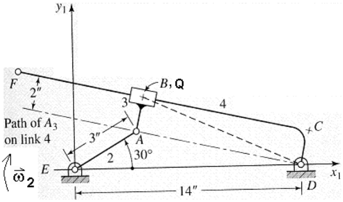
- Solving the equation graphically results in

$$|\vec{V}_{Q/D}| = 7.3 \text{ ft/s}$$

$$\omega_4 = \frac{V_{Q/D}}{R_{Q/D}} = \frac{7.3}{11.6/12} = 7.55 \text{ rad/s ccw}$$

- To complete the velocity image, note that since Link 3 is always $\perp$ to Link 4, $\vec{\omega}_3 = \vec{\omega}_4 = 7.55 \text{ rad/s ccw}$. Then, $V_{B/A} = (7.55 \text{ rad/s})(2/12 \text{ ft}) = 1.26 \text{ ft/s}$. This information can then be used to locate Point B.





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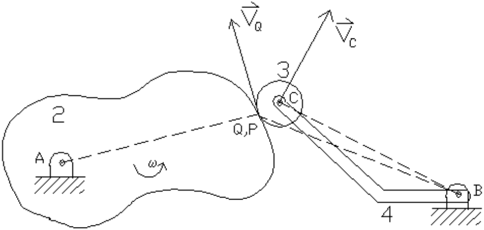
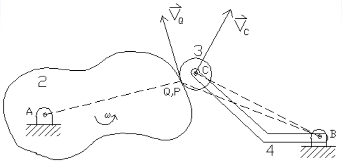
Example 6.4

Direct Contact Problem
Consider the cam and follower arrangement shown.



- P is defined as a point on Link 3 and located at the point of contact.
- Q is defined as a point on Link 2 (the cam) coincident with Point P.



- The apparent velocity expression is written between points P and Q

$$\vec{V}_P = \vec{V}_Q + \vec{V}_{P/Q}$$

- Rearrange and split into components normal and tangential to the line of sliding contact

$$\vec{V}_{P/Q} = \vec{V}_P - \vec{V}_Q \Longrightarrow \begin{array}{rcl} \left(\vec{V}_{P/Q}\right)_n & = & \left(\vec{V}_P\right)_n - \left(\vec{V}_Q\right)_n \\ \left(\vec{V}_{P/Q}\right)_t & = & \left(\vec{V}_P\right)_t - \left(\vec{V}_Q\right)_t \end{array}$$

where

$$\vec{V}_{P/Q} : \begin{array}{rcl} \left(\vec{V}_{P/Q}\right)_n & = & 0 \text{ (to avoid separation/interference of links)} \\ \left(\vec{V}_{P/Q}\right)_t & = & \text{sliding velocity (also zero if no sliding)} \end{array}$$

- If $\left(\vec{V}_{P/Q}\right)_t = 0$, then there is no slipping (i.e., minimal wear). This leads to rolling contact. In this case, the *rolling contact condition* can be expressed

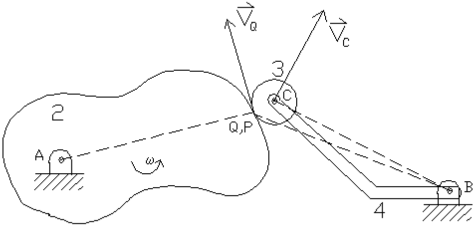
$$\vec{V}_{P/Q} = \vec{V}_P - \vec{V}_Q = 0 \Rightarrow \overset{\surd\surd}{\vec{V}_P} = \overset{\surd\surd}{\vec{V}_Q} = \overset{o\surd}{\vec{V}_C} + \overset{o\surd}{\vec{V}_{P/C}}$$

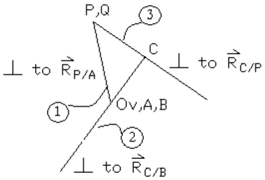
$$\vec{V}_{P/C} \Rightarrow \omega_3$$

And on link 4 by:

$$\vec{V}_C = \cancelto{0}{\vec{V}_B} + \vec{V}_{C/B} = \vec{V}_{C/B}$$

$$\vec{V}_{C/B} \Rightarrow \omega_4$$





Summary of Velocity Equations

Points are	*Coincident*	*Separated*
On same body	*Trivial Case:* $\vec{V}_P = \vec{V}_Q$	*Velocity Difference:* $\vec{V}_P = \vec{V}_Q + \vec{V}_{P/Q}$ $\vec{V}_{P/Q} = \vec{\omega}_j x \vec{R}_{P/Q}$
On different bodies	*Apparent Velocity:* $\vec{V}_{P_i} = \vec{V}_{P_j} + \vec{V}_{P_{i/j}}$ where path $P_{i/j}$ is known *Pure Rolling Contact* $\vec{V}_{P_i} = \vec{V}_{P_j}$ and $\vec{V}_{P_{i/j}} = 0$	*Too general: use two steps.*

Instantaneous Centres of Velocity

Definitions

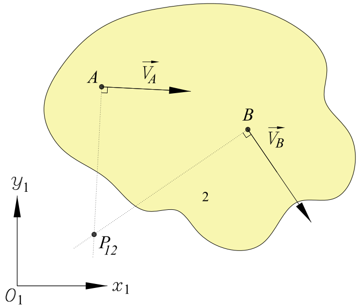
Instantaneous centres of velocity (ICV) are defined as:

1. The instantaneous coincident location of two different points located on two different rigid bodies in general relative plane motion where the **absolute velocities** of the two different points are instantaneously identical.
2. It can also be the instantaneously coincident location of a pair of points on two different rigid bodies where the **relative velocity** of the two points is instantaneously zero.
3. A point on one rigid body about which some other rigid body is rotating at the instant being considered.

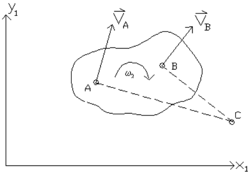
- The property of being an ICV is only valid at the instant in time being considered.
- A new pair of coincident points becomes the ICV at the next instant.
- The ICVs lie on well defined curves, called *centrodes*, as the motions occur over time.

Locating an ICV for a Moving and Fixed Rigid Body Pair

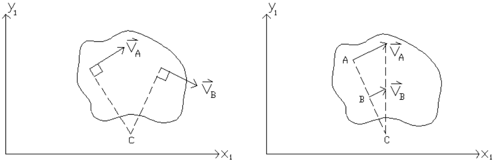
- Consider rigid body 2 at the instant shown moving relative to fixed rigid body 1 where points A and B have velocities $\vec{V}_A$ and $\vec{V}_B$.
- Construct a line segment from the tail of $\vec{V}_A$ perpendicular to the direction of $\vec{V}_A$.
- Construct another line segment from the tail of $\vec{V}_B$ perpendicular to the direction of $\vec{V}_B$.
- The intersection of these two line segments, labelled P_{12}, is the ICV between link 2 and the fixed rigid body, link 1.
- The order of the indices is not important: $P_{12} = P_{21}$.



ICV for a Moving and Fixed Rigid Body Pair



ICV for a moving body



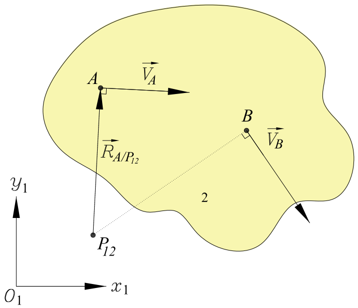
Determination of ICVs

Computing the Location of an ICV

- If the angular velocity of link 2 were known relative to link 1, then P_{12} is the instantaneous centre of ω_2, and the velocity of every point on link 2 can be computed.
- We can abstractly consider point P_{12} to be a coincident point located on both links 1 and 2.
- The relative position vector from P_{12} to any point on link 2 expressed in the coordinate system attached to link 1, let's say point A, is given by

$$\vec{R}_{A/P_{12}} = \frac{\vec{\omega}_2 \times \vec{V}_A}{\omega_2^2}$$

- Of course, we need to know the velocity of point A, $\vec{V}_A$, and the angular velocity of link 2, $\vec{\omega}_2$.

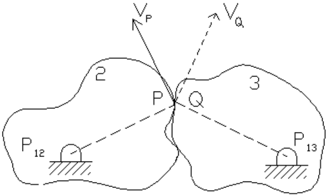


Aronhold-Kennedy Theorem of Three Centres

- ICVs are labelled P_{ij} where i and j are the labels of the two links for which the particular ICV applies.
- $P_{ij} \equiv P_{ji}$
- The number of ICVs N that exist for an n-link mechanism is given by

$$N = \frac{n(n-1)}{2} \tag{26}$$

- *The Aronhold-Kennedy theorem* states that the three ICVs shared by three rigid bodies in relative motion to one another all lie on the same straight line. This is proved by contradiction; any two coincident points P and Q that are not on the line $P_{12}P_{13}$ cannot be an ICV because $\vec{V}_P \neq \vec{V}_Q$.



- The Aronhold-Kennedy theorem was discovered independently by Siegfried Heinrich Aronhold (a German mathematician) in 1872 and by Alexander Blackie William Kennedy (an English civil and electrical engineer) in 1886.
- It is often called the Kennedy theorem, but Kennedy was second to the show, so that is just not right!

- The Kennedy circle method, combined with Aronhold-Kennedy's theorem, is used to determine all possible ICVs in a mechanism.
- R-joint centres are ICVs that can always be identified by inspection.
- These are called *primary* ICVs.
- ICVs that are determined by the Aronhold-Kennedy theorem are called *secondary* ICVs.
- For a planar four link mechanism, links 1, 2, 3, 4 can be grouped in four distinct sets of three, each set defining a distinct line of three ICVs

$$\left.\begin{array}{c} 1+2+3 \\ 1+2+4 \\ 1+3+4 \\ 2+3+4 \end{array}\right\} \Rightarrow \left\{\begin{array}{lll} P_{12}, & P_{23}, & P_{13} \\ P_{12}, & P_{24}, & P_{14} \\ P_{13}, & P_{34}, & P_{14} \\ P_{23}, & P_{34}, & P_{24} \end{array}\right.$$

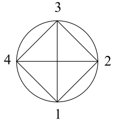
Example 6.5

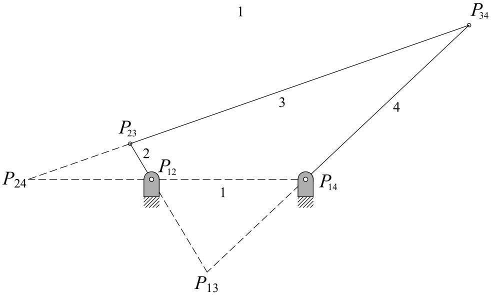
Consider the four-bar linkage shown.
Locate all ICVs.

Solution:

$$N = \frac{4(4-1)}{2} = 6$$

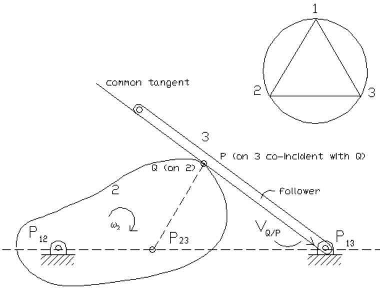
Kennedy Circle





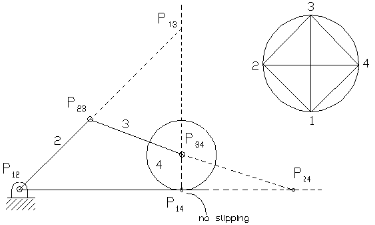
Examples of Finding ICVs

Direct Contact with Sliding



- The ICV must lie along a line perpendicular to the sliding direction
- Apply Kennedy's theorem

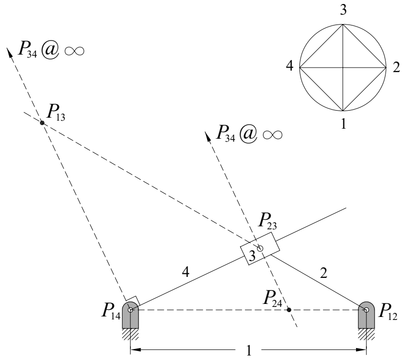
Rolling Contact



- Point of rolling without slipping is an ICV

Sliding Pair (P-Pair)

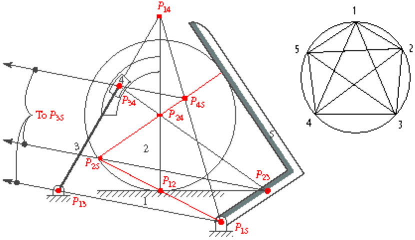
- $N = \frac{4(4-1)}{2} = 6$
- The R-pair ICVs are easily located
- P_{24} is on a line $\perp$ to the instantaneous direction of sliding containing P_{23}
- P_{13} is where the centreline of link 2 intersects a line $\perp$ to the instantaneous direction of sliding containing P_{14}
- P_{34} is on a line $\perp$ to the instantaneous direction of sliding with $|\vec{R}| = \infty$.
- Note that P_{34} is the intersection of the parallel lines containing P_{13} and P_{14}, as well as P_{23} and P_{24}



Example 6.6

Locate all the instant centres of the mechanism shown assuming rolling contact between links 1 and 2.

- The instant centres P_{13}, P_{34} and P_{15}, being pinned joints, are located by inspection.
- P_{12} is located at the point of rolling contact.
- The instant centre P_{24} may be noticed by the fact that this is the centre of the apparent rotation between links 2 and 4
- P_{23} and P_{14} can be found using the Aronhold-Kennedy theorem.
- One line for the instant centre P_{25} comes from noticing the direction of slipping between links 2 and 5.
- The other comes from the line $P_{12}P_{15}$.
- After these, all other instant centres can be found by repeated application the Aronhold-Kennedy theorem.



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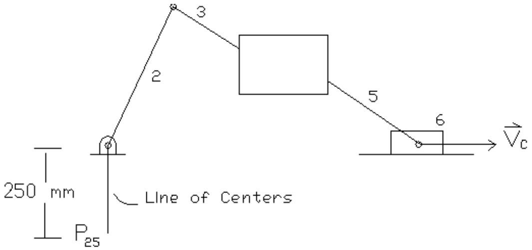
Velocity Analysis using ICVs Algorithm Steps

1. Identify the three link numbers that correspond to the given velocity, the unknown velocity of interest, and the reference link (usually the fixed ground link since absolute velocities are usually specified and requested).
2. Identify the three ICVs associated with the links identified in the previous step.
3. Determine the velocity of the common ICV by considering it as a point on the link whose velocity is given.
4. Using the velocity of the common ICV just determined, consider the common ICV as a point on the link with the unknown velocity and compute the velocity of the desired point.

Example 6.7

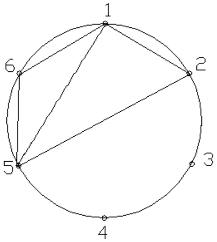
Slider mechanism with hidden links

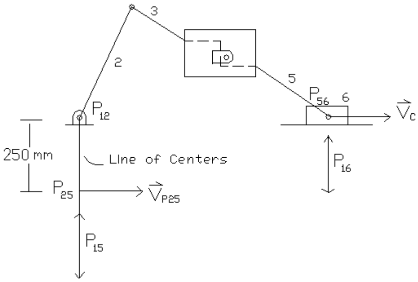
Consider the mechanism shown that includes an enclosed housing with unknown internal configuration. At the instant shown, the location of ICV P_{25} is known as well as $\vec{V}_C = 10$ m/s. Determine the angular velocity $\vec{\omega}_2$ using instant centres.



Solution:

- Since we are given $\vec{V}_{C5/1}$ and we are looking for $\vec{\omega}_{2/1}$, we need ICVs associated with links 1, 2, and 5; P_{12}, P_{15} and P_{25}.
- P_{12} is easily identified; P_{25} is given; therefore we need P_{15}. Consider using the circle method as shown.
- From the diagram it is apparent that we could obtain P_{15} if we knew P_{16} and P_{56}. P_{56} is known from inspection and P_{16} must be located at infinity $\perp$ to $\vec{V}_C$ due to sliding contact between links 1 and 6.
- Aronhold-Kennedy's theorem locates P_{15} as being at the intersection of lines $P_{12}P_{25}$ and $P_{16}P_{56}$. $\Longrightarrow$ P_{15} is located at infinity.



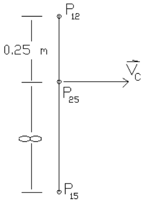


- Since P_{15} is at infinity, link 5 must be in pure translation (i.e., $\omega_5 = 0$). Therefore, $\vec{V}_{P25} = \vec{V}_{P56} = \vec{V}_C = 10$ m/s.
- Next, consider P_{25} as a point on link 2:

$$\vec{V}_{P_{25}} = \vec{\omega}_2 \times \vec{R}_{P_{25}/P_{12}}$$

Therefore

$$\omega_2 = \frac{V_{P25}}{R_{P25/P12}} = \frac{10 \text{ m/s}}{0.25} = 40 \text{ rad/s ccw}$$



The Angular Velocity Ratio Theorem

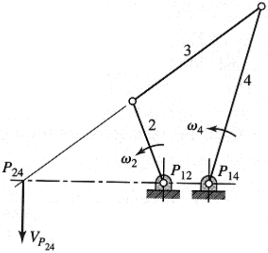
In the figure below, P_{24} is the instant centre common to links 2 and 4. Its absolute velocity $\vec{V}_{P_{24}}$ is the same whether P_{24} is considered as a point of link 2 or of link 4. Considering it each way, we can write

$$\vec{V}_{P_{24}} = \vec{\omega}_{2/1} \times \vec{R}_{P_{24}/P_{12}} = \vec{\omega}_{4/1} \times \vec{R}_{P_{24}/P_{14}} \tag{27}$$

Considering the magnitudes only, we can rearrange the equation to read

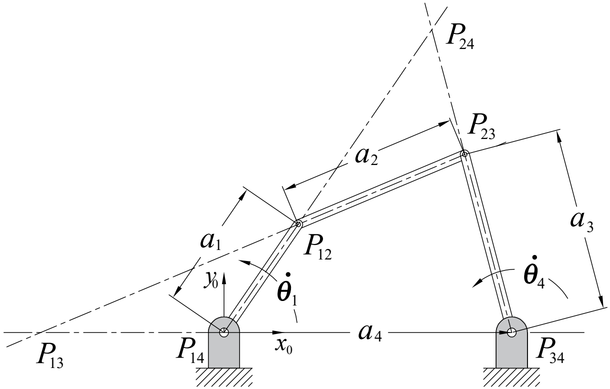
$$\frac{\vec{\omega}_{4/1}}{\vec{\omega}_{2/1}} = \frac{\vec{R}_{P_{24}/P_{12}}}{\vec{R}_{P_{24}/P_{14}}} \tag{28}$$

The angular velocity ratio theorem states that the angular velocity ratio of any two bodies in planar motion with respect to a third body is inversely proportional to the distances between the associated ICVs along the line of centres.



Generalised Velocity Ratio Theorem

- Nouns concerning joint centres are arbitrary.
- Let's restate and generalise the velocity ratio theorem, also known as Freudenstein's theorem, in terms of the algebraic IO equations.
- Because the link directed line segments (length vectors) are labelled a_1, a_2, a_3, and a_4, the ICVs must also be relabelled as in the figure.



Generalised Velocity Ratio Theorem

Recall the six algebraic IO equations for a planar 4R linkage:

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{29}$$

where

$$\begin{aligned} A &= A_1A_2 = (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4),\\ B &= B_1B_2 = (a_1 - a_2 + a_3 + a_4)(a_1 + a_2 + a_3 + a_4),\\ C &= C_1C_2 = (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4),\\ D &= D_1D_2 = (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4),\\ v_1 &= \tan\tfrac{\theta_1}{2}, \quad v_4 = \tan\tfrac{\theta_4}{2}. \end{aligned}$$

$$A_1B_1v_1^2v_2^2 + A_2B_2v_1^2 + C_1D_2v_2^2 + 8a_2a_4v_1v_2 + C_2D_1 = 0, \tag{30}$$

$$A_2B_1v_1^2v_3^2 + A_1B_2v_1^2 + C_1D_1v_3^2 + C_2D_2 = 0, \tag{31}$$

$$B_1C_1v_2^2v_3^2 + A_1D_2v_2^2 + A_2D_1v_3^2 - 8a_1a_3v_2v_3 + B_2C_2 = 0, \tag{32}$$

$$A_1C_1v_2^2v_4^2 + B_1D_2v_2^2 + A_2C_2v_4^2 + B_2D_1 = 0, \tag{33}$$

$$A_2C_1v_3^2v_4^2 + B_1D_1v_3^2 + A_1C_2v_4^2 - 8a_2a_4v_3v_4 + B_2D_2 = 0. \tag{34}$$

Generalised Velocity Ratio Theorem

The equation relating the time rates of change of the joint angle parameters v_1 and v_4 can be determined as the first time derivative of Equation (29):

$$\left((Av_4^2 + B)v_1 - 4a_1a_3v_4\right)\frac{d}{dt}v_1 + \left((Av_1^2 + C)v_4 - 4a_1a_3v_1\right)\frac{d}{dt}v_4. \qquad (35)$$

Because Equation (35) equates to zero, the velocity parameter ratio is

$$\frac{\frac{d}{dt}v_4}{\frac{d}{dt}v_1} = \frac{\dot{v}_4}{\dot{v}_1} = -\frac{(Av_4^2 + B)v_1 - 4a_1a_3v_4}{(Av_1^2 + C)v_4 - 4a_1a_3v_1}. \qquad (36)$$

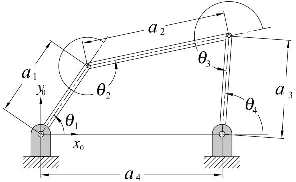
It is **VERY IMPORTANT** to note the negative sign!

It is also important to note that for the i^{th} link, $\dot{v}_i \neq \dot{\theta}_i$ since $v_i = \tan(\theta_i/2)$

$$\dot{v}_i = \frac{\dot{\theta}_i(1 + v_i^2)}{2},$$

and that

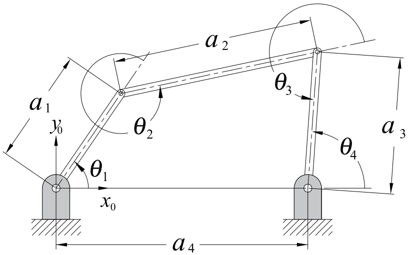
$$\dot{\theta}_i = \frac{2\dot{v}_i}{(1 + v_i^2)}.$$



Generalised Velocity Ratio Theorem

Hence, the reciprocal of the mechanical advantage is

$$\frac{\dot{\theta}_4}{\dot{\theta}_1} = -\frac{((Av_4^2+B)v_1-4a_1a_3v_4)(1+v_1^2)}{((Av_1^2+C)v_4-4a_1a_3v_1)(1+v_4^2)}. \tag{37}$$



- This is a ratio of relative angular velocities, but in this case $\dot{\theta}_1$ and $\dot{\theta}_4$ are both measured relative to stationary coordinate systems. This is not true for the other five ratios!
- We will denote the **relative** angular velocities with the symbol $\dot{\theta}$ and the **absolute** angular velocities with the symbol ω.

Generalised Velocity Ratio Theorem

The five remaining relative angular velocity ratios are

$$\frac{\dot\theta_2}{\dot\theta_1} = -\frac{\left((A_1B_1v_2^2 + A_2B_2)v_1 + 4a_2a_4v_2\right)(1+v_1^2)}{\left((A_1B_1v_1^2 + C_1D_2)v_2 + 4a_2a_4v_1\right)(1+v_2^2)}, \tag{38}$$

$$\frac{\dot\theta_3}{\dot\theta_1} = -\frac{\left((A_2B_1v_3^2 + A_1B_2)v_1\right)(1+v_1^2)}{\left((A_2B_1v_1^2 + C_1D_1)v_3\right)(1+v_3^2)}, \tag{39}$$

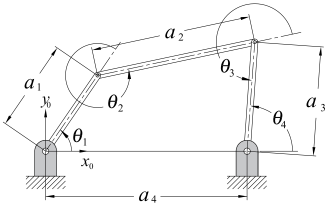
$$\frac{\dot\theta_3}{\dot\theta_2} = -\frac{\left((B_1C_1v_3^2 + A_1D_2)v_2 - 4a_2a_4v_3\right)(1+v_2^2)}{\left((B_1C_1v_2^2 + A_2D_1)v_3 - 4a_2a_4v_2\right)(1+v_3^2)}, \tag{40}$$

$$\frac{\dot\theta_4}{\dot\theta_2} = -\frac{\left((A_1C_1v_4^2 + B_1D_2)v_2\right)(1+v_2^2)}{\left((A_1C_1v_2^2 + A_2C_2)v_4\right)(1+v_4^2)}, \tag{41}$$

$$\frac{\dot\theta_4}{\dot\theta_3} = -\frac{\left((A_2C_1v_4^2 + B_1D_1)v_3 - 4a_2a_4v_4\right)(1+v_3^2)}{\left((A_2C_1v_3^2 + A_1C_2)v_4 - 4a_2a_4v_3\right)(1+v_4^2)}. \tag{42}$$

Example 6.8

Consider a planar 4R linkage similar to the one in the figure whose configuration is



Parameter	Dimension	Parameter	Dimension
a_1	5	θ_1	45°
a_2	6	θ_2	308.0304°
a_3	8	θ_3	207.5141°
a_4	2	θ_4	20.5445°

Determine the six possible angular velocity ratios

Example 6.8 Continued

We transform the joint angles to angle parameters using $v_i = \tan\theta_i/2$ and substituting the appropriate angle parameters and link lengths into Equations (37) to (42) and evaluating we obtain all six angular velocity ratios:

$$\frac{\dot\theta_4}{\dot\theta_1} = 1.0657; \qquad \frac{\dot\theta_3}{\dot\theta_2} = -1.2595;$$

$$\frac{\dot\theta_2}{\dot\theta_1} = -0.2532; \qquad \frac{\dot\theta_4}{\dot\theta_2} = -4.2085;$$

$$\frac{\dot\theta_3}{\dot\theta_1} = 0.3189; \qquad \frac{\dot\theta_4}{\dot\theta_3} = 3.3419.$$

If any of the $\dot\theta_i$ is specified it is a simple matter to determine the three other relative angular velocities using the ratios!

Example 6.8 Continued

- We must always remember that the $\dot{\theta}_i$ are all relative to some other $\dot{\theta}_j$, and they are not absolute angular velocities.
- If, for example, we specify $\dot{\theta}_1 = 10$ rads/s, and

$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = -0.2532,$$

then the angular velocity of $\dot{\theta}_2 =$ -25.32 rads/s is relative to $\dot{\theta}_1$.

- The absolute angular velocity of a_2 is $\omega_2 =$ -15.32 rads/s.
- The difference, in this case, is $\dot{\theta}_2 + \dot{\theta}_1$.
- Only $\dot{\theta}_1$ and $\dot{\theta}_4$ can be thought of as absolute angular velocities since they are measured relative to a non-moving coordinate system.

Example 6.8 Continued

- We must always remember that the $\dot{\theta}_i$ are all relative to some other $\dot{\theta}_j$, and they are not absolute angular velocities.

- If, for example, we specify $\dot{\theta}_1 = 10$ rads/s, and

$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = -0.2532,$$

  then the angular velocity of $\dot{\theta}_2 =$ -25.32 rads/s is relative to $\dot{\theta}_1$.

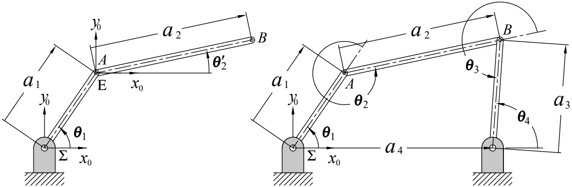
- The absolute angular velocity of a_2 is $\omega_2 =$ -15.32 rads/s.

- The difference, in this case, is $\dot{\theta}_2 + \dot{\theta}_1$.

- Only $\dot{\theta}_1$ and $\dot{\theta}_4$ can be thought of as absolute angular velocities since they are measured relative to a non-moving coordinate system.

Relatively Translating Coordinate System

- Consider the open kinematic chain on the left in the image.



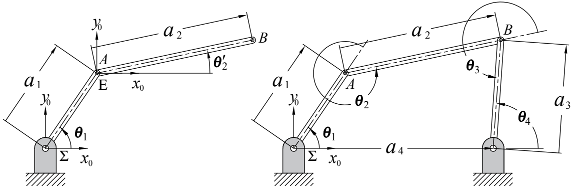
- The input link, a_1, is rotating with an absolute angular velocity of ω_1.
- To determine the absolute linear velocity of Point B on the coupler link, a_2, we attach a moving coordinate system to point A, and we consider point A to be located on the coupler.
- Point A is located on the centre of the joint connecting a_1 and a_2, so it can be considered to be located on either link.
- Since the moving coordinate system translates with a_2, but does not rotate, we can express the absolute linear velocity of Point B as

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega}_2 \times \vec{a}_2,$$

where $\vec{v}_A$ and $\vec{v}_B$ are absolute linear velocities and $\vec{\omega}_2$ is the absolute angular velocity of $\vec{a}_2$.

Relatively Translating and Rotating Coordinate System

- Consider the closed kinematic chain on the right in the image.



- The input link, a_1, is rotating with an absolute angular velocity of $\omega_1 = \dot{\theta}_1$.
- To determine the absolute linear velocity of Point B on the coupler link, a_2, we attach a moving coordinate system to point A, and we consider point A to now be located on the input link.
- Now, the moving coordinate system translates and rotates with a_1, we can express the absolute linear velocity of Point B as

$$\vec{v}_B \quad = \quad \vec{v}_A + \vec{\Omega}_1 \times \vec{a}_2 + \vec{\dot{\theta}}_2 \times \vec{a}_2,$$

where $\vec{v}_A$ and $\vec{v}_B$ are absolute linear velocities and $\vec{\Omega}_1$ is the absolute angular velocity the moving coordinate system, which is $\vec{\omega}_1$ of $\vec{a}_1$, and $\vec{\dot{\theta}}_2$ is the angular velocity of $\vec{a}_2$ measured in the rotating coordinate system.

Computing Maximum Angular Velocities and Critical Input Angles

- Now we will compute the critical values of the input angle required for the maximum values of output angular velocities.

- This method works for any of the six IO angular velocity equations, but we will illustrate the method using the v_1-v_4 equation listed as Equation (29). However, the equations become quite large so some computational software, like Maple, is needed.

- The method will be laid out algorithmically.

1. Convert the two variable angle parameters v_1 and v_4 to angles as $v_i = \tan \theta_i/2$.

2. Take the first time derivative of the resulting equation.

3. You now have an equation for $\dot{\theta}_4$ in terms of $\dot{\theta}_1$.

Computing Maximum Angular Velocities and Critical Input Angles

4. If values for a_1, a_2, a_3, and a_4 are specified and the input angular velocity is a constant specified value, we need to determine the critical values $\theta_{1_{\mathrm{crit}}}$ that result in $\dot{\theta}_{4_{\mathrm{max/min}}}$, so we need to eliminate θ_4 from both the position and angular velocity IO equations.

   4.1 Solve the IO equation for θ_4 and substitute the result into the angular velocity IO equation.

   4.2 Since the value for $\dot{\theta}_1$ is specified, the resulting equation expresses $\dot{\theta}_4$ in terms of θ_1.

   4.3 Solve the equation obtained in Step 4.2 for $\dot{\theta}_4$, giving $\dot{\theta}_4 = f(\theta_1)$.

   4.4 Take the derivative of this angular velocity equation determined in Step 4.3 with respect to θ_1.

   4.5 The values of $\theta_{1_{\mathrm{crit}}}$ that determine the extreme values of $\dot{\theta}_{4_{\mathrm{max/min}}}$ are those that make the derivative from the previous step equal zero.

Example 6.9

- Consider a planar 4R mechanisms where the following has been specified

$$\boxed{\begin{aligned} a_1 &= 2 \\ a_2 &= 6 \\ a_3 &= 7 \\ a_4 &= 5 \\ \dot{\theta}_1 &= 10 \text{ rad/sec, constant} \end{aligned}}$$

Determine $\theta_{1_{\text{crit}}}$ and the extreme values of $\dot{\theta}_4$ in both assembly modes of the mechanism.

SOLUTION

1. The specified link lengths mean that the mechanism is a Grashof crank-rocker.
2. When the substitution $v_i = \tan \theta_i/2$ has been made in Equation (29) the resulting θ_1-θ_4 equation is

$$\left(-36\left(\tan\frac{\theta_4}{2}\right)^2+160\right)\left(\tan\frac{\theta_1}{2}\right)^2-\left(112\tan\frac{\theta_1}{2}\tan\frac{\theta_4}{2}\right)-20\left(\tan\frac{\theta_4}{2}\right)^2+64=0. \quad (43)$$

Example 6.9 Continued

3. Solve Equation (43) for θ_4. You will obtain two values since the equation is of degree 2 in θ_4, one solution for each assembly mode.
4. Now substitute

$$\dot{v}_i = \frac{\dot{\theta}_i \left(1 + \left(\tan \frac{\theta_i}{2}\right)^2\right)}{2}$$

   into Equation (35).
5. We now have an equation that expresses $\dot{\theta}_4$ in terms of θ_1, θ_4, and $\dot{\theta}_1$. Solve this equation for $\dot{\theta}_4$.
6. Obtain two angular velocity equations, one for each assembly mode, by substituting the expression for θ_4, obtained in Step 3 above, into the angular velocity equation obtained in Step 4.
7. We now have two angular velocity expressions, one for each assembly mode.
8. When the specified value for $\dot{\theta}_1$ is substituted into each angular velocity equation we determine two equations for $\dot{\theta}_4$ in terms of θ_1.

Example 6.9 Continued

9. Symbolically solve each equation obtained in Step 8 for $\dot{\theta}_4$, which gives $\dot{\theta}_4 = f(\theta_1)$.

10. Due to this substitution technique, the angular velocity equations will contain many terms and become prohibitively large to manage by hand and a calculator, especially for the last and next steps.

    10.1 A symbolic computer algebra software such as Maple, or the MatLAB Symbolic Toolbox, is required to solve these equations.

    10.2 You will find several instructional videos on Brightspace to show you how to perform the symbolic computations in Maple.

11. To determine the critical values for θ_1 which result in maximum and minimum values for $\dot{\theta}_4$ in each assembly mode we must take the derivative of each angular velocity equation, which we have denoted as $\dot{\theta}_4 = f(\theta_1)$, and identify the values of $\theta_{1_{\mathrm{crit}}}$ cause the following to be true:

$$\frac{d\left(\dot{\theta}_4 = f(\theta_1)\right)}{d\theta_1} = 0. \tag{44}$$

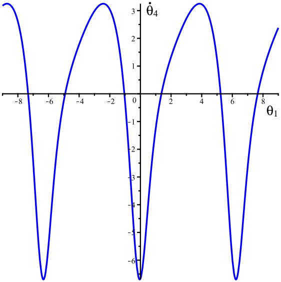
Example 6.9 Assembly Mode 1

12. For the specified link lengths and input angular velocity of $a_1 = 2$, $a_2 = 6$, $a_3 = 7$, $a_4 = 5$, and $\dot{\theta}_1 = 10$ rad/s, constant, substituted into the derivative obtained in Step 11 the equation is of degree 14 in θ_1.
13. Hence, Maple will reveal all 14 solutions for $\theta_{1_{\text{crit}}}$.
14. Shouldn't there be only two distinct real values for $\theta_{1_{\text{crit}}}$? Let's look at all the real solutions Maple gives us.
15. For the specified link lengths and input angular velocity we obtain the following four real critical values for θ_1 in Assembly Mode 1 and the corresponding maximum (most positive) and minimum (most negative) values for $\dot{\theta}_{4_{\text{max/min}}}$:

$$
\begin{aligned}
\theta_{1_{\text{crit}_1}} &= 357.3190^\circ, & \dot{\theta}_{4_{\text{min}}} &= -6.6959 \text{ rad/s} \\
\theta_{1_{\text{crit}_2}} &= 221.0362^\circ, & \dot{\theta}_{4_{\text{max}}} &= 3.2493 \text{ rad/s} \\
\theta_{1_{\text{crit}_3}} &= -2.6810^\circ, & \dot{\theta}_{4_{\text{min}}} &= -6.6959 \text{ rad/s} \\
\theta_{1_{\text{crit}_4}} &= -138.9638^\circ, & \dot{\theta}_{4_{\text{max}}} &= 3.2493 \text{ rad/s}
\end{aligned}
$$

Example 6.9 Assembly Mode 1

16. It is clear that two of the four real solutions are repeated. This is an artifact of the range selected for the input angle, and the fact that the $\dot{\theta}_4 = f(\theta_1)$ equation is periodic.



17. There are only two distinct real critical values for θ_1 and corresponding extreme values for $\dot{\theta}_4$ in Assembly Mode 1:

$$\begin{aligned} \theta_{1_{\mathrm{crit}_1}} &= 357.3190^\circ, & \dot{\theta}_{4_{\mathrm{min}}} &= \text{-6.6959 rad/s} \\ \theta_{1_{\mathrm{crit}_2}} &= 221.0362^\circ, & \dot{\theta}_{4_{\mathrm{max}}} &= \text{3.2493 rad/s} \end{aligned}$$

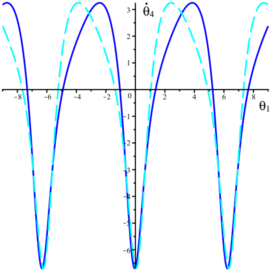
Example 6.9 Assembly Mode 2

18. For the specified link lengths and input angular velocity of $a_1 = 2$, $a_2 = 6$, $a_3 = 7$, $a_4 = 5$, and $\dot{\theta}_1 = 10$ rad/s, constant, we obtain the following four real critical values for θ_1 in Assembly Mode 2 and the corresponding maximum (most positive) and minimum (most negative) values for $\dot{\theta}_{4_{\mathrm{max/min}}}$:

$$\begin{array}{lclclcl}
\theta_{1_{\mathrm{crit}_1}} & = & 2.6810^\circ, & \dot{\theta}_{4_{\mathrm{min}}} & = & \text{-6.6959 rad/s} \\
\theta_{1_{\mathrm{crit}_2}} & = & 138.9638^\circ, & \dot{\theta}_{4_{\mathrm{max}}} & = & \text{3.2493 rad/s} \\
\theta_{1_{\mathrm{crit}_3}} & = & -357.3190^\circ, & \dot{\theta}_{4_{\mathrm{min}}} & = & \text{-6.6959 rad/s} \\
\theta_{1_{\mathrm{crit}_4}} & = & -221.0362^\circ, & \dot{\theta}_{4_{\mathrm{max}}} & = & \text{3.2493 rad/s}
\end{array}$$

Example 6.9 Assembly Mode 2

19. It is clear that two of the four real solutions are repeated. This is an artifact of the fact that the $\dot{\theta}_4 = f(\theta_1)$ equation is periodic.



20. We can see that the velocity profile curve for Assembly Mode 2, plotted with the dashed line-type, is reflected in the vertical $\dot{\theta}_4$-axis.

21. There are only two distinct real critical values for θ_1 and corresponding extreme values for $\dot{\theta}_4$ in Assembly Mode 2:

$$\begin{aligned} \theta_{1_{\mathrm{crit}_1}} &= 2.6810^\circ, & \dot{\theta}_{4_{\mathrm{min}}} &= \text{-6.6959 rad/s} \\ \theta_{1_{\mathrm{crit}_2}} &= 138.9638^\circ, & \dot{\theta}_{4_{\mathrm{max}}} &= \text{3.2493 rad/s} \end{aligned}$$