MAAE 3004
Dynamics of Machinery

Lecture Slide Set 7

Acceleration Analysis

Department of Mechanical and Aerospace Engineering
Carleton University

Outline

Linear Acceleration

Linear Acceleration

For change of velocity of point A:

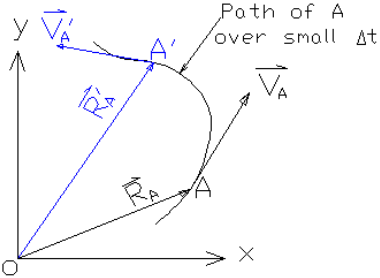
$$\triangle \vec{V}_A = \vec{V}_{A'} - \vec{V}_A$$

We define

$$\text{average acceleration} = \frac{\triangle \vec{V}_A}{\triangle t}$$

and, instantaneous acceleration or simply the acceleration as:

$$\vec{A} = \lim_{\triangle t \to 0} \frac{\triangle \vec{V}_A}{\triangle t} = \frac{d\vec{V}_A}{dt}$$



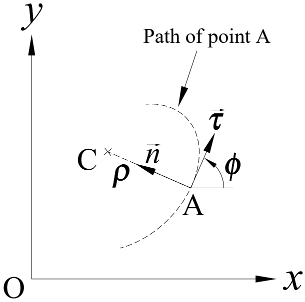
As with velocity:

-linear acceleration is defined at a point

-depends on the motion of the observer's coordinate system but not its position

-is absolute only if observed in an absolute i.e.,inertial coordinate system

Acceleration Components



s : distance traveled by point A along the path in time increment Δt

$\dot{s}$: instantaneous speed of point A along its path

$\vec{\tau}$: unit vector locally tangent to the path and directed toward increasing s

$\vec{n}$: unit vector normal to $\vec{\tau}$ and directed toward the instantaneous centre of rotation between point A and the body on which the path is defined

Then the velocity of point A is

$$\vec{V}_A = \dot{s}\vec{\tau} \tag{1}$$

From Equation (1)

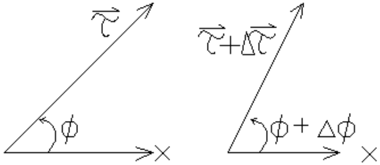
$$\vec{A}_A = \frac{d}{dt}\vec{V}_A = \frac{d}{dt}(\dot{s}\vec{\tau}) = \underbrace{\ddot{s}\vec{\tau}}_{Change\ \ in\ \ magnitude} + \underbrace{\dot{s}\frac{d\vec{\tau}}{dt}}_{Change\ \ in\ \ direction} \tag{2}$$

where,

$$\frac{d\vec{\tau}}{dt} = \frac{d\vec{\tau}}{d\phi}\frac{d\phi}{ds}\frac{ds}{dt} \tag{3}$$

Consider each derivative in the R.H.S. of Equation (3)

$$\frac{d\vec{\tau}}{dt} = \frac{d\vec{\tau}}{d\phi}\frac{d\phi}{ds}\frac{ds}{dt} \qquad (3)$$



- Over $\triangle$t the direction of $\vec{\tau}$ changes from $\vec{\tau}$ to $\vec{\tau}+\triangle\vec{\tau}$

Starting from a common origin we have:

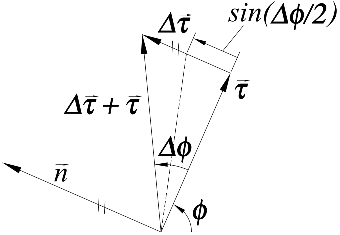
$$\mid \triangle\vec{\tau} \mid = 2(1)\sin\left(\frac{\triangle\phi}{2}\right)$$

$\triangle\vec{\tau}$ must be $\perp$ to $\vec{\tau}$,i.e., $\parallel$ to $\vec{n}$, therefore:

$$\triangle\vec{\tau} = 2(1)\sin\left(\frac{\triangle\phi}{2}\right)\vec{n}$$

Then for small $\triangle\phi$,

$$\frac{d\vec{\tau}}{d\phi} = \lim_{\triangle\phi\to 0}\frac{\triangle\vec{\tau}}{\triangle\phi} = \lim_{\triangle\phi\to 0}\frac{2\sin\left(\frac{\triangle\phi}{2}\right)\vec{n}}{\triangle\phi} = \vec{n}$$



- Next note that in Equation (3)

$\frac{d\phi}{ds} \equiv$ rate of change of the direction of the path with distance travelled $\equiv$ curvature of path $(\kappa) \equiv$ the reciprocal of the radius of curvature $= \frac{1}{\rho}$

$$\therefore \frac{d\phi}{ds} = \frac{1}{\rho}$$

- Also in Equation (3)

$$\frac{ds}{dt} = \dot{s}$$

Hence Equation (3) becomes

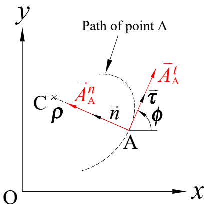
$$\frac{d\vec{\tau}}{dt} = \frac{d\vec{\tau}}{d\phi}\frac{d\phi}{ds}\frac{ds}{dt} = (\vec{n})(\dot{s})\left(\frac{1}{\rho}\right) = \frac{\dot{s}}{\rho}\vec{n} \tag{4}$$

Using (3) and (2):

$$\vec{A}_A = \underbrace{\ddot{s}\vec{\tau}}_{\text{tangential component}} + \underbrace{\frac{\dot{s}^2}{\rho}\vec{n}}_{\text{normal component}}$$

or,

$$\vec{A}_A = \vec{A}_A^t + \vec{A}_A^n$$



$\vec{A}_A^t$: has magnitude $\ddot{s}$ and is directed along the tangent to the path

$\vec{A}_A^n$: has a magnitude $\frac{\dot{s}^2}{\rho}$ and is directed towards the centre of curvature

Angular Acceleration

- Applies to motion of a rigid body.
- The change in angular velocity over time increment $\triangle$t is :

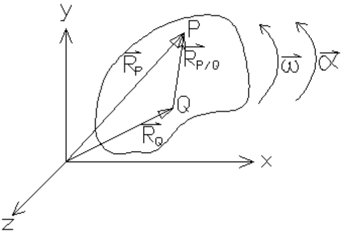
$$\triangle\vec{\omega} = \vec{\omega'} - \vec{\omega}$$

then the angular acceleration
$\vec{\alpha} \equiv$ time rate of change of the angular velocity of a rigid body
and,

$$\vec{\alpha} = \lim_{\triangle t \to 0} \frac{\triangle\vec{\omega}}{\triangle t} = \frac{d\omega}{dt} = \dot{\vec{\omega}}$$

Acceleration Difference Vector

Consider planar analysis



Position: $\vec{R}_P = \vec{R}_Q + \vec{R}_{P/Q}$

Velocity: $\vec{V}_P = \frac{d}{dt}(\vec{R}_p) = \dot{\vec{R}}_Q + \dot{\vec{R}}_{P/Q}$

$$\vec{V}_P = \vec{V}_Q + \underset{\substack{\downarrow \\ \text{velocity difference } = \vec{\omega} \times \vec{R}_{P/Q}}}{\vec{V}_{P/Q}}$$

(rel. velocity of P wrt Q)

Acceleration: $\vec{A}_P = \frac{d\vec{V}_P}{dt} = \underset{\substack{\downarrow \\ \vec{A}_Q}}{\dot{\vec{V}}_Q} + \underset{\substack{\downarrow \\ \vec{\alpha}}}{\dot{\vec{\omega}}} \times \vec{R}_{P/Q} + \vec{\omega} \times \underset{\substack{\downarrow \\ \vec{V}_{P/Q} = \vec{\omega} \times \vec{R}_{P/Q}}}{\dot{\vec{R}}_{P/Q}}$

$$\therefore \quad \vec{A}_P = \vec{A}_Q + \vec{\alpha} \times \vec{R}_{P/Q} + \vec{\omega} \times (\vec{\omega} \times \vec{R}_{P/Q}) \tag{5}$$

where,

- $\vec{A}_P$ is the absolute linear acceleration of point P
- $\vec{A}_Q$ is the absolute linear acceleration of point Q
- $\vec{\alpha} \times \vec{R}_{P/Q} \equiv$ tangential component of acceleration difference

  $|\, \vec{\alpha} \times \vec{R}_{P/Q} \,| = \alpha R_{P/Q}$

- To calculate $\vec{\omega} \times (\vec{\omega} \times \vec{R}_{P/Q})$: recall that for three vectors we have

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

then for $\vec{A} = \vec{\omega}$, $\vec{B} = \vec{\omega}$ & $\vec{C} = \vec{R}_{P/Q}$

we get $\vec{\omega} \times (\vec{\omega} \times \vec{R}_{P/Q}) = (\vec{\omega} \cdot \vec{R}_{P/Q})\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\vec{R}_{P/Q}$

where

$$(\vec{\omega} \cdot \vec{R}_{P/Q}) = 0, \quad \text{because } \vec{\omega} \perp \vec{R}_{P/Q}$$

$$(\vec{\omega} \cdot \vec{\omega}) = \omega^2$$

$$\therefore |\, \vec{\omega} \times \vec{\omega} \times \vec{R}_{P/Q} \,| = -\, \underbrace{\omega^2 \vec{R}_{P/Q}}_{\text{normal component of the acceleration difference}}$$

Thus for two points on the same rigid body:

$$\vec{A}_P = \vec{A}_Q + \underbrace{\underbrace{\vec{\alpha} \times \vec{R}_{P/Q}}_{\vec{A}^t{}_{P/Q}} + \underbrace{\vec{\omega} \times \vec{\omega} \times \vec{R}_{P/Q}}_{\vec{A}^n{}_{P/Q}}}_{\vec{A}_{P/Q}}$$

$$or, \qquad \vec{A}_P = \vec{A}_Q + \vec{A}_{P/Q} \tag{6}$$

Example 7.1

Consider a simple pendulum at the instant shown,
Determine $\vec{A}_A$.

<u>Solution:</u>

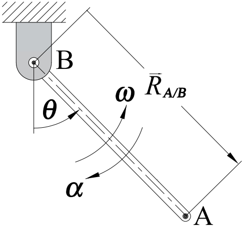
$$\vec{A}_A = \underbrace{\vec{A}_B}_{0} + \vec{A}_{A/B}$$

$$= \vec{A}^n{}_{A/B} + \vec{A}^t{}_{A/B}$$

$$= -\omega^2 \vec{R}_{A/B} + \vec{\alpha} \times \vec{R}_{A/B}$$

$$\Downarrow \qquad\qquad \Downarrow$$

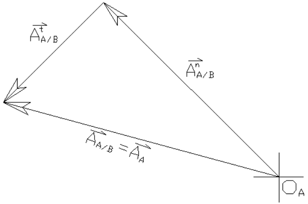
$$\mid \omega^2 \vec{R}_{A/B} \mid \nwarrow \qquad \swarrow \mid \alpha \vec{R}_{A/B} \mid$$



$$\mid \vec{A}_{A/B} \mid = \sqrt{(A^n{}_{A/B})^2 + (A^t{}_{A/B})^2}$$

$$= \sqrt{\omega^4 R_{A/B}{}^2 + \alpha^2 R_{A/B}{}^2}$$

$$= R_{A/B}\sqrt{\omega^4 + \alpha^2}$$



Acceleration Polygons

Acceleration polygons can be constructed in a manner similar to velocity polygons.

- The acceleration difference (relative acceleration) between two points P and Q on the same rigid body consists of two orthogonal components $\vec{A^n}_{P/Q}$ and $\vec{A^t}_{P/Q}$. The normal component, $(-\omega^2 \vec{R}_{P/Q})$, is completely determined from the velocity analysis.
- The orientation of the acceleration image depends on the angular accelerations and velocities of the link, i.e., the velocity image of the link is a constant rotation angle of $\pm 90^\circ$ from the link geometry.
- Whereas the rotation angle of the acceleration image is

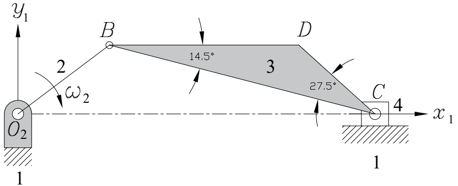
$$\delta = 180^o - \tan^{-1}\left(\frac{\alpha}{\omega^2}\right)$$

  where the positive sense of δ is counter-clockwise.
- The acceleration image is scaled from the geometry of the linkage by the factor $\sqrt{\omega^4 + \alpha^2}$ (see Example 7.1).
- The other properties of acceleration polygons remain the same as those for velocity polygons.

Example 7.2

For the slider crank mechanism determine the linear velocities and accelerations of points C & D and the angular acceleration of link 3 at the instant shown.

$O_2B = 2.5$ in
BC $= 6.0$ in
$\vec{\omega}_2 = 1800$ rpm
(constant)



Velocity analysis:

$$\omega_2 = (1800 \text{ rev/min})(2\pi \text{rad/rev})(1\text{min}/60\text{sec}) = 188.5 \text{ rad/sec cw}$$

$$\vec{V}_B = \vec{V}_{O_2} + \vec{V}_{B/O_2} = \vec{\omega_2} \times \vec{R}_{B/O_2} = (188.5 \text{ rad/sec})(2.5 \text{ in}/12 \text{ in/ft}) = 39.3 \text{ ft/s} \searrow (\perp \text{ to } O_2B)$$

$$\overset{o\surd}{\vec{V}}_C = \overset{\surd\surd}{\vec{V}}_B + \overset{o\surd}{\vec{V}}_{C/B}$$

$$\overset{o\surd}{\vec{V}}_C = \overset{\surd\surd}{\vec{V}}_B + \overset{o\surd}{\vec{V}}_{C/B}$$

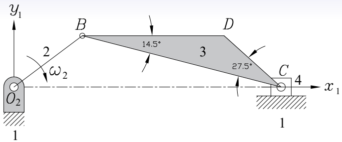
From velocity polygon:
$\vec{V}_C = 2.7 \text{ in}(10\text{ft/s/in}) = 27 \text{ ft/s} \rightarrow$

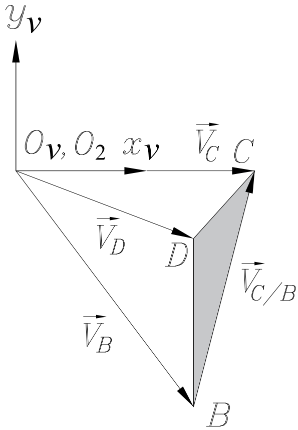
The velocity of D can be obtained by forming the image of link 3 by:

-measuring $\vartheta_{BC/BD}$ & $\vartheta_{CD/CB}$ and drawing link 3's image
-using the direction of $\vec{V}_{D/B}$ & $\vec{V}_{D/C}$ to draw the image.

From polygon:
$V_D = (2.1\text{in})(10\text{ft/s}) = 21 \text{ ft/s } (\searrow 27.5^o)$

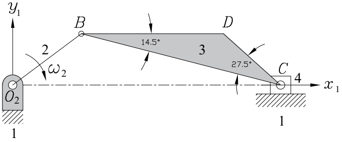
For ω_3:
$$\vec{V}_{C/B} = \vec{\omega}_3 \times \vec{R}_{C/B}$$
$$\therefore \ \omega_3 = \frac{V_{C/B}}{R_{C/B}} = \frac{3.44in(10ft/s/in)}{\frac{6.0in}{12in/ft}} = 68.8 \text{ rad/s ccw}$$





Acceleration Analysis:

$$\vec{A}_B = \underbrace{\vec{A}_O}_{0} + \vec{A}_{B/O} = \vec{A^n}_{B/O} + \vec{A^t}_{B/O},$$



$$\vec{A}_B = -\omega^2 \vec{R}_{B/O} + \underbrace{\alpha_2 \times \vec{R}_{B/O}}_{0}$$

$$\vec{A}_B = (188.5 rad/s)^2 \left(\frac{2.5in}{12in/ft}\right)$$

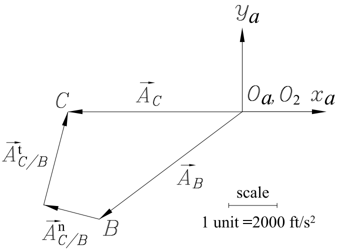
$$\vec{A}_B = 7403\text{ft/s} \swarrow$$

$$\vec{A}_C = \vec{A}_B + \vec{A}_{C/B}$$

or, $\overset{o\surd}{\vec{A}_C} = \overset{\surd\surd}{\vec{A}_B} + \overset{\surd\surd}{\vec{A^n}_{C/B}} + \overset{o\surd}{\vec{A^t}_{C/B}}$

where,

$$\vec{A^n}_{C/B} = -\omega^2 \vec{R}_{C/B} = (68.8 rad/s)^2 (0.5 ft)$$
$$= 2367 \text{ ft}/s^2 \nwarrow \; (\parallel \text{ to } \vec{R}_{C/B})$$
$$\vec{A^t}_{C/B} = \vec{\alpha}_3 \times \vec{R}_{C/B} \; (\perp \text{ to } \vec{R}_{C/B})$$



From the diagram,

$$\vec{A}_C = (4\text{in})(2000\text{ft}/s^2/\text{in})$$
$$= 8000 \text{ ft}/s^2 \leftarrow$$

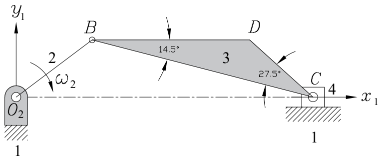
By using

$\vec{A^t}_{C/B} = \vec{\alpha}_3 \times \vec{R}_{C/B}$

$\Rightarrow \alpha_3 = \frac{A^t_{C/B}}{R_{C/B}}$

$= \frac{1.66in(2000ft/s^2/in)}{0.5ft}$

$= 6640 \text{ rad}/s^2 \text{ ccw}$



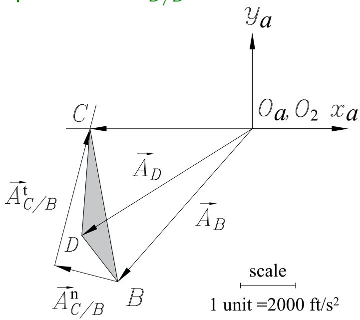
To determine $\vec{A}_D$:

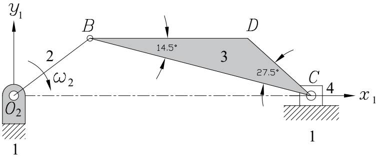
either,

- Write $\vec{A}_D = \vec{A}_B + \vec{A}_{D/B}$ and calculate components of $\vec{A}_{D/B}$ and add to acceleration diagram;

or,

- Draw acceleration image of link 3 using angles $\vartheta_{BC/BD}$ & $\vartheta_{CD/CB}$ (this is what has been done here)



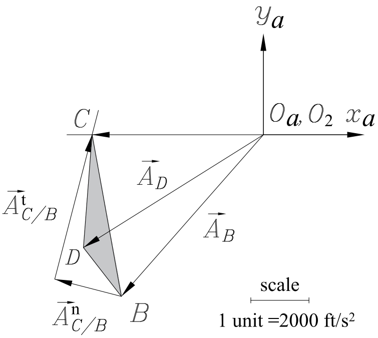


Determine the angle of rotation of BCD by:

$$\delta = 180 - tan^{-1}\left(\frac{\alpha}{\omega^2}\right)$$

Then from the diagram we get:

$$|\vec{A}_D| = 4.13in(2000ft/s^2/in) = 8260ft/s^2$$



Acceleration Analysis Using Complex Polar Notation

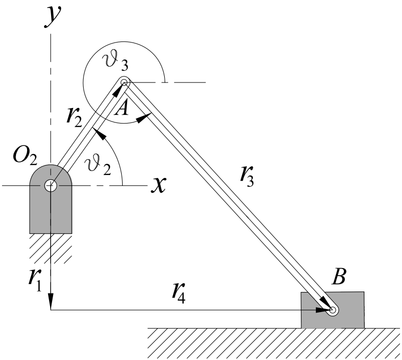
- Consider a general crank-slider linkage (RRRP).
- The loop closure equation can be expressed as

$$r_2 e^{j\vartheta_2} + r_3 e^{j\vartheta_3} \quad = \quad r_1 e^{j3\pi/2} + r_4 e^{j0}$$

- After using the Euler identity

$$e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$$

separate the real (x) and imaginary (y) components leading to



$$\begin{aligned} r_2\cos\vartheta_2 + r_3\cos\vartheta_3 &= r_1\cos 3\pi/2 + r_4\cos 0 = r_4 \\ r_2\sin\vartheta_2 + r_3\sin\vartheta_3 &= r_1\sin 3\pi/2 + r_4\sin 0 = -r_1 \end{aligned}$$

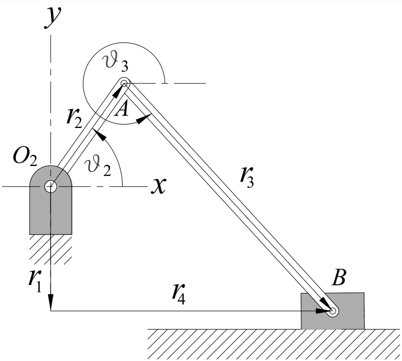
- Given r_1, r_2, r_3, and input angle ϑ_2, it is a simple matter to determine r_4 and ϑ_3.

- Differentiating the loop closure equation with respect to time determines the velocity level kinematics with the substitutions

$$\begin{aligned} \dot{\vartheta}_2 &= \omega_2, \\ \dot{\vartheta}_3 &= \omega_3, \\ \dot{r}_4 &= v_4 \end{aligned}$$

giving

$$j r_2 \omega_2 e^{j\vartheta_2} + j r_3 \omega_3 e^{j\vartheta_3} = v_4 \quad (7)$$

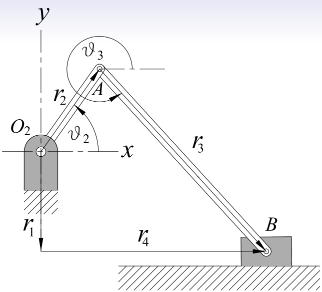


- Apply the Euler identity to obtain

$$j r_2 \omega_2 (\cos\vartheta_2 + j \sin\vartheta_2) + j r_3 \omega_3 (\cos\vartheta_3 + j \sin\vartheta_3) = v_4$$

- Separate into real and imaginary components to reveal

$$\begin{aligned} -r_2\omega_2 \sin\vartheta_2 - r_3\omega_3 \sin\vartheta_3 &= v_4, \\ r_2\omega_2 \cos\vartheta_2 + r_3\omega_3 \cos\vartheta_3 &= 0 \Rightarrow \omega_3 = -\frac{r_2\omega_2 \cos\vartheta_2}{r_3 \cos\vartheta_3} \end{aligned}$$



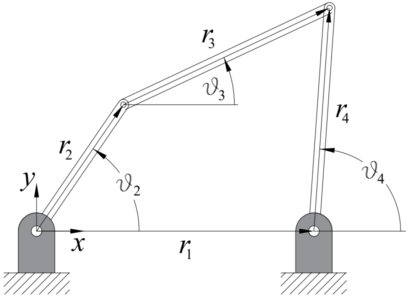
- Differentiating the velocity closure equation, Equation (7), with respect to time determines the acceleration.
- Proceeding as for the velocity equation, after separating the result into real and imaginary components yields two linearly independent equations that can be solved for two unknowns, typically the coupler angular and slider linear accelerations α_3 and a_4:

$$\begin{aligned} \alpha_3 &= \frac{r_2\omega_2^2 \sin\vartheta_2 - r_2\alpha_2 \cos\vartheta_2 + r_3\omega_3^2 \sin\vartheta_3}{r_3 \cos\vartheta_3}, \\ a_4 &= -r_2\omega_2^2 \cos\vartheta_2 - r_2\alpha_2 \sin\vartheta_2 - r_3\omega_3^2 \cos\vartheta_3 - r_3\alpha_3 \sin\vartheta_3 \end{aligned}$$

4*R* Complex Polar Acceleration Analysis

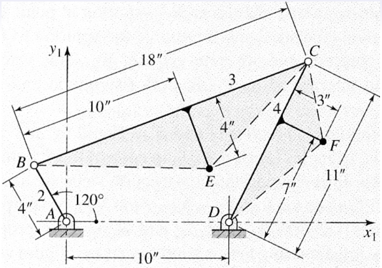
- The same procedure can be carried out for planar 4*R* linkages.
- Start with the loop closure equation in any convenient form:

$$r_2 e^{j\vartheta_2} + r_3 e^{j\vartheta_3} \quad = \quad r_1 e^{j0} + r_4 e^{j\vartheta_4}$$



Example 7.3

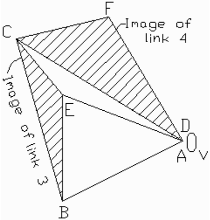
Consider the 4-bar linkage as shown. Determine the acceleration of points E and F, and angular accelerations of links 3 and 4.



J J.Uicker. *Theory of Machines and Mechanisms*. Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.

The velocities were analyzed in Examples 6.1 and 6.2, where the velocity polygon was developed as shown below from which we had:

$\vec{V}_B = 31.4$ ft/s
$\vec{V}_C = 46.6$ ft/s
$\vec{V}_E = 27.6$ ft/s
$\vec{V}_F = 31.8$ ft/s
$\vec{\omega}_2 = 94.2$ rad/s ccw
$\vec{\omega}_3 = 25.5$ rad/s ccw
$\vec{\omega}_4 = 49.6$ rad/s ccw



Now the acceleration equation is:

$$\overset{\surd\surd}{\vec{A}_B} = \overbrace{\vec{A}_A}^{0} + \vec{A}_{B/A} = \vec{A}^n{}_{B/A} + \overbrace{\vec{A}^t{}_{B/A}}^{0(\alpha=0)} = -\omega^2 \vec{R}_{B/A};$$

$$\therefore \quad \vec{A}_B = -(94.2\ \text{red}/s^2)(4/12\text{ft}) = 2958\ \text{ft}/s^2 \searrow$$

Now,

$$\vec{A}_C = \vec{A}_B + \vec{A}_{C/B} = \vec{A}_B + \vec{A}^n{}_{C/B} + \vec{A}^t{}_{C/B}$$

Also,

$$\vec{A}_C = \underbrace{\vec{A}_D}_{0} + \vec{A}_{C/D} = \vec{A}^n{}_{C/D} + \vec{A}^t{}_{C/D}$$

$$\therefore \quad \overset{\surd\surd}{\vec{A}_B} + \overset{\surd\surd}{\vec{A}^n{}_{C/B}} + \overset{o\surd}{\vec{A}^t{}_{C/B}} = \overset{\surd\surd}{\vec{A}^n{}_{C/D}} + \overset{o\surd}{\vec{A}^t{}_{C/D}}$$

where,

$$A^n{}_{C/B} = \omega_3{}^2 R_{C/B} = (25.6)^2(18/12) = 983\ \text{ft}/s^2 \swarrow$$

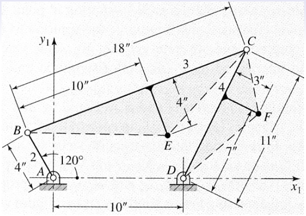
$$A^n{}_{C/D} = \omega_4{}^2 R_{C/D} = (49.6)^2(11/12) = 2255\ \text{ft}/s^2 \swarrow$$

By using,

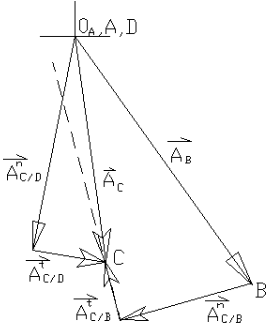
$$\vec{A}^t{}_{C/B} = \vec{\alpha} \times \vec{R}_{C/B}$$

$$\Rightarrow \alpha_3 = \frac{A^t{}_{C/B}}{R_{C/B}} = \frac{160ft/s^2}{18/12ft}$$

$$\alpha_3 = 107\ \text{rad}/s^2\ \text{ccw}$$



J J.Uicker. *Theory of Machines and Mechanisms*. Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.



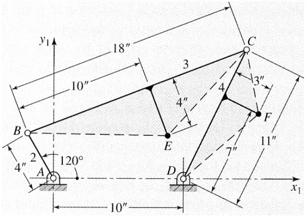
Similarly:

$$\vec{\alpha}_4 = \frac{A^t{}_{C/D}}{R_{C/D}} = \frac{1670 ft/s^2}{11/12 ft} = 1822 \text{ rad}/s^2 \text{ cw}$$

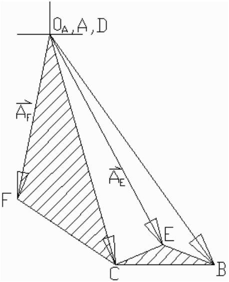
The linear accelerations at E and F can be determined in one of three ways:

1. Apply the same method used for evaluating $\vec{A}_C$ above; e.g., $\vec{A}_E = \vec{A}_B + \vec{A}_{E/B} = \vec{A}_C + \vec{A}_{E/C}$
2. Evaluate $\vec{A}_E$ and $\vec{A}_F$ analytically knowing $\vec{\alpha}_3$, $\vec{\alpha}_4$ (and $\vec{\omega}_3$ and $\vec{\omega}_4$)
3. From the acceleration images of links 3 and 4 as shown below

Scaling from acceleration image, we read:
$\vec{A}_E = 2580 \text{ ft}/s^2$
$\vec{A}_F = 1960 \text{ ft}/s^2$



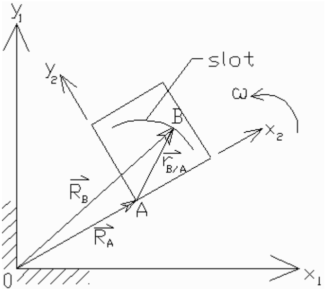
J J.Uicker. *Theory of Machines and Mechanisms*. Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.



Apparent Acceleration of a Point in a Moving Coordinate System

Consider a rigid body with attached $x_2y_2z_2$ coordinate system that is moving relative to the fixed $x_1y_1z_1$. The $x_2y_2z_2$ is fixed to the body, whereas $x_1y_1z_1$ system is fixed in space.

Recall that the general expression for the derivative of a vector $\vec{r} = x_2\vec{i} + y_2\vec{j}$ in a moving coordinate system with unit vectors $\vec{i}$ and $\vec{j}$, that is also rotating with an $\omega\vec{k}$, so we must account for change in both magnitude and direction, i.e.,



$$\frac{d}{dt}(\vec{r}) = \frac{d}{dt}(x_2\vec{i} + y_2\vec{j}) = (\dot{x}_2\vec{i} + \dot{y}_2\vec{j}) + (x_2\frac{d\vec{i}}{dt} + y_2\frac{d\vec{j}}{dt}) \tag{8}$$

$$= \dot{\vec{r}} + (x_2\vec{\omega} \times \vec{i} + y_2\vec{\omega} \times \vec{j}) = \dot{\vec{r}} + \vec{\omega} \times (x_2\vec{i} + y_2\vec{j}) = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$$

Then in the figure above assume that motion of B relative to A, i.e.,"apparent" motion is possible, i.e.,

$$\vec{R}_B = \vec{R}_A + \vec{r}_{B/A}$$

$$\vec{R}_B = \vec{R}_A + \vec{r}_{B/A}$$

$$\therefore \vec{V}_B = \frac{d}{dt}(\vec{R}_B) = \dot{\vec{R}}_A + \underbrace{\dot{\vec{r}}_{B/A}}_{\Delta\ magnitude} + \underbrace{\vec{\omega} \times \vec{r}_{B/A}}_{\Delta\ direction} = \vec{V}_A + \underbrace{\vec{V}_{rel}}_{velocily} + \vec{\omega} \times \vec{r}_{B/A}$$

of B as measured

from $x_2 y_2$ frame

and:

$$\vec{A}_B = \frac{d}{dt}(\vec{V}_B) = \dot{\vec{V}}_A + \dot{\vec{V}}_{rel} + \vec{\omega} \times \dot{\vec{r}}_{B/A} + \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times (\dot{\vec{r}}_{B/A} + \vec{\omega} \times \vec{r}_{B/A})$$

$$\vec{A}_B = \frac{d}{dt}(\vec{V}_B) = \dot{\vec{V}}_A + \dot{\vec{V}}_{rel} + \vec{\omega} \times \vec{V}_{rel} + \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times \vec{V}_{rel} + \vec{\omega} \times (\vec{\omega} \times r_{B/A})$$

$$\vec{A}_B = \underbrace{\dot{\vec{V}}_A}_{\vec{A}_A} + \underbrace{\dot{\vec{V}}_{rel}}_{\vec{A}_{rel}} + \underbrace{\dot{\vec{\omega}}}_{\vec{\alpha}} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times r_{B/A}) + 2\vec{\omega} \times \vec{V}_{rel}$$

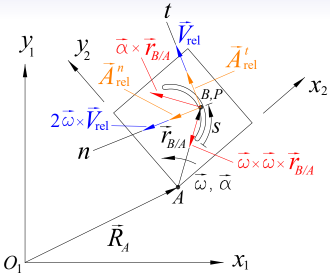
$$\Rightarrow \quad \vec{A}_B = \vec{A}_A + \vec{A}_{rel} + \underbrace{\vec{\alpha} \times \vec{r}_{B/A}}_{\vec{A}^t_{B/A}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})}_{\vec{A}^n_{B/A}} + \underbrace{2\vec{\omega} \times \vec{V}_{rel}}_{\vec{A}^c_{B/A}} \qquad (9)$$

$$\vec{A}_B = \vec{A}_A + \vec{A}_{rel} + \underbrace{\vec{\alpha} \times \vec{r}_{B/A}}_{\vec{A}^t_{B/A}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})}_{\vec{A}^n_{B/A}} + \underbrace{2\vec{\omega} \times \vec{V}_{rel}}_{\vec{A}^c_{B/A}}$$

where,

$\vec{A}_B$ = absolute acceleration of point B relative to the fixed frame
$\vec{A}_A$ = absolute acceleration of point A relative to the fixed frame
$\vec{A}_{rel}$ = acceleration of B relative to A in the moving frame
$\vec{\alpha} \times \vec{r}_{B/A}$ = tangential acceleration due to angular acceleration of moving frame
$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$ = normal acceleration due to angular velocity of the moving frame
$2\vec{\omega} \times \vec{V}_{rel}$ = Coriolis acceleration; this component of acceleration results from change in length of $\vec{\omega} \times \vec{r}_{B/A}$ and change in direction of $\vec{r}_{B/A}$

- it depends on quantities obtained from velocity analysis
- it is rotated 90^o from $\vec{V}_{rel}$ in the direction of $\vec{\omega}$



$$\vec{A}_B = \vec{A}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) + 2\vec{\omega} \times \vec{V}_{rel} + \vec{A}_{rel} \tag{10}$$

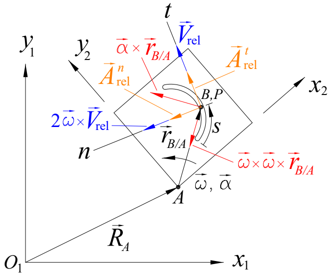
As observed from a non-rotating frame at A:

$|\,\vec{\alpha} \times \vec{r}_{B/A}\,| = \alpha r_{B/A}$ is perpendicular to $\vec{r}_{B/A}$ in the direction of α with centre at A
$|\,\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})\,| = \omega^2 r_{B/A}$ is directed from B to A

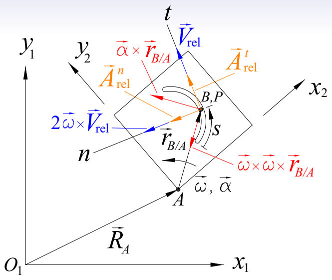
$\therefore$ in terms of the coincident point P we can interpret:

$\vec{\alpha} \times \vec{r}_{B/A}$: as the tangential component of $\vec{A}_{P/A}$ of point P in its circular motion about A

$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$: normal component of $\vec{A}_{P/A}$ of point P in its circular motion about A



$\mid (\vec{A}_{rel})_t \mid = \ddot{s}$:	change of speed of B relative to A or P
$\mid (\vec{A}_{rel})_n \mid = \frac{\vec{V}^2_{rel}}{\rho}$	towards the centre of the path
$2\vec{\omega} \times \vec{V}_{rel}$:	Coriolis acceleration: this is the difference between the $\vec{A}_{B/P}$ as measured from non-rotating axes and from rotating axes. It is always perpendicular to $\vec{V}_{B/A}$ and its direction is rotated 90^o from $\vec{V}_{rel}$ in the direction of $\vec{\omega}$ according to the right-hand-rule.



Hence

$$\begin{aligned}\vec{A}_B &= \vec{A}_A + \underbrace{\vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})}_{\vec{A}_{P/A}} + 2\vec{\omega} \times \vec{V}_{rel} + \vec{A}_{rel} \\ &= \underbrace{\vec{A}_A + \vec{A}_{P/A}}_{\vec{A}_P} + 2\vec{\omega} \times \vec{V}_{rel} + \vec{A}_{rel}\end{aligned}$$

$$\therefore \qquad \vec{A}_B = \vec{A}_P + 2\vec{\omega} \times \vec{V}_{rel} + \vec{A}_{rel} \qquad (11)$$

Note that the apparent acceleration expression reduces to the acceleration differences expression if A & B are points on the same rigid body, i.e., $\ddot{\vec{r}}_{B/A} = \dot{\vec{r}}_{B/A} = 0$.

Example 7.4

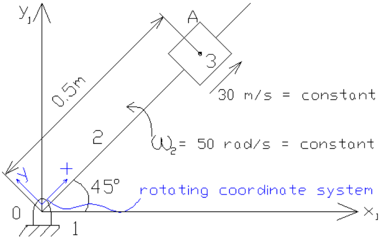
Given the rod with slider as shown. Determine acceleration of the slider at the instant shown.

Solution: Apply Equation (10):

$$\vec{A}_A = \vec{A}_O + \vec{A}_{rel} + \vec{\alpha} \times \vec{r}_{A/O} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/O}) + 2\vec{\omega} \times \vec{V}_{rel}$$

where,

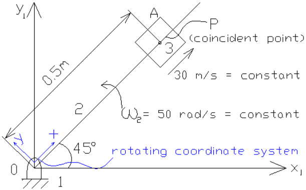
$$\vec{A}_O = 0$$
$$\vec{A}_{rel} = 0$$
$$\vec{\alpha} \times \vec{r}_{A/O} = 0$$



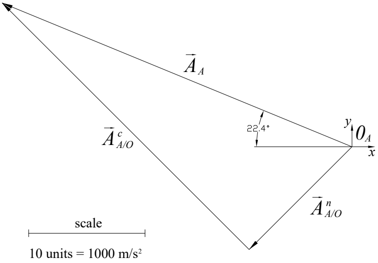
$$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/O}) = -\omega^2 \vec{r}_{A/O} = (50rad/s)^2(0.5m) = 1250\ m/s^2 \swarrow$$
$$2\vec{\omega} \times \vec{V}_{rel} = 2(50\text{ rad/s})(30\text{ m/s}) = 3000\ m/s^2 \nwarrow$$

Note that $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/O})$ may be thought of differently in terms of the coincident point P that is attached to the rod and the rotating coordinate system xy. Then we have : $\vec{\omega} \times \vec{r}_{P/O} = \vec{V}_{P/O}$ and $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O}) = (\vec{A}_{P/O})_n$

Acceleration Polygon:



Scale 1000 m/s^2 = 10 units



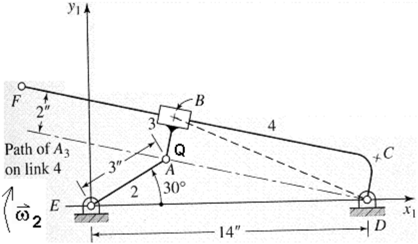
The resultant acceleration $\vec{A}_A$ can be scaled or calculated and found to be 3250 m/s^2 and at 22.4^o with the horizontal ↖

Example 7.5

Given the inverted slider-crank mechanism as shown below.
Where AE = 3.0 in, ED = 14.0 in, AB = 2.0 in and $\omega_2 = 36$ rad/s cw constant.
Determine the angular acceleration of link 4 at the instant shown.

Solution:
Let point Q be coincident with point A but attached to link 3, which has the same angular acceleration as link 4.
The velocity analysis was performed in Example 6.3 with the results:

$\vec{V}_A = 9$ ft/s $\searrow \perp$ to AE
$\vec{V}_{Q/D} = 7.24$ ft/s $\swarrow \perp$ to QD
$\vec{V}_{A/Q} = \vec{V}_{rel} = 5.52$ ft/s, $\parallel$ to FC $\searrow$
$\vec{\omega}_2 = 36$ rad/s cw (constant)
$\vec{\omega}_3 = \vec{\omega}_4 = 7.55$ rad/s ccw



J J.Uicker. *Theory of Machines and Mechanisms*. Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.

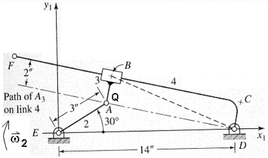
Continuing with acceleration analysis, we apply Equation (10) to links 3 & 4:

$$\therefore \quad \vec{A}_A = \vec{A}_Q + 2\vec{\omega}_4 \times \vec{V}_{rel} + \vec{A}_{rel} \qquad \text{(E1)}$$

Where from link 2:

$$\vec{A}_A = \vec{A}_E + \vec{A}_{A/E} = \vec{O} + \vec{A}_{A/E}$$

$$\vec{A}_A = \vec{A}^t_{A/E} + \vec{A}^n_{A/E} = \vec{O} + \vec{A}^n_{A/E}$$



J J.Uicker. *Theory of Machines and Mechanisms.* Oxford University Press Inc., New York, New York, United States, 3rd edition, 2003.

$$\vec{A}_A = -{\omega_2}^2 \vec{R}_{A/E} = -(36)^2(3/12) = 324 \text{ ft/s}^2, \parallel \text{ to AE} \swarrow \qquad \text{(E2)}$$

and from link 4:

$$\vec{A}_Q = \vec{A}_D + \vec{A}_{Q/D} = \vec{O} + \underline{\vec{A}^n_{Q/D}} + \underline{\vec{A}^t_{Q/D}}$$

$$\Downarrow \qquad \Downarrow$$

$$= -{\omega_4}^2 \vec{R}_{Q/D} \searrow + \alpha_4 \vec{R}_{Q/D} \swarrow\!\!\!\nearrow$$
$$= -(7.55)^2(11.5/12) \searrow + \alpha_4 \vec{R}_{Q/D} \swarrow\!\!\!\nearrow$$

Or, $\vec{A}_Q$ = -54.6 $\searrow$ ($\parallel$ to QD) $+ \alpha_4 \vec{R}_{Q/D} \swarrow\!\!\!\nearrow$ ($\perp$ to QD) (E3)

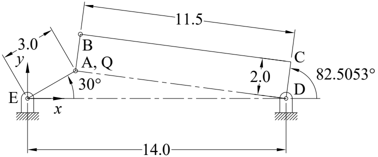
$$2\omega_4 \times \vec{V}_{rel} = 2(7.55)(5.52) = 83.4 \; ft/s^2 \nearrow (\perp \text{ to QD}) \qquad \text{(E4)}$$

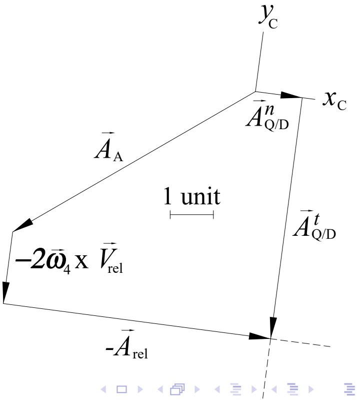
substitute E2, E3, E4 into E1 to get:

$$
\begin{array}{ccccccccc}
\vec{A}_A & = & \vec{A}^n_{Q/D} & + & \vec{A}^t_{Q/D} & + & 2\omega_4 \times \vec{V}_{rel} & + & \vec{A}_{rel} \\
324 & = & 54.6 & + & \vec{\alpha}_4 \times \vec{R}_{Q/D} & + & 83.4 & + & \vec{A}_{rel} \\
\swarrow(\parallel \text{ to AE}) & & \searrow(\parallel \text{ to QD}) & & \nearrow(\perp \text{ to QD}) & & \nearrow(\perp \text{ to QC}) & & \nwarrow(\parallel \text{ to QC})
\end{array}
$$

or

$$
\begin{array}{ccccccccc}
\vec{A}_A & - & 2\omega_4 \times \vec{V}_{rel} & - & \vec{A}_{rel} & = & \vec{A}^n_{Q/D} & + & \vec{A}^t_{Q/D} \\
324 & - & 83.4 & - & \vec{A}_{rel} & = & 54.6 & + & \vec{\alpha}_4 \times \vec{R}_{Q/D} \\
\swarrow(\parallel \text{ to AE}) & & \swarrow(\perp \text{ to QD}) & & \searrow(\parallel \text{ to QD}) & & \searrow(\parallel \text{ to QD}) & & \nearrow(\perp \text{ to QD})
\end{array}
$$



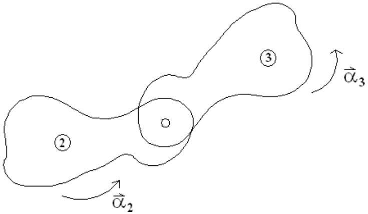


The acceleration polygon scale is 1 drawing unit $= 50\ ft/s^2$:

$(\vec{A}^t_{Q/D}) = 274\ ft/s^2$

$\therefore \quad |\,\vec{\alpha}_4\,| = \frac{274}{11.6/12} = 283\ rad/s^2$, ccw

Apparent Angular Acceleration



Let

α_2 : angular acceleration of link 2

α_3 : angular acceleration of link 3

then, we define

$\alpha_{3/2} = \alpha_3 - \alpha_2$: "apparent" angular acceleration of link 3 w.r.t. link 2

Alternatively:

$$\alpha_3 = \alpha_2 + \alpha_{3/2} \tag{12}$$

This concept is rarely used in solving problems.

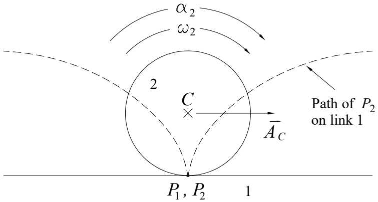
Apparent Acceleration at a point of Rolling Contact

Recall "Rolling Contact" implies no slip between links in contact.

- Points P_2 and P_1 are coincident points on the disc and ground
- The path of P_2 relative to link 1 is a *cycloid* of the disc
- When P_2 and P_1 are coincident, the tangent of the cycloid is perpendicular to both surfaces at the point of contact
- Since the path of P_2 is known, we have:

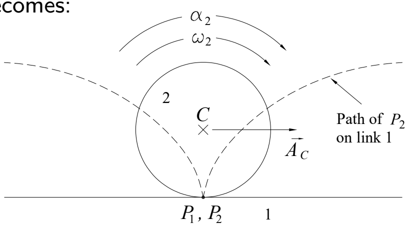
$$\vec{A}_{P_2} = \vec{A}_{P_1} + 2\vec{\omega} \times \vec{V}_{rel} + \vec{A}_{rel} \quad (13)$$

where:
$2\vec{\omega} \times \vec{V}_{rel} = 0$ (rolling contact & no slipping, $\vec{V}_{rel} = 0$)



$$\vec{A}_{rel} \equiv \vec{A}_{P_2/P_1} = \vec{A}^n_{P_2/P_1} + \vec{A}^t_{P_2/P_1} = \frac{\vec{V}^2_{P_2/P_1}}{\rho} + \vec{A}^t_{P_2/P_1} = 0 + \vec{A}^t_{P_2/P_1}$$

$\vec{A}^t_{P_2/P_1}$ is the acceleration component tangent to the path, i.e., $\perp$ to the contact surface. To avoid confusion, it is called rolling acceleration and denoted as $\vec{A}^r_{P_2/P_1}$ hence Equation (13) becomes:



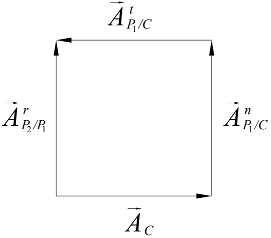
$$\vec{A}_{P_2} = \vec{A}_{P_1} + \vec{A}^r_{P_2/P_1} \qquad (14)$$

On the other hand from the disc:

$$\vec{A}_{P_1} = \vec{A}_C + \vec{A}_{P_1/C} = \vec{A}_C + \vec{A}^n_{P_1/C} + \vec{A}^t_{P_1/C} \qquad (15)$$

From Equations (14) & (15) we can write

$$\overset{o\checkmark}{\vec{A}^r}_{P_2/P_1} = \overset{\checkmark\checkmark}{\vec{A}_C} + \overset{\checkmark\checkmark}{\vec{A}^n}_{P_1/C} + \overset{o\checkmark}{\vec{A}^t}_{P_1/C}$$

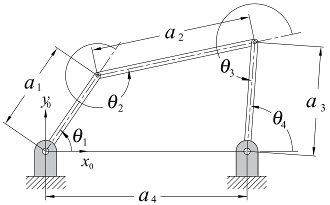


Instantaneous Centres of Acceleration (ICA)

- Defines as the instantaneous location of a pair of coincident points of two different links where absolute accelerations of the two points are equal
- ICAs are:
  - at different locations than ICVs
  - more difficult to locate than ICVs
  - not useful in analysis
  - not discussed further
- We will instead briefly turn our attention toward the algebraic input-output (IO) angular acceleration equations, in particular determining the configurations of a planar 4R linkage possessing angular acceleration extreme values for a constant angular velocity input

- The time derivative of the six angular velocity IO equations yields the six angular acceleration IO equations.
- Using these equations we can directly determine all the unknown angular accelerations occurring in a planar 4R linkage if one angular velocity and one angular acceleration are specified along with the configuration of the linkage at the instant considered.
- The parameter names and definitions in the figure are used in these equations where the coefficients are

$$A = A_1A_2 = (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4),$$
$$B = B_1B_2 = (a_1 - a_2 + a_3 + a_4)(a_1 + a_2 + a_3 + a_4),$$
$$C = C_1C_2 = (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4),$$
$$D = D_1D_2 = (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4),$$
$$v_1 = \tan\frac{\theta_1}{2}, \quad v_4 = \tan\frac{\theta_4}{2}.$$



First, recall the six algebraic IO equations for a planar 4R linkage:

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{16}$$

$$A_1B_1v_1^2v_2^2 + A_2B_2v_1^2 + C_1D_2v_2^2 + 8a_2a_4v_1v_2 + C_2D_1 = 0, \tag{17}$$

$$A_2B_1v_1^2v_3^2 + A_1B_2v_1^2 + C_1D_1v_3^2 + C_2D_2 = 0, \tag{18}$$

$$B_1C_1v_2^2v_3^2 + A_1D_2v_2^2 + A_2D_1v_3^2 - 8a_1a_3v_2v_3 + B_2C_2 = 0, \tag{19}$$

$$A_1C_1v_2^2v_4^2 + B_1D_2v_2^2 + A_2C_2v_4^2 + B_2D_1 = 0, \tag{20}$$

$$A_2C_1v_3^2v_4^2 + B_1D_1v_3^2 + A_1C_2v_4^2 - 8a_2a_4v_3v_4 + B_2D_2 = 0. \tag{21}$$

The six algebraic IO angular velocity ratio equations for a planar 4R linkage are

$$\frac{\dot{\theta}_4}{\dot{\theta}_1} = -\frac{((Av_4^2 + B)v_1 - 4a_1a_3v_4)(1 + v_1^2)}{((Av_1^2 + C)v_4 - 4a_1a_3v_1)(1 + v_4^2)}. \tag{22}$$

$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = -\frac{((A_1B_1v_2^2 + A_2B_2)v_1 + 4a_2a_4v_2)\,(1 + v_1^2)}{((A_1B_1v_1^2 + C_1D_2)v_2 + 4a_2a_4v_1)\,(1 + v_2^2)}, \tag{23}$$

$$\frac{\dot{\theta}_3}{\dot{\theta}_1} = -\frac{((A_2B_1v_3^2 + A_1B_2)v_1)\,(1 + v_1^2)}{((A_2B_1v_1^2 + C_1D_1)v_3)\,(1 + v_3^2)}, \tag{24}$$

$$\frac{\dot{\theta}_3}{\dot{\theta}_2} = -\frac{((B_1C_1v_3^2 + A_1D_2)v_2 - 4a_2a_4v_3)\,(1 + v_2^2)}{((B_1C_1v_2^2 + A_2D_1)v_3 - 4a_2a_4v_2)\,(1 + v_3^2)}, \tag{25}$$

$$\frac{\dot{\theta}_4}{\dot{\theta}_2} = -\frac{((A_1C_1v_4^2 + B_1D_2)v_2)\,(1 + v_2^2)}{((A_1C_1v_2^2 + A_2C_2)v_4)\,(1 + v_4^2)}, \tag{26}$$

$$\frac{\dot{\theta}_4}{\dot{\theta}_3} = -\frac{((A_2C_1v_4^2 + B_1D_1)v_3 - 4a_2a_4v_4)\,(1 + v_3^2)}{((A_2C_1v_3^2 + A_1C_2)v_4 - 4a_2a_4v_3)\,(1 + v_4^2)}, \tag{27}$$

The six algebraic angular acceleration IO equations for a planar 4R are

$$\begin{aligned}((Av_4^2+B)v_1-4a_1a_3v_4)\ddot{v}_1+((Av_1^2+C)v_4-4a_1a_3v_1)\ddot{v}_4+\\ (Av_4^2+B)\dot{v}_1^2+(Av_1^2+C)\dot{v}_4^2+(4Av_1v_4-8a_1a_3)\dot{v}_1\dot{v}_4,\end{aligned} \tag{28}$$

$$\begin{aligned}((A_1B_1v_2^2+A_2B_2)v_1+4a_2a_4v_2)\ddot{v}_1+((A_1B_1v_1^2+C_1D_2)v_2+4a_2a_4v_1)\ddot{v}_2+\\ (A_1B_1v_2^2+A_2B_2)\dot{v}_1^2+(A_1B_1v_1^2+C_1D_2)\dot{v}_2^2+(4A_1B_1v_1v_2+8a_2a_4)\dot{v}_1\dot{v}_2,\end{aligned} \tag{29}$$

$$\begin{aligned}(A_2B_1v_3^2+A_1B_2)v_1\ddot{v}_1+(A_2B_1v_1^2+C_1D_1)v_3\ddot{v}_3+\\ (A_2B_1v_3^2+A_1B_2)\dot{v}_1^2+(A_2B_1v_1^2+C_1D_1)\dot{v}_3^2+4A_2B_1v_1v_3\dot{v}_1\dot{v}_3,\end{aligned} \tag{30}$$

$$\begin{aligned}((B_1C_1v_3^2+A_1D_2)v_2-4a_1a_3v_3)\ddot{v}_2+((B_1C_1v_2^2+A_2D_1)v_3-4a_1a_3v_2)\ddot{v}_3+\\ (B_1C_1v_3^2+A_2D_1)\dot{v}_2^2+(B_1C_1v_2^2+A_2D_1)\dot{v}_3^2+(4B_1C_1v_2v_3-8a_1a_3)\dot{v}_2\dot{v}_3,\end{aligned} \tag{31}$$

$$\begin{aligned}(A_1C_1v_4^2+B_1D_2)v_2\ddot{v}_2+(A_1C_1v_2^2+A_2C_2)v_4\ddot{v}_4+\\ (A_1C_1v_4^2+B_1D_2)\dot{v}_2^2+(A_1C_1v_2^2+A_2C_2)\dot{v}_4^2+4A_1C_1v_2v_4\dot{v}_2\dot{v}_4,\end{aligned} \tag{32}$$

$$\begin{aligned}((A_2C_1v_4^2+B_1D_1)v_3+4a_2a_4v_4)\ddot{v}_3+((A_2C_1v_3^2+A_1C_2)v_4+4a_2a_4v_3)\ddot{v}_4+\\ (A_2C_1v_4^2+B_1D_1)\dot{v}_3^2+(A_2C_1v_3^2+A_1C_2)\dot{v}_4^2+(4A_2C_1v_3v_4+8a_2a_4)\dot{v}_3\dot{v}_4.\end{aligned} \tag{33}$$

It is important to remember that for the i^{th} link, $\dot{v}_i \neq \dot{\theta}_i$ and $\ddot{v}_i \neq \ddot{\theta}_i$ since $v_i = \tan(\theta_i/2)$

For angular velocity

$$\dot{v}_i = \frac{\dot{\theta}_i(1+v_i^2)}{2}, \qquad (34)$$

and

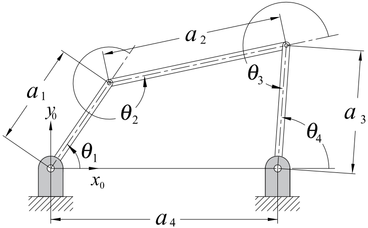
$$\dot{\theta}_i = \frac{2\dot{v}_i}{(1+v_i^2)}. \qquad (35)$$

For angular acceleration

$$\ddot{v}_i = \frac{1}{2}\left(\ddot{\theta}_i + \dot{\theta}_i^2 v_i\right)(1+v_i^2), \qquad (36)$$

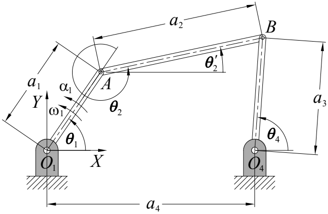
and

$$\ddot{\theta}_i = \frac{2\ddot{v}_i}{(1+v_i^2)} - \dot{\theta}_i^2 v_i. \qquad (37)$$



Example 7.6

Consider a planar 4R linkage similar to the one in the figure whose configuration is



Given

Parameter	Dimension
a_1	8
a_2	14
a_3	9
a_4	17
θ_1	60°
$\dot{\theta}_1$	10 rads/sec
$\ddot{\theta}_1$	100 rads/sec^2

Determine

θ_2	θ_2'	θ_4
$\dot{\theta}_2$	ω_2	$\dot{\theta}_4$
$\ddot{\theta}_2$	α_2	$\ddot{\theta}_4$

Example 7.6 Solution

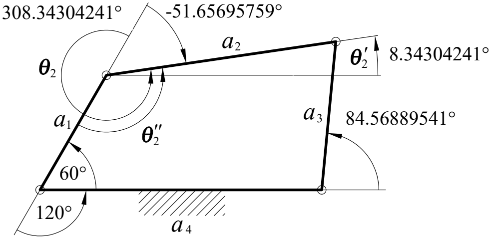
- To determine θ_2' we can start by determining θ_2 using Equation (17), the v_1-v_2 equation.
- Maple provides the following solution:

$$v_2 = -0.4840, \quad -1.8988 \Rightarrow \theta_2 = -51.6569^\circ, \quad -124.4528^\circ$$

- We can choose to represent θ_2 as a positive value, so we can add 360° to the negative angles giving

$$\theta_2 = 308.3430^\circ, \quad 235.5472^\circ$$

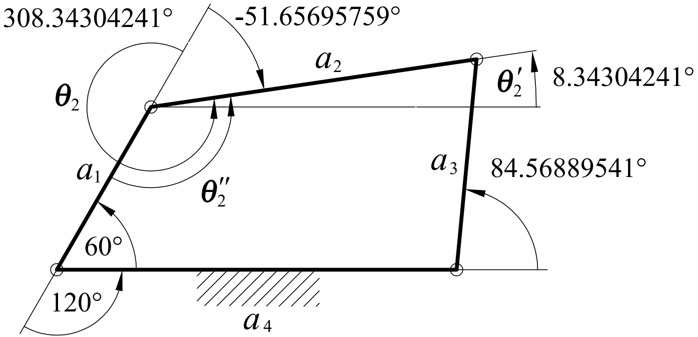
- Let's select the first value for θ_2 and construct a scaled drawing for reference, which looks like



- To compute θ_2', consider the figure which is drawn to scale.

Example 7.6 Solution

- The interior angle between a_1 and a_2 is 128.3430°.



- We additionally observe that the angle between a_1 and horizontal reference line from which θ_2' is measured, is 120°.
- Hence,

$$\begin{aligned} \theta_2'' &= 308.3430^\circ - 180^\circ &= 128.3430^\circ, \\ \theta_2' &= 128.3430^\circ - 120^\circ &= 8.3430^\circ \end{aligned}$$

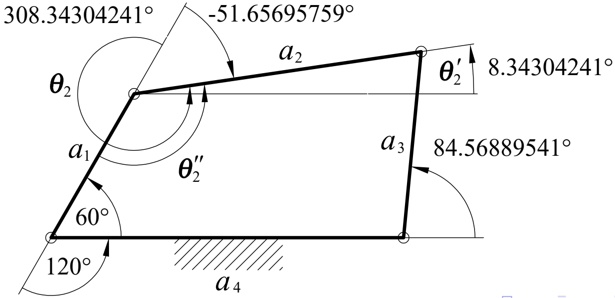
Example 7.6 Solution

- To determine θ_4 we can use Equation (16), the v_1-v_4 equation.
- Maple provides the following solution:

$$v_4 = -2.7989, \quad 0.9094 \Rightarrow \theta_4 = -140.6787^\circ, \quad 84.5689^\circ$$

- Since we have chosen the upper assembly mode then θ_4 must be the second value

$$\theta_4 = 84.5689^\circ.$$

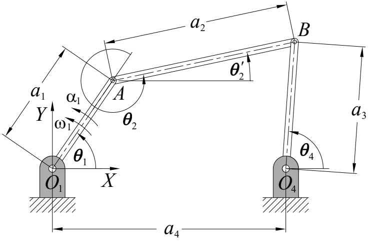


Example 7.6 Solution

- To determine $\dot{\theta}_4$ we can use Equation (22), the $\dot{\theta}_1$-$\dot{\theta}_4$ angular velocity ratio.
- After substituting all known quantities we directly obtain

$$\dot{v}_4 = 6.5574 \Rightarrow \dot{\theta}_4 = \omega_4 = 7.1781 \text{ rad/s}$$

- Compare this to the traditional trigonometric method for determining ω_4 with an observer attached to the coupler a_2 at Point A who only translates with a_2, but does not rotate.



$$\vec{V}_B = \vec{\omega}_4 \times \vec{a}_3 = \vec{V}_A + \vec{V}_{B/A}$$
$$\vec{V}_{B/A} = \vec{\omega}_2 \times \vec{a}_2$$

The x and y components of the vectors yield two linearly independent equations in terms of $\vec{\omega}_2$ and $\vec{\omega}_4$ which can be solved simultaneously giving

$$\omega_2 = -2.4463 \text{ rad/s}, \quad \omega_4 = 7.1781 \text{ rad/s}$$

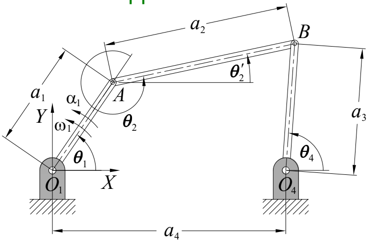
Example 7.6 Solution

- To determine $\dot{\theta}_2$ we can use Equation (23), the $\dot{\theta}_1$-$\dot{\theta}_2$ angular velocity ratio.
- After substituting all known quantities we directly obtain

$$\dot{v}_2 = -7.6811 \Rightarrow \dot{\theta}_2 = -12.4463 \text{ rad/s}$$

and $\omega_2 = \dot{\theta}_2 + \dot{\theta}_1 = -2.4463$ rad/s

- Compare this to the traditional trigonometric method for determining ω_4 with an observer attached to a_1 at Point A who translates and rotates with a_1, and observes the motion of a_2 who perceives the effects of the angular velocity of a_1 as a component of the angular velocity of a_2, but in the opposite sense.



$$\vec{V}_B = \vec{\omega}_4 \times \vec{a}_3 = \vec{V}_A + \vec{V}_{B/A}$$
$$\vec{V}_{B/A} = \vec{\Omega}_1 \times \vec{a}_2 + \vec{\dot{\theta}}_2 \times \vec{a}_2$$

The x and y components of the vectors yield two linearly independent equations in terms of $\dot{\theta}_2$ and ω_4 which can be solved simultaneously giving

$$\dot{\theta}_2 = -12.4463 \ \text{rad/s}, \ \ \omega_4 = 7.1781 \ \text{rad/s}$$

Example 7.6 Solution

- To determine $\ddot{\theta}_4$ we can use Equation (28), the $\ddot{v}_1$-$\ddot{v}_4$ angular acceleration parameter equation.
- After substituting all known quantities we directly obtain

$$\ddot{v}_4 = 157.4654 \Rightarrow \ddot{\theta}_4 = \alpha_4 = 125.5109 \text{ rad/s}^2$$

- Compare this to the traditional trigonometric method for determining α_4 using a translating observer attached to Point A on the coupler a_2

$$\begin{aligned} \vec{A}_B &= \vec{A}_A + \vec{A}_{B/A}, \\ \vec{\omega}_4 \times \vec{V}_B + \vec{\alpha}_4 \times \vec{a}_3 &= \vec{\omega}_1 \times \vec{V}_A + \vec{\alpha}_1 \times \vec{a}_1 + \vec{\omega}_2 \times \vec{V}_{B/A} + \vec{\alpha}_2 \times \vec{a}_2 \end{aligned}$$

- The x and y components of the vectors yield two linearly independent equations in terms of $\vec{\alpha}_2$ and $\vec{\alpha}_4$ which can be solved simultaneously yielding the angular accelerations relative to the non-moving coordinate system:

$$\alpha_2 = -3.5916 \text{ rad/s}^2, \quad \alpha_4 = 125.5109 \text{ rad/s}^2$$

Example 7.6 Solution

- To determine $\ddot{\theta}_2$ we can use Equation (29), the $\ddot{v}_1$-$\ddot{v}_2$ angular acceleration parameter equation.
- After substituting all known quantities we directly obtain

$$\ddot{v}_2 = -110.2051 \Rightarrow \ddot{\theta}_2 = -103.5916 \text{ rad/s}^2$$

and $\alpha_2 = \ddot{\theta}_2 + \ddot{\theta}_1 = -3.5916$ rad/s^2

- Compare this to the traditional trigonometric method for determining α_2 using a translating and rotating observer attached to Point A on a_1

$$\begin{aligned}\vec{A}_B &= \vec{A}_A + \vec{A}_{B/A},\\ (\vec{\omega}_4 \times \vec{V}_B + \vec{\alpha}_4 \times \vec{a}_3) &= (\vec{\omega}_1 \times \vec{V}_A + \vec{\alpha}_1 \times \vec{a}_1) + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{a}_2) +\\ &\quad 2(\vec{\omega}_1 \times (\vec{\dot{\theta}}_2 \times \vec{a}_2)) + \vec{\dot{\theta}}_2 \times (\vec{\dot{\theta}}_2 \times \vec{a}_2) + \vec{\ddot{\theta}}_2 \times \vec{a}_2\end{aligned}$$

- The x and y components of the vectors yield two linearly independent equations in terms of $\vec{\alpha}_2$ and $\vec{\alpha}_4$ which can be solved simultaneously yielding the angular accelerations relative to the non-moving coordinate system:

$$\ddot{\theta}_2 = -103.5916 \text{ rad/s}^2, \quad \alpha_4 = 125.5109 \text{ rad/s}^2$$

Computing Extreme Angular Accelerations and Critical Input Angles

- Now we will compute the critical values of the input angle required for the extreme values of output angular accelerations given a specified set of feasible link lengths and specified constant value for the input angular velocity $\dot{\theta}_i$.
- This method works for any of the six IO angular acceleration equations, but we will illustrate the method using the v_1-v_4 equation listed as Equation (16). However, as for computing extreme values of the output angular velocity, the equations become quite large so some computational software, like Maple, is needed.

1. Convert the two variable angle parameters v_1 and v_4 to angles as $v_i = \tan \theta_i/2$ and solve for θ_4, there will be two solutions.
2. Substitute each expression for θ_4 into the $\dot{\theta}_1$-$\dot{\theta}_4$ equation and solve the resulting for $\dot{\theta}_4$.
3. Substitute the expressions for θ_4 and $\dot{\theta}_4$ into the $\ddot{v}_1$-$\ddot{v}_4$ equation after you have substituted $v_i = \tan(\theta_i/2)$ and Equations (34) and (36).
4. Solve the resulting equation for $\ddot{\theta}_4$ and substitute the specified value for $\dot{\theta}_1$. You now have an equation for $\ddot{\theta}_4$ in terms of θ_1 as the only variable.

Computing Maximum Angular Accelerations and Critical Input Angles

5. Because $\dot{\theta}_1$ is constant means that $\ddot{\theta}_1 = 0$, so the equation determined in Step 4 is

$$\ddot{\theta}_4 = f(\theta_1)$$

6. To determine the critical values of the input angle $\theta_{1_{\mathrm{crit}}}$ where the output angular acceleration is an extreme value, either the greatest positive or negative value, determine the derivative of $\ddot{\theta}_4 = f(\theta_1)$ with respect to θ_1.
7. The extreme values of $\ddot{\theta}_4$ occur at the values of $\theta_{1_{\mathrm{crit}}}$ that cause the following equation to be satisfied:

$$\frac{d\left(\ddot{\theta}_4 = f(\theta_1)\right)}{d\theta_1} = 0.$$

8. If you obtain the solution using Maple, you will be faced with four real values for $\theta_{1_{\mathrm{crit}}}$, and 18 complex solutions.
9. Determine the corresponding values for $\ddot{\theta}_{4_{\mathrm{max/min}}}$ and you should see that there are only two distinct values, one corresponding to a pair of values for $\theta_{1_{\mathrm{crit}}}$, the other pair corresponding to the other extreme value of $\ddot{\theta}_4$.

Example 7.7

- We will now continue with Example 6.9 in Lecture Slide Set 6
- Consider a planar 4R mechanisms where the following has been specified

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 6 \\ a_3 &= 7 \\ a_4 &= 5 \\ \dot{\theta}_1 &= 10 \text{ rad/sec, constant} \end{aligned}$$

Determine $\theta_{1_{\text{crit}}}$ and the extreme values of $\ddot{\theta}_4$ in both assembly modes of the mechanism.

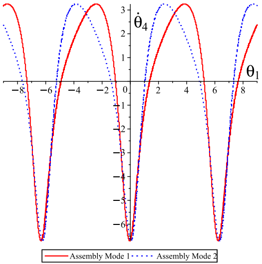
SOLUTION

1. The specified link lengths mean that the mechanism is a Grashof crank-rocker.
2. When the substitution $v_i = \tan\theta_i/2$ has been made in Equation (16) the resulting θ_1-θ_4 equation is

$$\left(-36\left(\tan\frac{\theta_4}{2}\right)^2+160\right)\left(\tan\frac{\theta_1}{2}\right)^2-\left(112\tan\frac{\theta_1}{2}\tan\frac{\theta_4}{2}\right)-20\left(\tan\frac{\theta_4}{2}\right)^2+64=0. \quad (38)$$

Example 7.7 Continued

3. Solve Equation (38) for θ_4 and substitute into the $\dot{\theta}_1$-$\dot{\theta}_4$ equation, then solve this equation for $\dot{\theta}_4$ leading to the angular velocity profile as $\dot{\theta}_4 = f(\theta_1)$



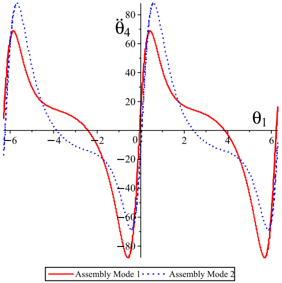
4. We can see that the velocity profile curve for Assembly Mode 2, plotted with the dashed line-type, is reflected in the vertical $\dot{\theta}_4$-axis.

5. There are only two distinct real critical values for θ_1 and corresponding extreme values for $\dot{\theta}_4$ in Assembly Modes 1 and 2:

$$\begin{aligned} \theta_{1_{\mathrm{crit}_1}} &= 2.6810^\circ, & \dot{\theta}_{4_{\mathrm{min}}} &= -6.6959 \text{ rad/s} \\ \theta_{1_{\mathrm{crit}_2}} &= 138.9638^\circ, & \dot{\theta}_{4_{\mathrm{max}}} &= 3.2493 \text{ rad/s} \end{aligned}$$

Example 7.7 Continued

6. Using Equation (28), the $\ddot{v}_1$-$\ddot{v}_4$ equation, we similarly obtain the angular acceleration profile in both assembly modes as $\ddot{\theta}_4 = f(\theta_1)$ with $\ddot{\theta}_4$ in the vertical axis and θ_1 on the horizontal axis in radians



7. We now follow the same method as for the extreme values of $\dot{\theta}_4$.
8. Substitute the following into the $\ddot{v}_1$-$\ddot{v}_4$ equation in the order given

$$\dot{\theta}_i = \frac{2\dot{v}_i}{(1+v_i^2)}, \quad \ddot{\theta}_i = \frac{2\ddot{v}_i}{(1+v_i^2)} - \dot{\theta}_i^2 v_i, \quad v_i = \tan(\theta_i/2).$$

Example 7.7 Assembly Mode 1

9. Solve the resulting equation for $\ddot{\theta}_4$ giving $\ddot{\theta}_4 = f(\theta_1)$
10. Take the derivative of this equation with respect to θ_1 and solve for the critical values of $\theta_{1_{\text{crit}}}$ that cause the following to be true

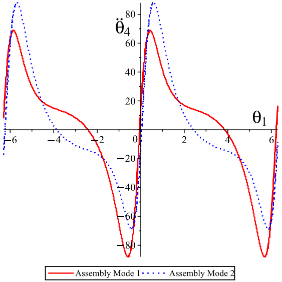
$$\frac{d\left(\ddot{\theta}_4 = f(\theta_1)\right)}{d\theta_1} = 0.$$

11. The resulting equation in Maple has degree 22, leading to four real values for $\theta_{1_{\text{crit}}}$, but as for the extreme values of the output angular velocity, only two of the solutions correspond to the largest positive and negative values of $\ddot{\theta}_4$ for Assembly Mode 1:

$$\begin{aligned}
\theta_{1_{\text{crit}_1}} &= 24.6475^\circ, & \ddot{\theta}_{4_{\max}} &= 68.7731 \text{ rad/s}^2 \\
\theta_{1_{\text{crit}_2}} &= 326.1382^\circ, & \ddot{\theta}_{4_{\min}} &= \text{-}87.9671 \text{ rad/s}^2 \\
\theta_{1_{\text{crit}_3}} &= \text{-}33.8618^\circ, & \ddot{\theta}_{4_{\min}} &= \text{-}87.9671 \text{ rad/s}^2 \\
\theta_{1_{\text{crit}_4}} &= \text{-}335.3525^\circ, & \ddot{\theta}_{4_{\max}} &= 68.7731 \text{ rad/s}^2
\end{aligned}$$

Example 7.7 Assembly Mode 1

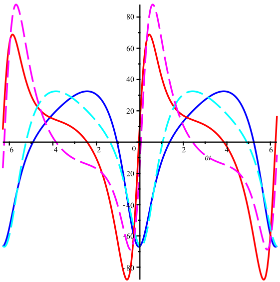
11. It is clear that two of the four real solutions are repeated. This is an artifact of the range selected for the input angle, and the fact that the $\dot{\theta}_4 = f(\theta_1)$ equation is periodic.



12. There are only two distinct real critical values for θ_1 and corresponding extreme values for $\ddot{\theta}_4$ in Assembly Mode 1:

$$\begin{aligned} \theta_{1_{\mathrm{crit}_1}} &= 24.6475^\circ, & \ddot{\theta}_{4_{\max}} &= 68.7731 \text{ rad/s}^2 \\ \theta_{1_{\mathrm{crit}_2}} &= 326.1382^\circ, & \ddot{\theta}_{4_{\min}} &= \text{-}87.9671 \text{ rad/s}^2 \end{aligned}$$

Example 7.7 Assembly Mode 2



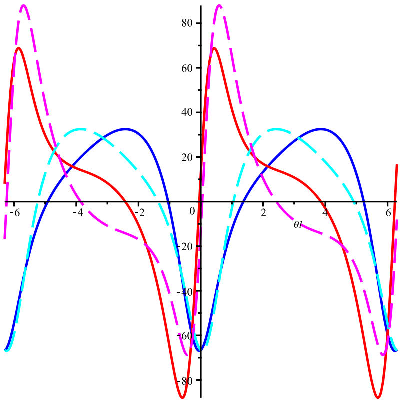
13. There are only two distinct real critical values for θ_1 and corresponding extreme values for $\ddot{\theta}_4$ in Assembly Mode 2:

$$\begin{aligned} \theta_{1_{\mathrm{crit}_1}} &= 33.8616^\circ, & \ddot{\theta}_{4_{\max}} &= 87.9671 \text{ rad/s}^2 \\ \theta_{1_{\mathrm{crit}_2}} &= 335.3525^\circ, & \ddot{\theta}_{4_{\min}} &= \text{-}68.7731 \text{ rad/s}^2 \end{aligned}$$

14. The amplitude of the angular velocity profile has been magnified and superimposed on the angular acceleration profile.

15. It is to be seen that at extreme values of $\dot{\theta}_4$ we have $\ddot{\theta}_4 = 0$.

Example 7.7 Continued



16. It is not clear why the values of $\theta_{1_{\text{crit}}}$ or the extreme values of $\ddot{\theta}_4$ occur in the identified configurations.

17. More research here is required to fully understand these new results.