# MAAE 3004
# Dynamics of Machinery

## Lecture Slide Set 8

## Force Analysis of Mechanisms

Department of Mechanical and Aerospace Engineering
Carleton University

# Outline

# Introduction

In the design of mechanisms and machines, the required motions are often specified first. The dynamics task is essentially determining those forces that must be applied and/or those forces that accompany the motions

The required process is to:

1. Determine masses, centres of mass, and mass moments of inertia, $I_G$, of the links
2. Perform kinematic analysis to determine linear and angular accelerations of all links (evaluate $\vec{a}_G$ at the centre of mass for all links as well as $\vec{\alpha}$)
3. Perform force analysis either by:
   - graphical analysis and superposition, or by
   - analytical force balances and matrix solution

# Preliminaries

## Newton's Laws

For each link in a mechanism, $\vec{a}_G$ and $\vec{\alpha}$ are found from kinematics where $G$ indicates the centre of mass

Newton's laws state:

- The actions of two bodies on each other are always equal and directly opposite
- The change in motion of a body is proportional to the moving force impressed upon it and occurs in the direction of the impressed force

$$\sum \vec{F} = m\vec{a}_G \quad \text{(translation)} \tag{1}$$

and

$$\underbrace{\sum \vec{M}_G}_{\substack{\text{net moment} \\ \text{about CM}}} = \underbrace{I_G}_{\substack{\text{mass moment} \\ \text{of inertia about the CM}}} \vec{\alpha} \quad \text{(rotation)} \tag{2}$$

Newton's laws are vector equations implying both magnitude and direction.

# D'Alembert's Principle

D'Alembert's principle states

*The vector sum of all external forces and inertia forces acting on a system of rigid bodies is zero.*

$$\sum \vec{F}_{applied} - m\vec{a}_G = 0 \tag{3}$$

*The vector sum of all external moments and inertia moments acting on a system of rigid bodies is also separately zero*

$$\sum \vec{M}_{Gapplied} - I_G\vec{\alpha} = 0 \tag{4}$$

The inertia force $\vec{F}_I$ and inertia moment $\vec{M}_I$ are defined such that

$$\sum \vec{F} + \vec{F}_I = 0 \quad \Rightarrow \quad \vec{F}_I = -m\vec{a}_G \tag{5}$$

$$\sum \vec{M} + \vec{M}_I = 0 \quad \Rightarrow \quad \vec{M}_I = -I_G\vec{\alpha} \tag{6}$$

Therefore

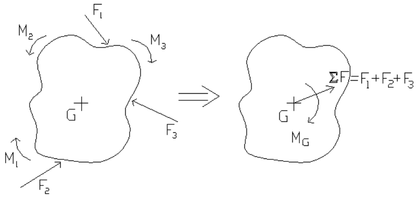
$$\sum \vec{F} = 0 \quad \text{(including } \vec{F}_I\text{)} \tag{7}$$

$$\sum \vec{M} = 0 \quad \text{(including } \vec{M}_I\text{)} \tag{8}$$

D'Alembert's principle allows dynamic problems to be treated similarly to static problems (i.e., quasi-static)

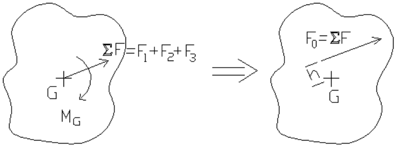
# Combining $\vec{F}_I$ and $\vec{M}_I$ into One Offset Force

The combined effect of all forces and moments acting on a rigid body can be represented by a single equipollent force and a single equipollent moment acting about the centre of mass




The equipollent force and moment can be replaced by a single offset force $\vec{F}_o$ that has the same magnitude and direction as $\Sigma\vec{F}$ but is shifted by $h$ to cause the same effect as $\vec{M}_G$

$$h = \frac{|\vec{M}_G|}{|\vec{F}_O|} \tag{9}$$




The offset direction is chosen such that the offset force results in the same moment direction as $\vec{M}_G$

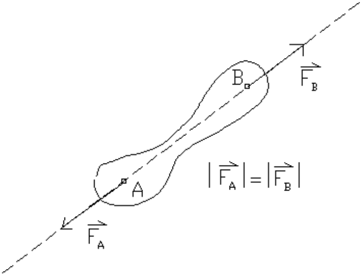
# Mass and Weight

- Mass
  - Defines the inertia of a system
  - Is unrelated to gravity
  - Should be considered when large mass and or accelerations (related to operating speeds) cause significant inertia forces
- Weight
  - Defines the force exerted by gravity such that $W = mg$ where $m$ is the link mass and $|g| = 9.8\ \mathrm{m/s^2} = 32.2\ \mathrm{ft/s^2}$
  - Is a static load
  - Should be considered when it is significant and acts in a relevant direction

# Forces Acting on Links

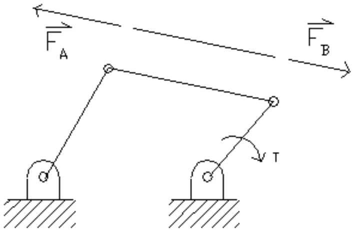
## Two-Force Members

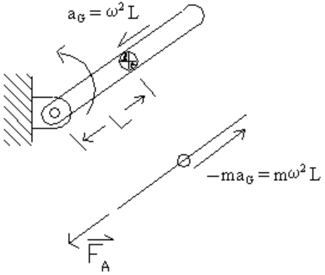
A two-force member has only 2 significant forces acting on it




- $m$ and $I$ often may be ignored due to small values
- the two forces must be colinear to satisfy $\sum \vec{M} = 0$ and must act along the line joining their points of application
- the two forces must have equal magnitudes but opposite directions to satisfy $\sum \vec{F} = 0$
- One of the two forces could be the inertia force.

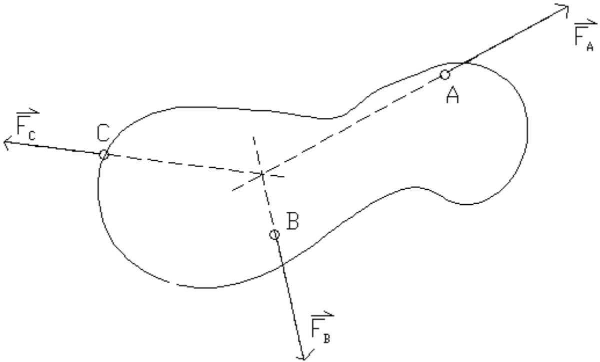
Other examples of two-force members:



## Three-Force Members

A three-force member has three significant forces acting on it:

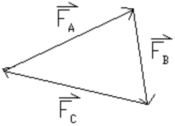



- the lines of action of all three forces must intersect at a point (necessary to satisfy $\sum \vec{M} = 0$)

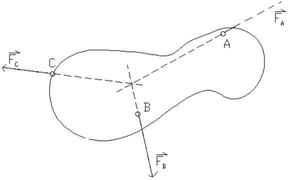
- the sum of the applied forces must balance (i.e., $\sum \vec{F} = 0$ including the inertia force)

  For the system shown below

$$\sum \vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C = 0 \qquad (10)$$

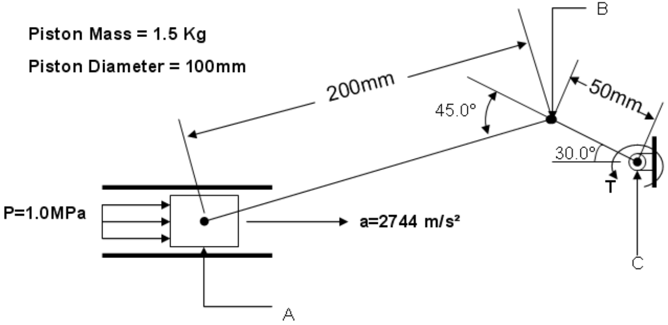



- two unknown force magnitudes can be solved for, provided the magnitude and direction of one force (e.g., the inertia force) and directions of the other two forces are known.

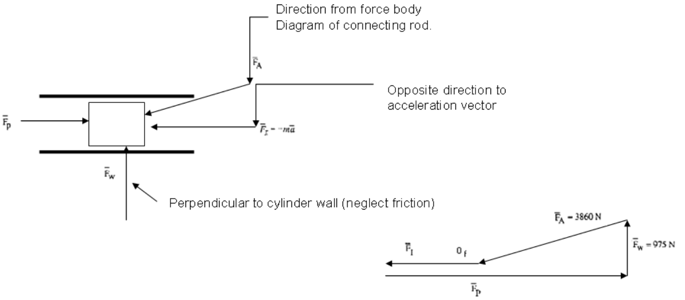


# Example 8.1
## Offset-Crank Engine

Determine the output torque $\tau$ that can be developed by the offset-crank engine in the configuration shown using graphical force analysis. Ignore the inertia force and moment acting on the connecting rod and crank.



# 1. Piston (three-force member)




$$|\vec{F_I}| = ma_G = (1.5kg)(2744m/s^2) = 4116 \text{ N}$$

$$|\vec{F_P}| = PA = (1.0x10^6 \frac{N}{m^2})(\frac{\pi(0.1)^2}{4}) = 7854 \text{ N}$$

since two known applied forces can be combined into a single force, then piston becomes a three-force member

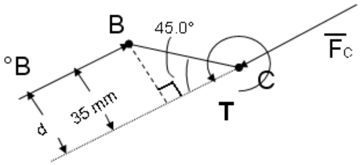
$$(\overset{\surd\surd}{\vec{F_I}} + \overset{\surd\surd}{\vec{F_P}}) + \overset{o\surd}{\vec{F_W}} + \overset{o\surd}{\vec{F_A}} = \overset{\surd\surd}{\vec{0}}$$

then from the force polygon $|\vec{F_A}| = 3860$ N; direction as shown

2. Connecting Rod (two-force member neglecting inertia)


- $\vec{F}_B = -\vec{F}_A \qquad (\overset{\surd\surd}{\vec{F}_A} + \overset{o\surd}{\vec{F}_B} = \overset{\surd\surd}{\vec{0}}\ )$
- $|\vec{F}_B| = 3860$ N; direction as shown

3. Crank




- $\Sigma \vec{F} = m\vec{a} \Rightarrow \Sigma \vec{F} = 0$

  $\vec{F_C} = -\vec{F_B}$

- $\Sigma \vec{M_C} = I_G \vec{\alpha} = 0$

  $\tau - dF_B = 0$

  where d is the perpendicular distance from $F_B$ to a parallel line passing through point C, then

  $\tau = dF_B = (35\text{mm})(3860\text{N}) = 135.1\text{Nm ccw}$

# Analytical Force Balances and Matrix Solution

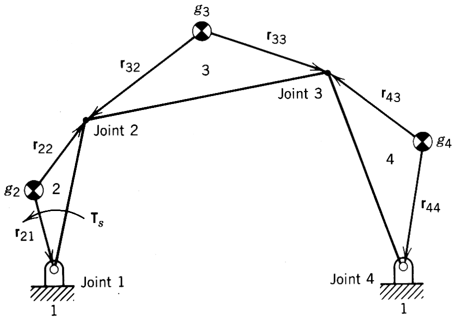
The general procedure with this approach is to:

1. Draw free body diagrams for each link;
2. Generate equilibrium equations for each link;
3. Assemble all equilibrium equations into a single linear matrix equation; and
4. Solve the matrix equation for all unknown force components.

Note that for *static* moment equilibrium, any point can be used as the reference point; with *dynamic* equilibrium, in general, moments should be summed about the link centres of mass.

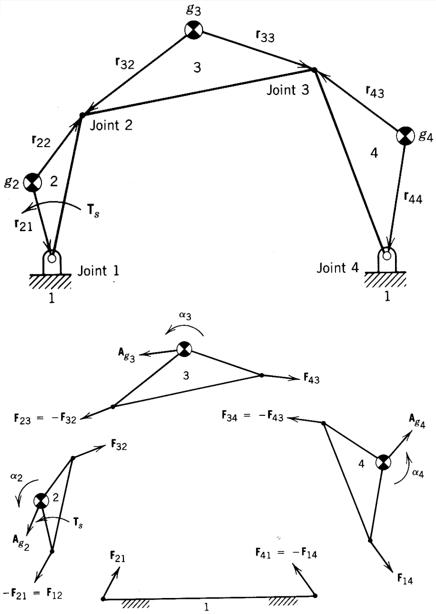
# Example 8.3

Outline how the matrix solution method can be used to determine the required applied driving torque $\mathbf{T}_s$ and corresponding joint reaction forces in the four-bar mechanism for a set of specified linear and angular accelerations of each of the links determined by a previous kinematic analysis of the linkage.

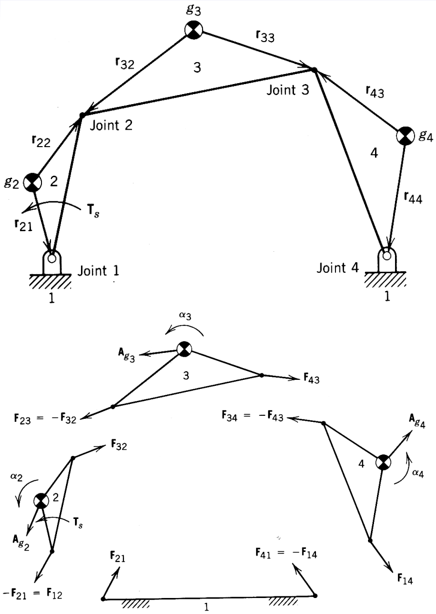



Solution Method:

1. Draw free-body diagrams
2. Write equilibrium equations
3. Form matrix equation
4. Solve

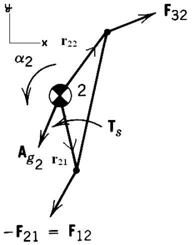



- $\mathbf{r}_{ij}$ vector from CG of link $i$ to joint $j$
- $\mathbf{F}_{ik}$ force link $i$ exerts on link $k$
- $g_i$ CG of link $i$
- $\mathbf{A}_{g_i}$ acceleration of CG $g_i$
- $\alpha_i$ angular acceleration of link $i$
- $M_i$ mass of link $i$
- $I_i$ mass moment of inertia of link $i$ about $g_i$
- $\mathbf{T}_s$ driving torque applied to link 2




- $\mathbf{F}_{ik} = -\mathbf{F}_{ki}$
- An independent set of solution variables must be chosen.
- $\mathbf{F}_{14}$, $\mathbf{F}_{21}$, $\mathbf{F}_{32}$, and $\mathbf{F}_{43}$ are selected as solution variables in this example.
- Other selections are possible.

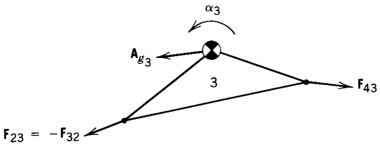
Equilibrium equations are written for link 2.




$$F_{32x} - F_{21x} = M_2 A_{g_2 x}$$
$$F_{32y} - F_{21y} = M_2 A_{g_2 y}$$

$$r_{22x} F_{32y} - r_{22y} F_{32x} - r_{21x} F_{21y}$$
$$+ r_{21y} F_{21x} + T_s = I_2 \alpha_2$$

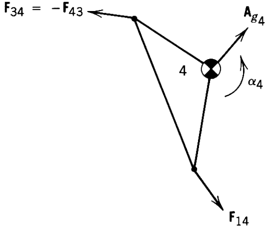
Equilibrium equations are written for link 3.




$$F_{43x} - F_{32x} = M_3 A_{g_3 x}$$
$$F_{43y} - F_{32y} = M_3 A_{g_3 y}$$

$$r_{33x} F_{43y} - r_{33y} F_{43x} - r_{32x} F_{32y} + r_{32y} F_{32x} = I_3 \alpha_3$$

Equilibrium equations are written for link 4.




$$F_{14x} - F_{43x} = M_4 A_{g_4 x}$$
$$F_{14y} - F_{43y} = M_4 A_{g_4 y}$$

$$r_{44x} F_{14y} - r_{44y} F_{14x} - r_{43x} F_{43y}$$
$$+ r_{43y} F_{43x} = I_4 \alpha_4$$

Each of the nine equilibrium equations are combined into a single 9 by 9 matrix equation where the unknowns are the eight unknown joint reaction force components and the applied torque $\mathbf{T}_s$.

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ r_{21y} & -r_{21x} & -r_{22y} & r_{22x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & r_{32y} & -r_{32x} & -r_{33y} & r_{33x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & r_{43y} & -r_{43x} & -r_{44y} & r_{44x} & 0 \end{bmatrix} \begin{Bmatrix} F_{21x} \\ F_{21y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ T_s \end{Bmatrix} = \begin{Bmatrix} M_2 A_{g_2 x} \\ M_2 A_{g_2 y} \\ I_2 \alpha_2 \\ M_3 A_{g_3 x} \\ M_3 A_{g_3 y} \\ I_3 \alpha_3 \\ M_4 A_{g_4 x} \\ M_4 A_{g_4 y} \\ I_4 \alpha_4 \end{Bmatrix}$$

This equation can be solved either numerically or symbolically as it is in the familiar form for simultaneous linear equations

$$[A]\{x\} = \{b\}$$

and

$$\{x\} = [A]^{-1}\{b\}$$