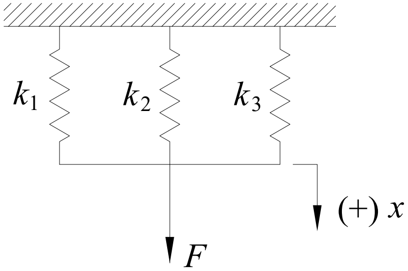
MAAE 3004
Dynamics of Machinery

Review Lecture

Department of Mechanical and Aerospace Engineering
Carleton University

Equivalent Springs

Springs in parallel



- The springs in a mechanical system can be in parallel, series, or in combination.
- When springs are in parallel, the deformation of each spring is the same for a given applied force.
- The reaction forces of the three springs are

$$\begin{aligned} F_1 &= k_1 x \\ F_2 &= k_2 x \\ F_3 &= k_3 x \end{aligned}$$

- The sum of these three forces must be equal in magnitude to the applied force, therefore

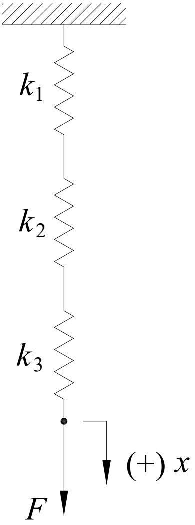
$$F = k_1 x + k_2 x + k_3 x = (k_1 + k_2 + k_3)x = k_{eq} x$$

- For n springs in parallel the equivalent spring constant is

$$k_{eq} = \sum_{i=1}^{n} k_i$$

Equivalent Springs

Springs in series



- When springs are in series, the force in each spring is the same as the given applied force.
- The total deformation x of the springs is the sum of the individual deformations.
- Thus, with

$$F = k_1 x_1 = k_2 x_2 = k_3 x_3$$

and

$$x = x_1 + x_2 + x_3$$

we find that

$$x = F\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)$$

- The equivalent spring constant for springs in series is

$$k_{eq} = \frac{1}{\sum_{i=1}^{n} \frac{1}{k_i}}, \quad \text{or} \quad \frac{1}{k_{eq}} = \sum_{i=1}^{n} \frac{1}{k_i}$$

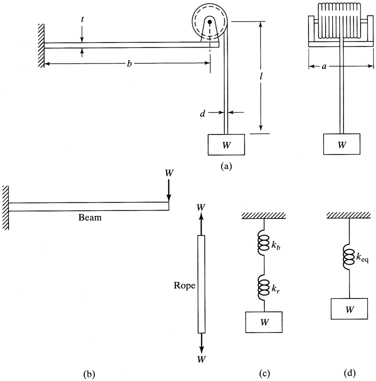
Example 1.1

Given the hoisting drum that is mounted at the end of a rectangular cross-section cantilever beam and carrying a steel wire cable, determine the k_{eq} of the system. The cable length $= \mathrm{l}$ and the beam and cable have a Young's modulus $=$ E.

For a cantilever beam:

$$\delta_{max} = \frac{Wb^3}{3EI} \Rightarrow k_b = \frac{W}{\delta_{max}}$$

$$k_b = \frac{3EI}{b^3} = \frac{3E}{b^3}\left(\frac{1}{12}at^3\right) = \frac{Eat^3}{4b^3}$$



For a cable: $k_c = \dfrac{AE}{\mathrm{l}} = \dfrac{\pi d^2 E}{4\mathrm{l}}$

k_b and k_c are in series,

$$\frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_c} = \frac{4b^3}{Eat^3} + \frac{4\mathrm{l}}{\pi d^2 E}$$

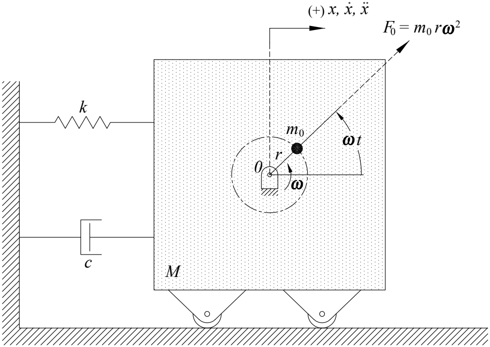
Therefore, $k_{eq} = \dfrac{E}{4}\left(\dfrac{\pi a t^3 d^2}{\pi d^2 b^3 + \mathrm{lat}^3}\right)$

Excitation Due to an Unbalanced Rotating Mass

- To illustrate how to use the real part of $e^{i\omega t}$, consider the horizontal mass M rolling on a horizontal surface that is excited by a rotating unbalanced mass m_0
- The differential equation of motion of this system is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\cos(\omega t)$$

where $m = M + m_0$

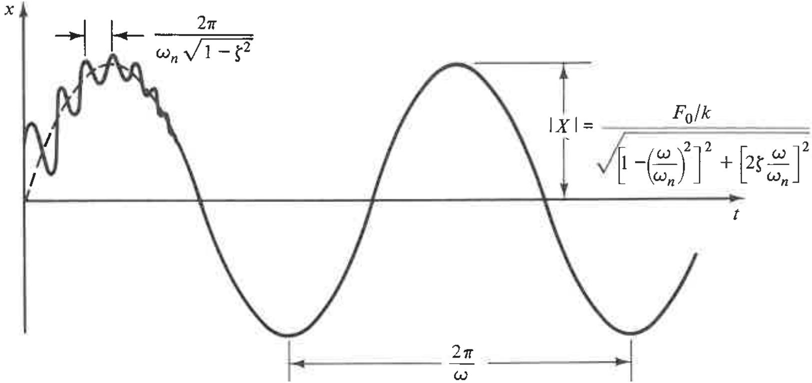


Excitation Due to an Unbalanced Rotating Mass

- Adding the transient homogeneous solution x_h to the steady-state solution x_p gives the *complete* solution of Equation (3)

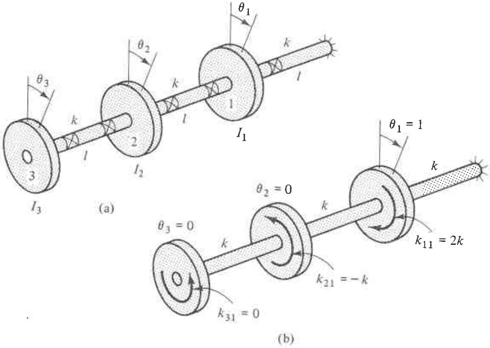
$$x = e^{-\zeta\omega_n t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right) + |X|\sin(\omega t - \phi) \quad (15)$$

- The combined transient and steady-state response is shown for $\omega < \omega_n\sqrt{1-\zeta^2}$, and recall that $\omega_d = \omega_n\sqrt{1-\zeta^2}$



Example 3.4

- A shaft with three evenly spaced discs rigidly attached to it is fixed at one end
- The mass moments of inertia of the discs are I_1, I_2, and I_3
- The torsional spring constant of each of the three shaft intervals of length l is $k = GJ/\mathrm{l}$ where G is the shear modulus and J is the polar mass moment of inertia of the shaft interval

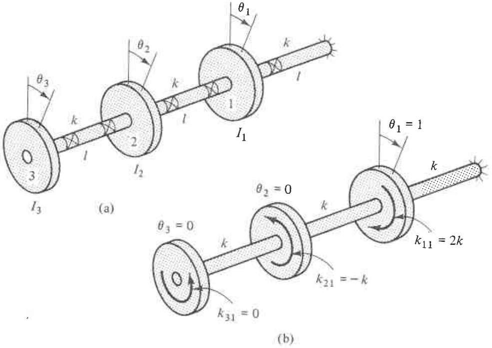


Example 3.4

- Using the angular displacements θ_1, θ_2, and θ_3 as generalised coordinates and the definition of k_{ij} determine

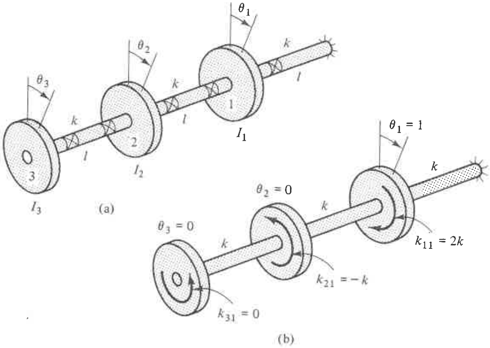
  a. The stiffness matrix $\mathbf{K}$

  b. The differential equations of motion of the undamped free vibration of the system in matrix form



Example 3.4 Solution

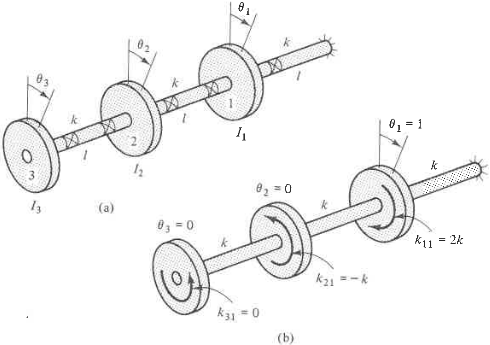
- a. To determine the first column of **K** disc 1 is given a unit angular displacement, $\theta_1 = 1$, with disks 2 and 3 held fixed so that $\theta_2 = \theta_3 = 0$
- The moment k_{11} required to rotate disc 1 through the angle $\theta_1 = 1$ is $2k$, and the moment k_{21} required to keep disc 2 fixed is $-k$
- There is no tendency for disc 3 to rotate, so $k_3 = 0$



Example 3.4 Solution

- Thus

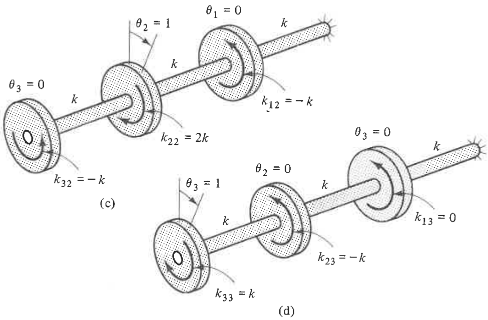
$$\left.\begin{aligned} k_{11} &= 2k \\ k_{21} &= -k \\ k_{31} &= 0 \end{aligned}\right\} \quad \text{first column of } \mathbf{K}$$



Example 3.4 Solution

- Similarly, giving disc 2 a unit angular displacement, $\theta_2 = 1$, with discs 1 and 3 held fixed so that $\theta_1 = \theta_3 = 0$, the moments required to maintain this configuration are

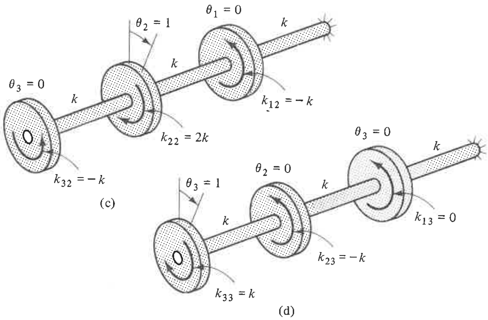
$$\left.\begin{aligned} k_{12} &= -k \\ k_{22} &= 2k \\ k_{32} &= -k \end{aligned}\right\} \quad \text{second column of } \mathbf{K}$$



Example 3.4 Solution

- Finally, with $\theta_3 = 1$ and discs 1 and 2 held fixed so that $\theta_1 = \theta_3 = 0$, the moments required to maintain this configuration are

$$\left.\begin{aligned} k_{13} &= 0 \\ k_{23} &= -k \\ k_{33} &= k \end{aligned}\right\} \quad \text{third column of } \mathbf{K}$$



Example 3.4 Solution

- b. The general form of the matrix equation expressing the equations of motion of the undamped free vibration is

$$\mathbf{M}\ddot{\mathbf{\Theta}} + \mathbf{K}\mathbf{\Theta} = \mathbf{0}$$

- Since the mass matrix **M** consists of the mass moments of inertia on the diagonal, the matrix equation expressing the differential equations of motion of the undamped free vibration of the disc-and-shaft mechanical system is

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \mathbf{0}$$

- The differential equations of motion for this disc-and-shaft mechanical system is analogous to the three story building from Example 3.3

Example 3.7

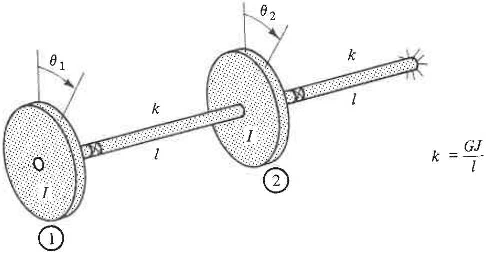
- Two identical discs each having centroidal mass moment of inertia I are rigidly attached to a steel shaft that is fixed at one end
- Each section of shaft has diameter d, segment length l, and a torsional spring constant k, where

$$\mathrm{l} = 0.6\ m \quad \text{(length of each segment of shaft)}$$

$$d = 30.0\ mm \quad \text{(shaft diameter)}$$

$$G = 800.0\ GPa \quad \text{(shaft shear modulus)}$$

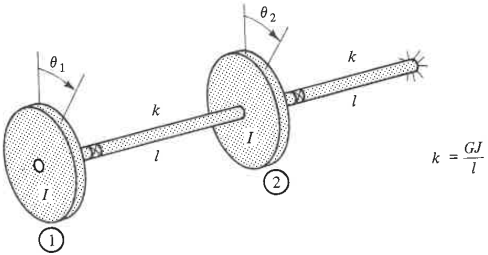
$$\mathrm{I} = 10.0\ kgm^2 \quad \text{(centroidal mass moment of inertia of each disc)}$$



Example 3.7 Solution

- Using the given system data, determine:
  a. the eigenvalues of the system;
  b. the natural frequencies of the system [Hz];
  c. the eigenvectors, i.e., the normal-mode shapes, of the system
- a. k_{11} is the moment required to give disc 1 a unit rotation, $\theta_1 = 1$
- k_{21} is the moment required to keep disc 2 stationary when k_{11} is applied, i.e., for $\theta_1 = 1$ and $\theta_2 = 0$

$$\begin{aligned} k_{11} &= k \\ k_{21} &= -k \end{aligned}$$



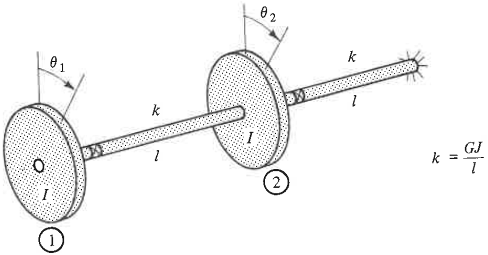
Example 3.7 Solution

- Similarly, for $\theta_1 = 0$ and $\theta_2 = 1$

$$\begin{aligned} k_{12} &= -k \\ k_{22} &= 2k \end{aligned}$$

- The stiffness and mass matrices are

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}$$



Example 3.7 Solution

- To determine the eigenvalues, we must first compute the characteristic (frequency) equation using the determinant of coefficient matrix from Equation (17)
- After setting $\omega^2 = \lambda$ we obtain

$$\det\left[\mathbf{M}^{-1}\mathbf{K} - \lambda\mathbf{I}\right] = \det\begin{bmatrix} \left(\frac{k}{\mathrm{I}} - \lambda\right) & \frac{-k}{\mathrm{I}} \\ \frac{-k}{\mathrm{I}} & \left(\frac{2k}{\mathrm{I}} - \lambda\right) \end{bmatrix} = 0$$

- The determinant yields the characteristic equation

$$\left(\frac{k}{\mathrm{I}} - \lambda\right)\left(\frac{2k}{\mathrm{I}} - \lambda\right) - \left(\frac{k}{\mathrm{I}}\right)^2 = 0$$

in which $\lambda = \omega^2$

- Expanding this equation leads to the quadratic characteristic equation

$$\lambda^2 - \frac{3k}{\mathrm{I}}\lambda + \left(\frac{k}{\mathrm{I}}\right)^2 = 0$$

Example 3.7 Solution

- The roots of the quadratic characteristic equation yield the two eigenvalues for the 2-DOF mechanical system

$$\lambda_1 = \omega_1^2 = \frac{k}{\mathrm{I}}\left[\frac{3-\sqrt{5}}{2}\right] \quad \text{and} \quad \lambda_2 = \omega_2^2 = \frac{k}{\mathrm{I}}\left[\frac{3+\sqrt{5}}{2}\right]$$

- b. Using the given data and the tables for elastic elements as springs tables in Lecture Slide Set 1, the torsional spring constant is computed as

$$k = \frac{GJ}{\mathrm{l}}, \quad \text{where } J \text{ is the polar area moment of inertia,} \quad J = \frac{\pi d^4}{32}$$

which gives

$$k = \frac{800.0(10)^9\ Pa}{0.6\ m}\left[\frac{\pi(0.030\ m)^4}{32}\right] = 106028.8\ Nm/rad$$

and

$$\frac{k}{\mathrm{I}} = \frac{106028.8\ Nm/rad}{10.0\ kgm^2} = 10602.88\ s^{-2}$$

Example 3.7 Solution

- substituting the value for k/I into the eigenvalue equations yields

$$\lambda_1 = \omega_1^2 = 4049.94\ s^{-2}$$
$$\lambda_2 = \omega_2^2 = 27758.70\ s^{-2}$$

- From which the two natural frequencies are

$$f_1 = \frac{\omega_1}{2\pi} = \frac{\sqrt{4049.94}}{2\pi} = 10.13\ Hz$$
$$f_2 = \frac{\omega_2}{2\pi} = \frac{\sqrt{27758.70}}{2\pi} = 26.52\ Hz$$

Example 3.7 Solution

- c. To determine the eigenvectors, which represent the normal mode shapes of this torsional vibration system, corresponding to each eigenvalue, we use the torsional version of Equation (17) to obtain

$$\begin{bmatrix} \left(\frac{k}{\mathrm{I}} - \lambda\right) & \frac{-k}{\mathrm{I}} \\ \frac{-k}{\mathrm{I}} & \left(\frac{2k}{\mathrm{I}} - \lambda\right) \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Performing the matrix-vector multiplication leads to

$$\left.\begin{aligned} \left(\frac{k}{\mathrm{I}} - \lambda\right)\Theta_1 - \frac{k}{\mathrm{I}}\Theta_2 = 0 \quad \text{(a)} \\ -\frac{k}{\mathrm{I}}\Theta_1 - \left(\frac{2k}{\mathrm{I}} - \lambda\right)\Theta_2 = 0 \quad \text{(b)} \end{aligned}\right\} \tag{19}$$

Example 3.7 Solution

- The relationships between the eigenvector components for the eigenvalues λ_1 and λ_2 can be determined from either Equation (19-a) or Equation (19-b)

- Without loss in generality, we select the first one, (a), to determine the ratio that depends on the eigenvalues

$$\frac{\Theta_2}{\Theta_1} = \frac{k/\mathrm{I} - \lambda}{k/\mathrm{I}} \tag{20}$$

- Substituting the values for λ_1 and λ_2 into Equation (20) gives

$$\left(\frac{\Theta_2}{\Theta_1}\right)_1 = \frac{10602.88 - 4049.94}{10602.88} = 0.62 \quad \text{(the first mode)}$$

and

$$\left(\frac{\Theta_2}{\Theta_1}\right)_2 = \frac{10602.88 - 27758.70}{10602.88} = -1.62 \quad \text{(the second mode)}$$

Example 3.7 Solution

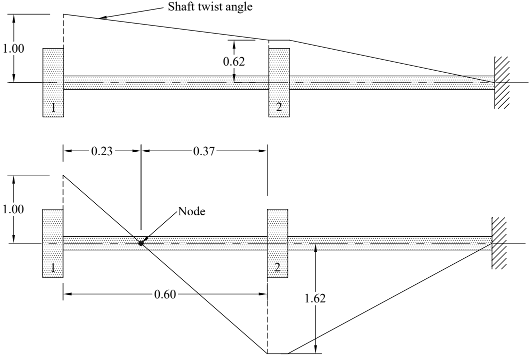
- Although there are an infinite number of values of Θ_1 and Θ_2 that will satisfy the ratios for the first and second mode, we can normalise the ratio by setting $\Theta_1 = 1$, meaning that for the first mode $\Theta_2 = 0.62$, and for the second mode $\Theta_2 = -1.62$, which gives the eigenvectors as

$$\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}_2 = \begin{bmatrix} 1 \\ -1.62 \end{bmatrix}$$

- The eigenvector for the first mode describes the undamped free vibration configuration of the system when it is vibrating at $f_1 = 10.13\ Hz$ where $\Theta_2 = 0.62\Theta_1$ at any instant in time, and since the ratio is positive, both discs are vibrating in phase with each other
- The eigenvector for the second mode describes the system configuration for the undamped free vibration mode when the system is vibrating at $f_2 = 26.52\ Hz$, with the discs vibrating at 180° out of phase with each other because the ratio is a negative number with $\Theta_2 = -1.62\Theta_1$

Example 3.7 Solution

- The figures illustrate the mode shapes, i.e., the twist configurations of the shaft where the vertical axis is the twist angle in radians and the horizontal axis is shaft length

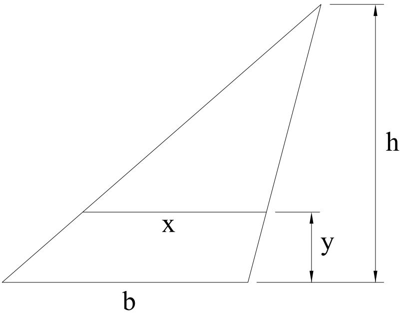


- The *node* for mode shape 2 is where the shaft has zero angular deflection, and the precise location along the length of the shaft segment between the two discs can be determined using similar triangles

Example 3.7 Solution

- In plane geometry, similar triangles are those with identical interior angles and proportional edge lengths
- The edge lengths are related according to the ratios

$$\frac{x}{b} = \frac{h-y}{h}$$



Modal Matrix $\mathbf{U}$

- Although modal analysis is extremely useful in determining the response of forced n-DOF systems, it can also be used in obtaining the free vibration response of systems that comes from initial conditions
- The discussion that follows will examine both free and forced n-DOF vibration response
- The *modal matrix*, $\mathbf{U}$, is required for the modal analysis and decoupling of the differential equations of motion
- The modal matrix is a square $n \times n$ matrix whose columns correspond to the n eigenvectors of the mechanical system where column 1 is mode 1, colum2 is mode 2, et c.
- For example, the 2×2 modal matrix for Example 3.7 where the two eigenvectors were found to be

$$\mathbf{X}_1 = \begin{bmatrix} 1.00 \\ 0.62 \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_2 = \begin{bmatrix} 1.00 \\ -1.62 \end{bmatrix},$$

so the corresponding modal matrix is

$$\mathbf{U} = \begin{bmatrix} 1.00 & 1.00 \\ 0.62 & -1.62 \end{bmatrix}$$

Decoupled Equations for Undamped Free Vibration

- The differential equations of motion of an n-DOF mechanical system are dynamically coupled if the generalised coordinates used lead to a non-diagonal mass matrix $\mathbf{M}$, and statically coupled if the stiffness matrix $\mathbf{K}$ is non-diagonal
- It is often possible to select generalised coordinates that eliminate dynamic coupling, but it is generally not possible to select generalised coordinates that eliminate static coupling
- As a result, as we have seen in all of our examples involving stiffness and flexibility coefficients, static coupling is typically always present in the differential equations of motion and the stiffness matrix $\mathbf{K}$ is symmetric, but not diagonal

Decoupled Equations for Undamped Free Vibration

- To decouple these equations of motion we use the *linear coordinate transformation*

$$\mathbf{X} = \mathbf{U}\boldsymbol{\nu} \tag{41}$$

from which

$$\ddot{\mathbf{X}} = \mathbf{U}\ddot{\boldsymbol{\nu}} \tag{42}$$

- In these equations $\mathbf{X}$ is the vector of x_i generalised coordinates, $\mathbf{U}$ is the modal matrix, while $\boldsymbol{\nu}$ is the vector of principal coordinates ν_i
- The ν_i principal coordinates of points are described in the orthogonal principal coordinate system, while the modal matrix transforms the principal coordinate system into the generalised coordinate system, and the generalised coordinates x_i are the coordinates of the **same** points, but now described in the orthogonal generalised coordinate system
- The ν_i are obtained from

$$\boldsymbol{\nu} = \mathbf{U}^{-1}\mathbf{X} \tag{43}$$

Decoupled Equations for Undamped Free Vibration

- A well known theorem in linear algebra states that if $\mathbf{D}$ is a $n \times n$, i.e. square matrix, then the following two statements are always true:

  a. $\mathbf{D}$ is diagonalisable, and

  b. $\mathbf{D}$ has n linearly independent eigenvectors

- And hence, the $n \times n$ mass $\mathbf{M}$ and stiffness $\mathbf{K}$ matrices are always diagonalisable

- This is accomplished for the undamped free n-DOF system by pre-multiplying $\mathbf{M}$ and $\mathbf{K}$ by $\mathbf{U}^T$ and post-multiplying them by $\mathbf{U}$, respectively

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\boldsymbol{\nu}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\boldsymbol{\nu} = \mathbf{0} \tag{44}$$

- This results in the diagonalisation of the mass and stiffness matrices giving the $\mathbf{M}_r$ and $\mathbf{K}_r$ elements for $r = 1, 2, \cdots, n$

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \begin{bmatrix} M_{r=1} & 0 & \cdots & 0 \\ 0 & M_{r=2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{r=n} \end{bmatrix}, \tag{45}$$

Decoupled Equations for Undamped Free Vibration

and

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \begin{bmatrix} K_{r=1} & 0 & \cdots & 0 \\ 0 & K_{r=2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{r=n} \end{bmatrix} \quad (46)$$

- Recalling that

$$\omega_r^2 \mathbf{M}_r = \mathbf{K}_r, \qquad r \in \{1, 2, \cdots, n\}$$

we can rewrite Equation (46) as

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \begin{bmatrix} \omega_1^2 M_1 & 0 & \cdots & 0 \\ 0 & \omega_2^2 M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 M_n \end{bmatrix} \quad (47)$$

where ω_r = undamped natural circular frequency of the r^{th} mode

$M_r = \sum_{i=1}^n m_i (X_i^2)_r$ = r^{th} mode generalised mass for diagonal $\mathbf{M}$

$M_r = \sum_{i=1}^n \sum_{j=1}^n m_{ij} (X_i^2)_r (X_j^2)_r$ = r^{th} mode generalised mass for non-diagonal $\mathbf{M}$

Decoupled Equations for Undamped Free Vibration

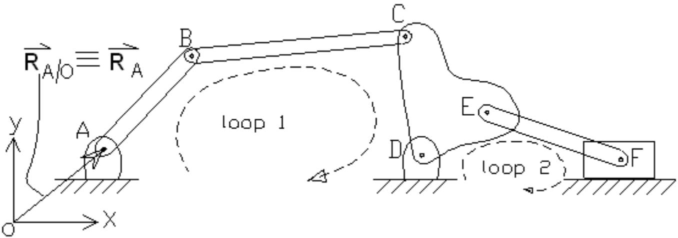
- Referring to Equations (45) and (47), the decoupled differential equations of motion for n-DOF free vibration in Equation (44) take on the pleasing form

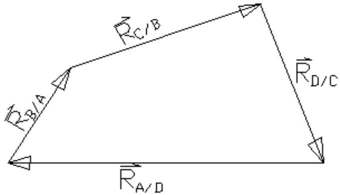
$$\begin{bmatrix} \nwarrow & & \\ & M_r & \\ & & \searrow \end{bmatrix} \ddot{\boldsymbol{\nu}} + \begin{bmatrix} \nwarrow & & \\ & \omega_r^2 M_r & \\ & & \searrow \end{bmatrix} \boldsymbol{\nu} = \mathbf{0} \tag{48}$$

- Equation (48) shows that the decoupled differential equations of motion for n-DOF free vibration, in terms of the principal coordinates, are linearly independent and each has the form

$$\left.\begin{aligned} \ddot{\nu}_1 + \omega_1^2 \nu_1 &= 0 \\ \ddot{\nu}_2 + \omega_2^2 \nu_2 &= 0 \\ &\vdots \\ \ddot{\nu}_r + \omega_r^2 \nu_r &= 0 \\ &\vdots \\ \ddot{\nu}_n + \omega_n^2 \nu_n &= 0 \end{aligned}\right\} \tag{49}$$

- Application of vector addition around closed loops results in the **Loop Closure Equation** that models a mechanism. e.g





loop 1: $\vec{R}_{B/A} + \vec{R}_{C/B} + \vec{R}_{D/C} + \vec{R}_{A/D} = 0$

loop 2: $\vec{R}_{E/D} + \vec{R}_{F/E} + \vec{R}_{D/F} = 0$

- Note that various relative position equations can be written. e.g. $\vec{R}_F = \vec{R}_D + \vec{R}_{E/D} + \vec{R}_{F/E} = \vec{R}_A + \vec{R}_{B/A} + \vec{R}_{C/B} + \vec{R}_{E/C} + \vec{R}_{F/E}$

$$\vec{A}_B = \vec{A}_A + \vec{A}_{rel} + \underbrace{\vec{\alpha} \times \vec{r}_{B/A}}_{\vec{A}^t_{B/A}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})}_{\vec{A}^n_{B/A}} + \underbrace{2\vec{\omega} \times \vec{V}_{rel}}_{\vec{A}^c_{B/A}}$$

where,

$\vec{A}_B$ = absolute acceleration of point B relative to the fixed frame

$\vec{A}_A$ = absolute acceleration of point A relative to the fixed frame

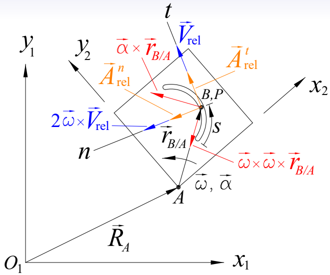
$\vec{A}_{rel}$ = acceleration of B relative to A in the moving frame

$\vec{\alpha} \times \vec{r}_{B/A}$ = tangential acceleration due to angular acceleration of moving frame

$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$ = normal acceleration due to angular velocity of the moving frame

$2\vec{\omega} \times \vec{V}_{rel}$ = Coriolis acceleration; this component of acceleration results from change in length of $\vec{\omega} \times \vec{r}_{B/A}$ and change in direction of $\vec{r}_{B/A}$

- it depends on quantities obtained from velocity analysis
- it is rotated 90^o from $\vec{V}_{rel}$ in the direction of $\vec{\omega}$



$$\vec{A}_B = \vec{A}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) + 2\vec{\omega} \times \vec{V}_{rel} + \vec{A}_{rel} \qquad (10)$$

As observed from a non-rotating frame at A:

$|\,\vec{\alpha} \times \vec{r}_{B/A}\,| = \alpha r_{B/A}$ is perpendicular to $\vec{r}_{B/A}$ in the direction of α with centre at A
$|\,\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})\,| = \omega^2 r_{B/A}$ is directed from B to A

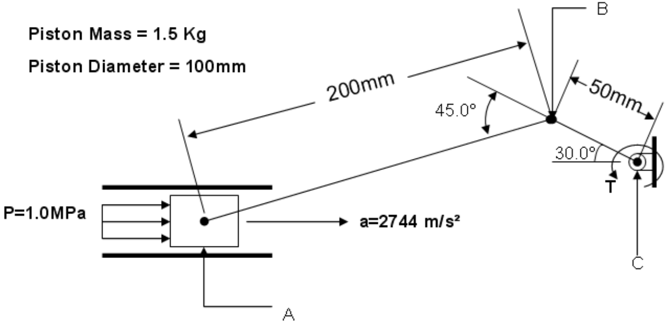
$\therefore$ in terms of the coincident point P we can interpret:

$\vec{\alpha} \times \vec{r}_{B/A}$: as the tangential component of $\vec{A}_{P/A}$ of point P in its circular motion about A

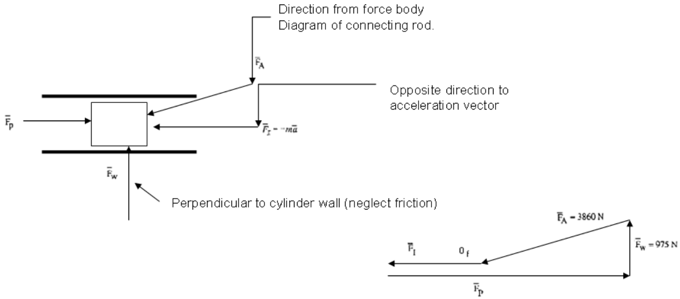
$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$: normal component of $\vec{A}_{P/A}$ of point P in its circular motion about A

Example 8.1
Offset-Crank Engine

Determine the output torque τ that can be developed by the offset-crank engine in the configuration shown using graphical force analysis. Ignore the inertia force and moment acting on the connecting rod and crank.



1. Piston (three-force member)



$$|\vec{F}_I| = ma_G = (1.5kg)(2744m/s^2) = 4116 \text{ N}$$

$$|\vec{F}_P| = PA = (1.0x10^6 \frac{N}{m^2})(\frac{\pi(0.1)^2}{4}) = 7854 \text{ N}$$

since two known applied forces can be combined into a single force, then piston becomes a three-force member

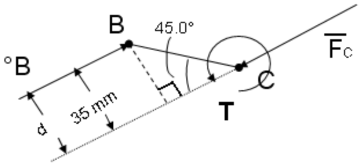
$$(\overset{\surd\surd}{\vec{F}_I} + \overset{\surd\surd}{\vec{F}_P}) + \overset{o\surd}{\vec{F}_W} + \overset{o\surd}{\vec{F}_A} = \overset{\surd\surd}{\vec{0}}$$

then from the force polygon $|\vec{F}_A| = 3860$ N; direction as shown

2. Connecting Rod (two-force member neglecting inertia)

- $\vec{F}_B = -\vec{F}_A$ $\quad (\overset{\surd\surd}{\vec{F}_A} + \overset{o\surd}{\vec{F}_B} = \overset{\surd\surd}{\vec{0}})$
- $|\vec{F}_B| = 3860$ N; direction as shown

3. Crank



- $\Sigma\vec{F} = m\vec{a} \Rightarrow \Sigma\vec{F} = 0$

  $\vec{F_C} = -\vec{F_B}$
- $\Sigma\vec{M_C} = I_G\vec{\alpha} = 0$

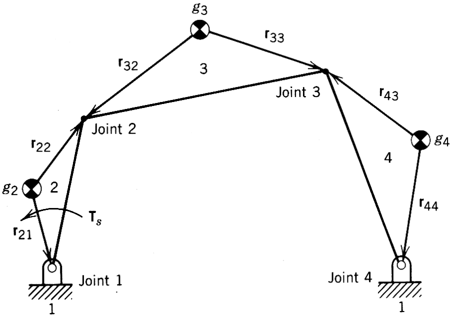
  $\tau - dF_B = 0$

  where d is the perpendicular distance from F_B to a parallel line passing through point C, then

  $\tau = dF_B = (35\text{mm})(3860\text{N}) = 135.1\text{Nm ccw}$

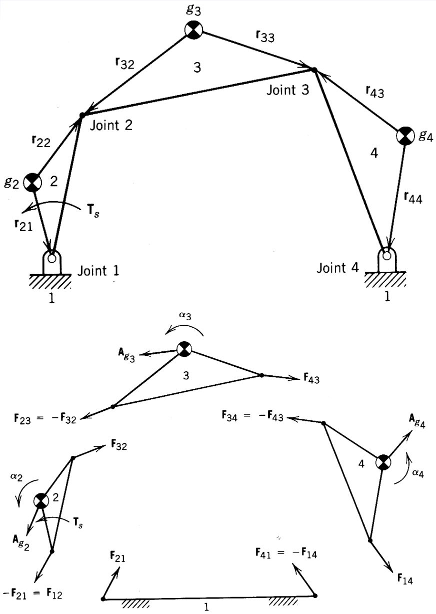
Example 8.2

Outline how the matrix solution method can be used to determine the required applied driving torque $\mathbf{T}_s$ and corresponding joint reaction forces in the four-bar mechanism for a set of specified linear and angular accelerations of each of the links determined by a previous kinematic analysis of the linkage.

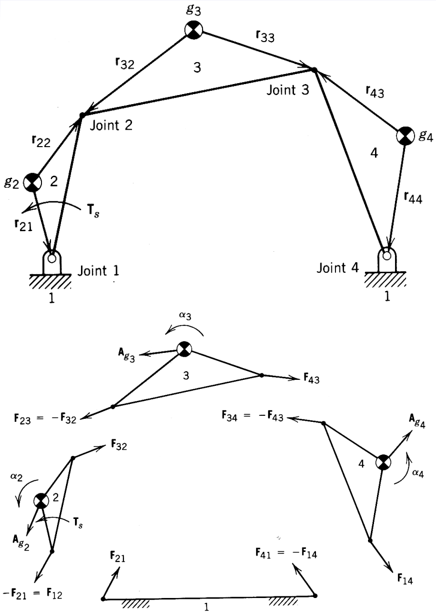


Solution Method:

1. Draw free-body diagrams
2. Write equilibrium equations
3. Form matrix equation
4. Solve

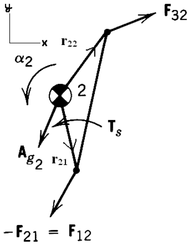


- $\mathbf{r}_{ij}$ vector from CG of link i to joint j
- $\mathbf{F}_{ik}$ force link i exerts on link k
- g_i CG of link i
- $\mathbf{A}_{g_i}$ acceleration of CG g_i
- α_i angular acceleration of link i
- M_i mass of link i
- I_i mass moment of inertia of link i about g_i
- $\mathbf{T}_s$ driving torque applied to link 2



- $\mathbf{F}_{ik} = -\mathbf{F}_{ki}$
- An independent set of solution variables must be chosen.
- $\mathbf{F}_{14}$, $\mathbf{F}_{21}$, $\mathbf{F}_{32}$, and $\mathbf{F}_{43}$ are selected as solution variables in this example.
- Other selections are possible.

Equilibrium equations are written for link 2.



$$
\begin{aligned}
F_{32x} - F_{21x} &= M_2 A_{g_2 x} \\
F_{32y} - F_{21y} &= M_2 A_{g_2 y}
\end{aligned}
$$

$$
r_{22x} F_{32y} - r_{22y} F_{32x} - r_{21x} F_{21y} \\
+ r_{21y} F_{21x} + T_s = I_2 \alpha_2
$$

Example 9.3

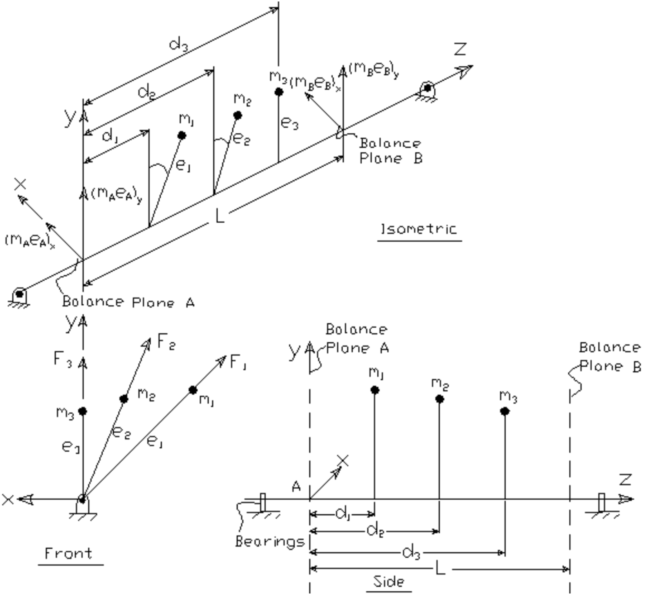
The rotor in an aircraft gas turbine, having a mass of m = 180 kg operates at 16000 rev/min. If the CM has an eccentricity of 0.025mm from the axis of rotation, determine the unbalance force.

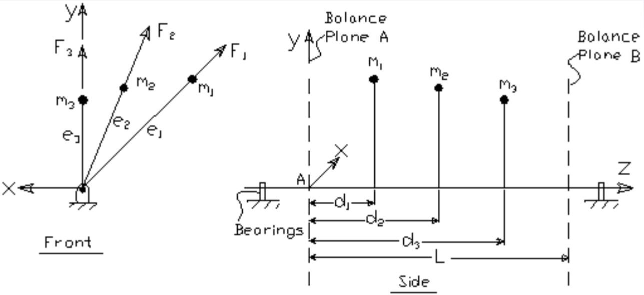
<u>Solution:</u>

This results in an unbalance of
$me = (180\text{ kg})(0.025\text{mm}) = 4.5\text{ kg-mm}$
that would in turn a cause a centrifugal force of
$F_{rot} = me\omega^2 = (180\text{kg})(0.025\times10^{-3}\text{m})(16000 \times \frac{2\pi}{60})^2 = 12630\text{ N} \approx 2800\text{ lbf}$

In this case centrifugal force is the unbalance force. Such a force could potentially damage the machine. It would be difficult and very time consuming to manufacture the rotor of a machine, so that the centre of mass would lie within 0.025mm from the axis of rotation. Therefore, the rotor in this case would be balanced after manufacture. This balancing is done experimentally.

General Unbalance





For complete balance of the rotor,

$$\sum \vec{F} = \vec{0}, \qquad \text{and} \qquad \sum \vec{M} = \vec{0}$$

$$\vec{F} = \sum(m_i \vec{e}_i \omega^2) = \sum(m_i \vec{e}_i)\omega^2 = \vec{0},$$

but since $\omega \neq \vec{0}$

$$\Rightarrow \qquad \sum(m_i \vec{e}_i) = \vec{0} \tag{5}$$

Also

$$\sum \vec{M} = \sum(\vec{d}_i \times m_i \vec{e}_i)\omega^2 = \vec{0}$$

but since $\omega \neq \vec{0}$

$$\Rightarrow \qquad \sum(\vec{d}_i \times m_i \vec{e}_i) = \vec{0} \tag{6}$$

Note:

- The moment arm of any given inertia force is d_i and it is always along the shaft, so it can be treated as a scalar, i.e., Eq (6) can be written as:

  $$\sum \vec{M} = \sum (m_i \vec{e}_i d_i)\omega^2 = \vec{0}$$

  Note that $\vec{d}_1 \times (m_1 \vec{e}_1 \omega^2) + \vec{d}_2 \times (m_2 \vec{e}_2 \omega^2) + \ldots = \vec{0}$

  $$\Rightarrow d_1 \vec{k} \times (m_1 \vec{e}_1 \omega^2) + d_2 \vec{k} \times (m_2 \vec{e}_2 \omega^2) + \ldots = \vec{0}$$

  $$\Rightarrow \vec{k} \times (d_1 m_1 \vec{e}_1 + d_2 m_2 \vec{e}_2 + \ldots) = \vec{0}$$

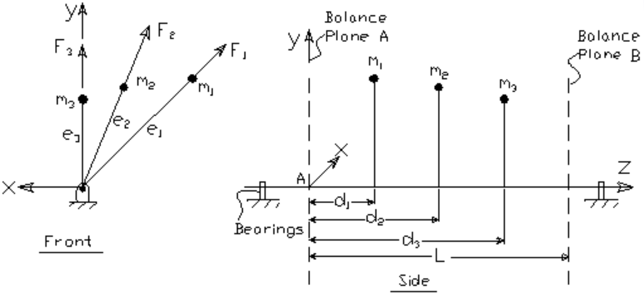
  $$\therefore d_1 m_1 \vec{e}_1 + d_2 m_2 \vec{e}_2 + \ldots = \vec{0}$$

  i.e.,

  $$\sum (m_i d_i) \vec{e}_i = \vec{0}$$

- For a rotor with masses which lie in multiple transverse axial planes, in order to satisfy both $\sum F = 0$ and $\sum M = 0$, a minimum of two balancing masses are required.

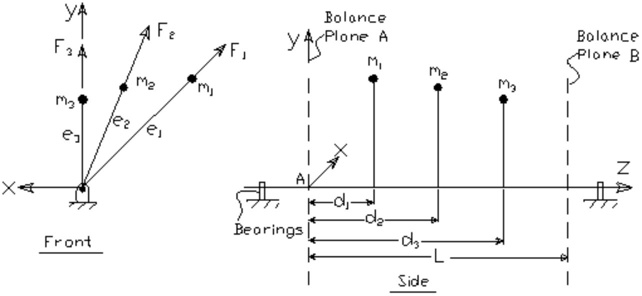
Analytical Method



- Determine the planes A and B as balance planes.
- Consider the equilibrium of moments about plane A in terms of the x and y components:

$$\sum M_{Ax} = 0 \;\Rightarrow\; \underbrace{(m_1 e_1)_y d_1}_{\text{known}} + \underbrace{(m_2 e_2)_y d_2}_{\text{known}} + \ldots + \underbrace{(m_B e_B)_y L}_{\text{solve for } m_B e_{B_y}} = 0$$

$$\sum M_{Ay} = 0 \;\Rightarrow\; \underbrace{(m_1 e_1)_x d_1}_{\text{known}} + \underbrace{(m_2 e_2)_x d_2}_{\text{known}} + \ldots + \underbrace{(m_B e_B)_x L}_{\text{solve for } m_B e_{B_x}} = 0$$



- Next, consider the equilibrium of forces:

$$\sum F_x = 0 \;\Rightarrow\; \underbrace{(m_1 e_1)_x}_{\text{known}} + \underbrace{(m_2 e_2)_x}_{\text{known}} + \ldots + \underbrace{(m_B e_B)_x}_{\text{found}} + \underbrace{(m_A e_A)_x}_{\text{solve for } m_A e_{A_x}} = 0$$

$$\sum F_y = 0 \;\Rightarrow\; \underbrace{(m_1 e_1)_y}_{\text{known}} + \underbrace{(m_2 e_2)_y}_{\text{known}} + \ldots + \underbrace{(m_B e_B)_y}_{\text{found}} + \underbrace{(m_A e_A)_y}_{\text{solve for } m_A e_{A_y}} = 0$$