DIMENSIONAL KINEMATIC SYNTHESIS OF ANGULAR VELOCITY FUNCTION GENERATORS

M. John D. Hayes, Mirja Rotzoll
Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, ON, K1S 5B6, Canada
Email: john.hayes@carleton.ca; mirja.rotzoll@carleton.ca

ABSTRACT

As a planar 4R mechanism moves the four relative angles between the four links change, implying angular velocity and acceleration. The input-output (IO) equation expresses one joint angle in terms of another as an implicit function scaled by the link lengths. There are six distinct angle pairings meaning there are six distinct IO equations for any given four-bar linkage. In this paper we present the six algebraic IO equations, and their first two time derivatives. We introduce novel algorithms for determining the extreme output angular velocities and accelerations given a specified constant input angular velocity, and present several detailed examples. Moreover, it is shown that any of the angular velocity IO equations can be expressed as an output angular velocity as a function of input angle for a specified constant input angular velocity. Using this, an example of dimensional kinematic synthesis for output angular velocity function generation is described in detail.

Keywords: Algebraic input-output equation; differential level kinematics; angular velocity function generation.

SYNTHÈSE CINÉMATIQUE DIMENSIONNELLE DE LA VITESSE ANGULAIRE DES GÉNÉRATEURS DE FONCTIONS

RÉSUMÉ

Au fur et à mesure qu'un quadrilatère articulé se déplace, les quatre angles relatifs entre les quatre liens changent, ce qui implique la vitesse et l'accélération. L'équation d'entrée-sortie (IO) exprime un angle de liaison par rapport à un autre. Il existe six paires d'angles distinctes, ce qui signifie qu'il existe six équations IO distinctes pour tous mécanisme à quatre barres données. Ces équations IO sont des fonctions implicites en termes de longueurs de liens. Dans cet article, nous présentons les six équations algébriques IO, et leurs deux premières dérivées par rapport au temps. Nous énonçons des algorithmes pour déterminer des vitesses et accélérations angulaires extrêmes et nous fournissons quelques exemples détaillés étant donné une vitesse angulaire d'entrée constante. De plus, il est montré que les équations de vitesse angulaire IO peuvent être exprimées de la forme vitesse angulaire de sortie en fonction de l'angle relative d'entrée pour une vitesse angulaire d'entrée constante spécifiée. Un exemple de la génération de fonction de vitesse angulaire de sortie est décrit en détail.

Mots-clés : Équations algébriques d'entrée-sortie ; cinématique de niveau différentiel ; vitesse angulaire des générateurs de fonctions .

1. INTRODUCTION

The algebraic v_i-v_j input-output (IO) equations, where $v = \tan(\theta/2)$, for planar and spherical four-bar linkages [1–3] have recently led to some elegant and useful applications for synthesis and analysis. There are six such equations for every four-bar kinematic architecture that relate the six distinct pairings of variable joint parameters taken two at a time. Because these are algebraic polynomials, it is straightforward to differentiate with respect to time, thereby revealing the velocity- and acceleration-level kinematics [4]. In this paper, we will discuss a novel approach to determining output angular velocity and acceleration extrema, as well as output angular velocity function generation synthesis in planar 4R mechanisms.

The first work investigating extreme output angular velocities is likely that of Kraus from 1939 [5, 6]. In that work, it was proposed that the angular velocity output/input ratio, $\dot{\theta}_4/\dot{\theta}_1$, of a double crank planar linkage reaches an extreme value when the coupler and follower, links a_2 and a_3 in Fig. 1, become mutually perpendicular. However, in 1944 Rosenauer [7] demonstrated that this is generally not true. Following Kraus, the extreme angular velocities and accelerations of four-bar mechanisms were later investigated in a methodical and more complete way by Ferdinand Freudenstein in 1956 [8]. In that work, Freudenstein used graphical methods to derive the $\dot{\theta}_4/\dot{\theta}_1$ angular velocity ratio based on a directed distance ratio along the Aronhold-Kennedy line of three collinear instantaneous centres of velocity (ICV), P_{13}, P_{14}, and P_{34}, see Fig. 1. This angular velocity ratio is the reciprocal of the mechanical advantage of the linkage [9], and hence an important index of merit. But, there are in fact six distinct angular velocity ratios that are revealed with the six planar 4R $\dot{v}_i$-$\dot{v}_j$ IO equations [4]. Recall that the collineation axis is the line joining the two secondary ICV, P_{13} and P_{24}. In [8], Freudenstein proposed that an extreme value of the angular velocity ratio $\dot{\theta}_4/\dot{\theta}_1$ occurs when the collineation axis is perpendicular to the coupler, now known as Freudenstein's Theorem 1. In an appendix to that same paper, A.S. Hall rigourously proved the theorem.

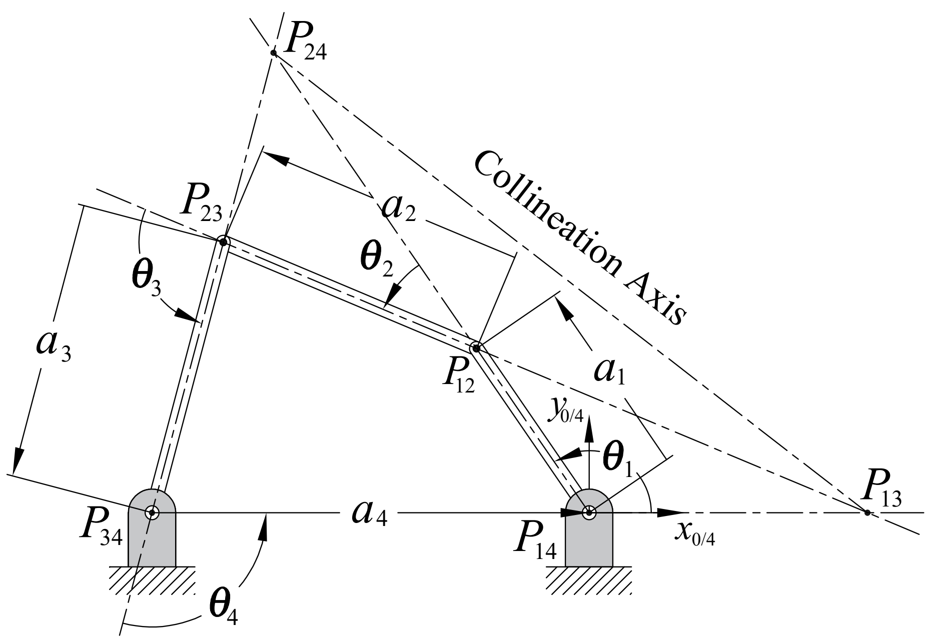


Fig. 1. The collineation axis and six instantaneous centres of velocity, P_{ij}, in a planar 4R linkage.

Consider the planar 4R linkage illustrated in Fig. 1. It is well known that as the 4R linkage moves it has four primary ICV, one at the centre of each R-pair, and two secondary ICV. The six ICV are known as *velocity poles* and the curves they move along are described as *polodes*, or *centrodes* as they are often called, see [9–12] for example. Two of the primary ICV, P_{12} and P_{23}, move on polodes defined by the link lengths, while P_{14} and P_{34} are stationary. The two secondary ICV, P_{13} and P_{24}, also move on polodes. By virtue of the Aronhold-Kennedy theorem, P_{13}, P_{14}, and P_{34} remain collinear as the motion evolves over time meaning that the polode for P_{13} is a segment of the line joining the two ground-fixed R-pairs, the x_0-axis, and is located at the point of intersection of the x_0-axis and the extension of the centreline of the coupler, a_2. Freudenstein's Theorem 1 implies that the value of the ratio of the output angular velocity and the input angular velocity, $\dot{\theta}_4$ and $\dot{\theta}_1$, can be expressed by the ratio of the values of the relative directed distances between the three ICV located on the x_0-axis in the following way [8, 13]:

$$\frac{\dot{\theta}_4}{\dot{\theta}_1} = \frac{d_{P_{13}P_{14}}}{d_{P_{13}P_{14}} + d_{P_{14}P_{34}}}, \tag{1}$$

where the directed distances $d_{P_{14}P_{34}}$ and $d_{P_{13}P_{14}}$ can be positive or negative depending on their relative directions. For example, $d_{P_{13}P_{14}}$ is the distance from P_{13} to P_{14}. The angular velocities are oppositely directed when the ratio is negative.

However, Freudenstein's Theorem 1 also applies to the ICV on each of the three other Aronhold-Kennedy lines of three collinear ICV with respect to a number line coincident with the line of three ICV having its origin on the central ICV. It seems that, to the best of the authors collective knowledge, with the exception of [14], this fact has not been discussed in the literature. Following Freudenstein's derivation logic, the three remaining velocity ratios expressed as ratios of the relative locations of the three ICV on the three other Aronhold-Kennedy lines, see Fig. 1, have never been stated explicitly as:

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{d_{P_{24}P_{12}}}{d_{P_{24}P_{12}} + d_{P_{12}P_{14}}}; \quad \frac{\dot{\theta}_3}{\dot{\theta}_2} = \frac{d_{P_{13}P_{12}}}{d_{P_{13}P_{12}} + d_{P_{12}P_{23}}}; \quad \frac{\dot{\theta}_4}{\dot{\theta}_3} = \frac{d_{P_{24}P_{23}}}{d_{P_{24}P_{23}} + d_{P_{23}P_{34}}}. \tag{2}$$

Regardless, the angular velocity ratios of $\dot{\theta}_1/\dot{\theta}_3$ and $\dot{\theta}_2/\dot{\theta}_4$ cannot be derived as ratios of collinear ICV relative directed distances.

In Freudenstein's work [8], the idea is first expressed of synthesising planar four-bar mechanisms to generate prescribed output angular accelerations, but not as functions of linkage configuration, and was not explored further. With the exception of one investigation on dimensional synthesis of planar four-bar function generators under velocity and acceleration constraints [15], there is no reported work, up to the authors' collective knowledge, on generating an angular velocity output that is a function of the input angle. Hence, one of the main contributions of this paper will be a novel algorithm for synthesising a planar 4R mechanism to generate an angular velocity output that is a function of mechanism configuration and a constant specified input angular velocity, for any of the six distinct IO angle pairs. The other main contribution will be algorithms for computing output angular velocity and acceleration extrema using the algebraic IO equations, and their time derivatives.

2. EXTREME OUTPUT ANGULAR VELOCITIES AND ACCELERATIONS

All moveable four-bar linkages generate six distinct functions between the four distinct joint variable parameters taken two at a time, which we abstractly call v_i and v_j. While this may or may not be common knowledge in the kinematics community, there is but one convenient and consistent way to determine and express these six functions using algebraic means to be found in the literature, see [3]. In this section we will list the six v_i-v_j IO equations, their time derivatives, algorithms for computing extreme values of angular velocity and acceleration, and give several examples.

2.1. The Six $\mathbf{v}_i$ - $\mathbf{v}_j$ Algebraic IO Equations and Their Time Derivatives

Let the tangent half-angle input parameter be $v_1 = \tan(\theta_1/2)$ and the output angle parameter be $v_4 = \tan(\theta_4/2)$. In [1] two elimination steps were applied to the Gröbner bases of the ideal generated by the

Study soma coordinates x_3, y_1 and y_2 to eliminate the angle parameters v_2 and v_3 from the equations yielding the algebraic IO equation relating the v_1 and v_4 angle parameters, which we call the v_1-v_4 IO equation. It has the form

$$Av_1^2v_4^2+Bv_1^2+Cv_4^2-8a_1a_3v_1v_4+D=0, \tag{3}$$

where

$$\begin{aligned}
A=A_1A_2&=(a_1-a_2+a_3-a_4)(a_1+a_2+a_3-a_4),\\
B=B_1B_2&=(a_1+a_2-a_3-a_4)(a_1-a_2-a_3-a_4),\\
C=C_1C_2&=(a_1-a_2-a_3+a_4)(a_1+a_2-a_3+a_4),\\
D=D_1D_2&=(a_1+a_2+a_3+a_4)(a_1-a_2+a_3+a_4),\\
v_1=\tan\left(\frac{\theta_1}{2}\right),&\quad v_4=\tan\left(\frac{\theta_4}{2}\right).
\end{aligned}$$

This algebraic equation is of degree 4 in the v_1 and v_4 variable parameters, while the coefficients labelled A, B, C, and D are each products of two bilinear factors which can be viewed as eight distinct planes treating the four a_i link lengths as homogeneous coordinates.

For the planar 4R, the five remaining v_i-v_j IO equations each contain all eight of the bilinear factors of the coefficients labelled A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 in Eq. (3), but in different permutations. The v_1-v_2, v_1-v_3, v_2-v_3, v_2-v_4, and v_3-v_4 IO equations are

$$A_1B_2v_1^2v_2^2+A_2B_1v_1^2+C_1D_2v_2^2-8a_2a_4v_1v_2+C_2D_1=0, \tag{4}$$

$$A_1B_1v_1^2v_3^2+A_2B_2v_1^2+C_2D_2v_3^2+C_1D_1=0, \tag{5}$$

$$A_1D_2v_2^2v_3^2+B_2C_1v_2^2+B_1C_2v_3^2-8a_1a_3v_2v_3+A_2D_1=0, \tag{6}$$

$$A_1C_1v_2^2v_4^2+B_2D_2v_2^2+A_2C_2v_4^2+B_1D_1=0, \tag{7}$$

$$A_1C_2v_3^2v_4^2+B_1D_2v_3^2+A_2C_1v_4^2+8a_2a_4v_3v_4+B_2D_1=0. \tag{8}$$

The first time derivative needs a few words of discussion. Because the angle parameter is $v=\tan(\theta/2)$, the time derivative is configuration dependent in the following way

$$\dot{v} \;=\; \frac{d}{dt}\tan(\theta/2) \;=\; \frac{\dot{\theta}}{2}\sec^2(\theta/2) \;=\; \frac{\dot{\theta}}{2}\left(\frac{\cos^2(\theta/2)+\sin^2(\theta/2)}{\cos^2(\theta/2)}\right) \;=\; \frac{\dot{\theta}}{2}\left(1+v^2\right). \tag{9}$$

Therefore the six angular velocity parameter IO equations are

$$\left(\left(A_1B_2v_2^2+A_2B_1\right)v_1-4a_2a_4v_2\right)\dot{v}_1+\left(\left(A_1B_2v_1^2+C_1D_2\right)v_2-4a_2a_4v_1\right)\dot{v}_2=0, \tag{10}$$

$$\left(A_1B_1v_3^2+A_2B_2\right)v_1\dot{v}_1+\left(A_1B_1v_1^2+C_2D_2\right)v_3\dot{v}_3=0, \tag{11}$$

$$\left(\left(A_1A_2v_4^2+B_1B_2\right)v_1-4a_1a_3v_4\right)\dot{v}_1+\left(\left(A_1A_2v_1^2+C_1C_2\right)v_4-4a_1a_3v_1\right)\dot{v}_4=0, \tag{12}$$

$$\left(\left(A_1D_2v_3^2+B_2C_1\right)v_2-4a_1a_3v_3\right)\dot{v}_2+\left(\left(A_1D_2v_2^2+B_1C_2\right)v_3-4a_1a_3v_2\right)\dot{v}_3=0, \tag{13}$$

$$\left(A_1C_1v_4^2+B_2D_2\right)v_2\dot{v}_2+\left(A_1C_1v_2^2+A_2C_2\right)v_4\dot{v}_4=0, \tag{14}$$

$$\left(\left(A_1C_2v_4^2+B_1D_2\right)v_3+4a_2a_4v_4\right)\dot{v}_3+\left(\left(A_1C_2v_3^2+A_2C_1\right)v_4+4a_2a_4v_3\right)\dot{v}_4=0. \tag{15}$$

Using Eq. (9), the six angular velocity ratios are trivially obtained as

$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = -\frac{\left((A_1B_2v_2^2+A_2B_1)v_1-4a_2a_4v_2\right)\left(1+v_1^2\right)}{\left((A_1B_2v_1^2+C_1D_2)v_2-4a_2a_4v_1\right)\left(1+v_2^2\right)}, \tag{16}$$

$$\frac{\dot{\theta}_3}{\dot{\theta}_1} = -\frac{\left(A_1B_1v_3^2+A_2B_2\right)\left(1+v_1^2\right)v_1}{\left(A_1B_1v_1^2+C_2D_2\right)\left(1+v_3^2\right)v_3}, \tag{17}$$

$$\frac{\dot{\theta}_4}{\dot{\theta}_1} = -\frac{\left((A_1A_2v_4^2+B_1B_2)v_1-4a_1a_3v_4\right)\left(1+v_1^2\right)}{\left((A_1A_2v_1^2+C_1C_2)v_4-4a_1a_3v_1\right)\left(1+v_4^2\right)}, \tag{18}$$

$$\frac{\dot{\theta}_3}{\dot{\theta}_2} = -\frac{\left((A_1D_2v_3^2+B_2C_1)v_2-4a_1a_3v_3\right)\left(1+v_2^2\right)}{\left((A_1D_2v_2^2+B_1C_2)v_3-4a_1a_3v_2\right)\left(1+v_3^2\right)}, \tag{19}$$

$$\frac{\dot{\theta}_2}{\dot{\theta}_4} = -\frac{\left(A_1C_1v_4^2+A_2C_2\right)\left(1+v_4^2\right)v_4}{\left(A_1C_1v_2^2+B_2D_2\right)\left(1+v_2^2\right)v_2}, \tag{20}$$

$$\frac{\dot{\theta}_4}{\dot{\theta}_3} = -\frac{\left((A_1C_2v_4^2+B_1D_2)v_3+4a_2a_4v_4\right)\left(1+v_3^2\right)}{\left((A_1C_2v_3^2+A_2C_1)v_4+4a_2a_4v_3\right)\left(1+v_4^2\right)}. \tag{21}$$

It is a straightforward exercise to show that Eqs. (1) and (18) yield identical results. These six angular velocity IO equations can be directly used for angular velocity level synthesis as will be demonstrated in Sect. 3. To do so with Eq. (1) requires additional kinematic models.

The six angular acceleration parameter IO equations are easily obtained as the time derivatives of the angular velocity parameter IO equations. The time derivative of $\dot{v}$ is a somewhat more complicated compound function requiring a combination of the chain and power rules from elementary differential calculus [16] to determine that

$$\ddot{v} = \frac{1}{2}\left(\ddot{\theta}+\dot{\theta}^2\right)\left(1+v^2\right), \tag{22}$$

which reveals that the angular acceleration parameter $\ddot{v}$ depends not only on angular acceleration, but on angular velocity and configuration as well. The six angular acceleration parameter equations are

$$\begin{aligned}&\left((A_1B_2v_2^2+A_2B_1)v_1-4a_2a_4v_2\right)\ddot{v}_1+\left((A_1B_2v_1^2+C_1D_2)v_2-4a_2a_4v_1\right)\ddot{v}_2+\\&\qquad(A_1B_2v_2^2+A_2B_1)\dot{v}_1^2+(A_1B_2v_1^2+C_1D_2)\dot{v}_2^2+4(A_1B_2v_1v_2-2a_2a_4)\dot{v}_1\dot{v}_2=0,\end{aligned} \tag{23}$$

$$(A_1B_1v_3^2+A_2B_2)v_1\ddot{v}_1+(A_1B_1v_1^2+C_2D_2)v_3\ddot{v}_3+(A_1B_1v_3^2+A_2B_2)\dot{v}_1^2+(A_1B_1v_1^2+C_2D_2)\dot{v}_3^2+4A_1B_1v_1v_3\dot{v}_1\dot{v}_3=0, \tag{24}$$

$$\begin{aligned}&\left((A_1A_2v_4^2+B_1B_2)v_1-4a_1a_3v_4\right)\ddot{v}_1+\left((A_1A_2v_1^2+C_1C_2)v_4-4a_1a_3v_1\right)\ddot{v}_4+\\&\qquad(A_1A_2v_4^2+B_1B_2)\dot{v}_1^2+(A_1A_2v_1^2+C_1C_2)\dot{v}_4^2+4(A_1A_2v_1v_4-2a_1a_3)\dot{v}_1\dot{v}_4=0,\end{aligned} \tag{25}$$

$$\begin{aligned}&\left((A_1D_2v_3^2+B_2C_1)v_2-4a_1a_3v_3\right)\ddot{v}_2+\left((A_1D_2v_2^2+B_1C_2)v_3-4a_1a_3v_2\right)\ddot{v}_3+\\&\qquad(A_1D_2v_3^2+B_2C_1)\dot{v}_2^2+(A_1D_2v_2^2+B_1C_2)\dot{v}_3^2+4(A_1D_2v_2v_3-2a_1a_3)\dot{v}_2\dot{v}_3=0,\end{aligned} \tag{26}$$

$$(A_1C_1v_4^2+B_2D_2)v_2\ddot{v}_2+(A_1C_1v_2^2+A_2C_2)v_4\ddot{v}_4+(A_1C_1v_4^2+B_2D_2)\dot{v}_2^2+(A_1C_1v_2^2+A_2C_2)\dot{v}_4^2+4A_1C_1v_2v_4\dot{v}_2\dot{v}_4=0, \tag{27}$$

$$\begin{aligned}&\left((A_1C_2v_4^2+B_1D_2)v_3+4a_2a_4v_4\right)\ddot{v}_3+\left((A_1C_2v_3^2+A_2C_1)v_4+4a_2a_4v_3\right)\ddot{v}_4+\\&\qquad(A_1C_2v_4^2+B_1D_2)\dot{v}_3^2+(A_1C_2v_3^2+A_2C_1)\dot{v}_4^2+4(A_1C_2v_3v_4+2a_2a_4)\dot{v}_3\dot{v}_4=0.\end{aligned} \tag{28}$$

2.2. Angular Velocity Extrema

We will determine the extreme output angular velocity given a specified set of link lengths and constant input angular velocity. We can use any of the six v_i-v_j algebraic IO equations. To identify extreme angular velocity and acceleration outputs for a constant input angular velocity requires that the angle parameters be transformed back into angles. While $\dot{\theta}_i$ may be constant, the corresponding parameter $\dot{v}_i$ is not, since it is configuration dependent. For this example, we will consider the v_2-v_4 and $\dot{v}_2$-$\dot{v}_4$ IO equations, Eqs. (7) and (20) respectively, since this angle pairing has never been found in the literature. The extreme angular velocities, along with the configurations in which they occur in both assembly modes, can be easily obtained computationally with the following algorithm.

Extreme planar 4R angular velocity algorithm.

If values for a_1, a_2, a_3, and a_4 are given and the input angular velocity is a constant specified value, we wish to determine the critical values $\theta_{i_{\text{crit}}}$ that result in $\dot{\theta}_{j_{\text{min/max}}}$, so θ_j must be eliminated from both the position and angular velocity IO equations.

1. Convert v_i and v_j in the IO equation to angles as $v = \tan(\theta/2)$ and solve for θ_j. There will be two solutions, one for each assembly mode.
2. Substitute the expression for θ_j from Step 1 into the $\dot{\theta}_i$-$\dot{\theta}_j$ equation and solve for $\dot{\theta}_j$, which gives $\dot{\theta}_j = f(\theta_i)$ since $\dot{\theta}_i$ is a specified constant.
3. Solve $\dfrac{d\dot{\theta}_j}{d\theta_i} = 0$ for $\theta_{i_{\text{crit}}}$ and determine the values of $\dot{\theta}_{j_{\text{min/max}}}$ corresponding to each distinct value of $\theta_{i_{\text{crit}}}$.

For this example we will consider v_4 and v_2 as the input and output angle parameters respectively, and let the constant input angular velocity and the link lengths of a planar 4R four-bar mechanism be $\dot{\theta}_4 = 10$ rad/s, $a_1 = 5$, $a_2 = 6$, $a_3 = 8$, $a_4 = 2$. Following the algorithm, we obtain an implicit equation of $\dot{\theta}_2 = f(\theta_4)$ and plot it, see Fig. 2. Solving $\dfrac{d\dot{\theta}_2}{d\theta_4} = 0$ for $\theta_{4_{\text{crit}}}$ we determine the values of $\dot{\theta}_{2_{\text{min/max}}}$ corresponding to each distinct value of $\theta_{4_{\text{crit}}}$, which are listed in Table 1. It should be noted that determining the points of inflection of a particular $\dot{\theta}_j = f(\theta_i)$ curve requires the second derivative, $\dfrac{d^2\dot{\theta}_j}{d\theta_i^2} = 0$.

Table 1. $\dot{\theta}_{2_{\text{min/max}}}$ and $\theta_{4_{\text{crit}}}$ for $\dot{\theta}_4 = 10$ rad/s.

Assembly mode	$\dot{\theta}_{2_{\text{min/max}}}$ rad/s	$\theta_{4_{\text{crit}}}$ rad
1	5.385202141	-1.481326671
	-5.385202141	1.4813266715
2	-5.385202141	-1.481326671
	5.385202141	1.481326671

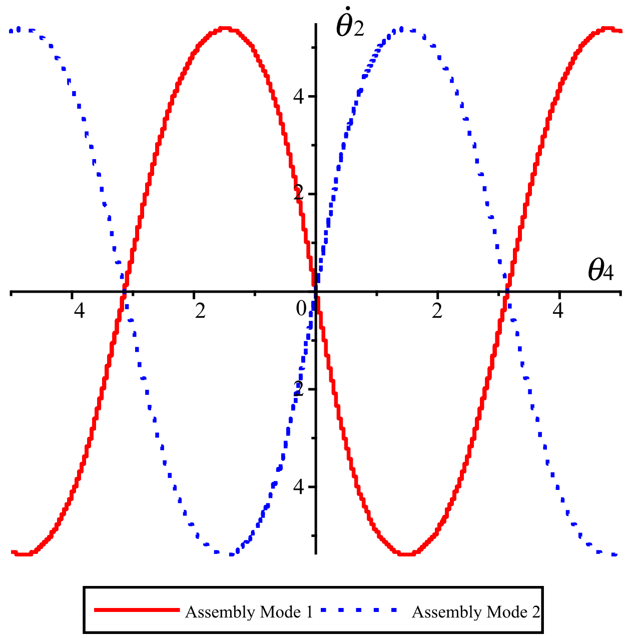


Fig. 2. The $\dot{\theta}_2 = f(\theta_4)$ angular velocity profile.

2.3. Angular Acceleration Extrema

According to Freudenstein in [8], one of the extreme output angular accelerations for a crank-rocker, assuming constant input angular velocity, is given by

$$\ddot{\theta}_{4_{\max}} = \frac{\dot{\theta}_1^2 a_1}{a_2 a_3}(a_1 + a_2), \tag{29}$$

where the link lengths are subject to the condition

$$(a_1 + a_2)^2 + a_3^2 = a_4^2. \tag{30}$$

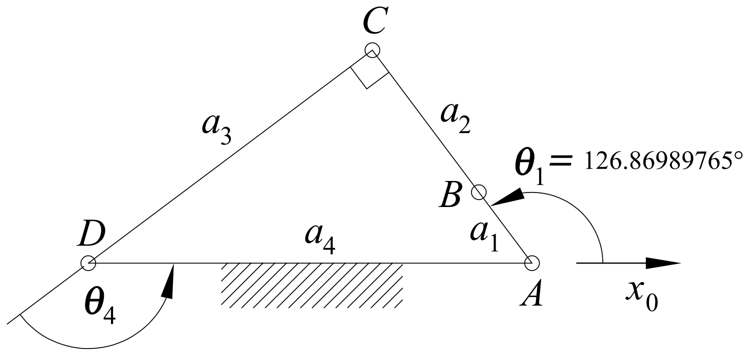
This extreme output angular acceleration occurs when the angle $\angle BCD = 90°$, see Fig. 3a. Freudenstein continues in the same paper to state an equation for one extreme output angular acceleration for drag-link mechanisms, which is given by

$$\ddot{\theta}_{4_{\max}} = \frac{\dot{\theta}_1^2 a_4}{a_2 a_3}(a_2 + a_3), \tag{31}$$

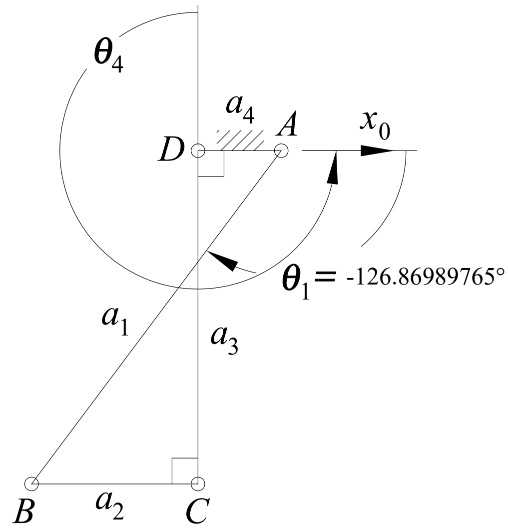
where the link lengths are subject to the condition

$$(a_2 + a_4)^2 + a_3^2 = a_1^2. \tag{32}$$

At an extreme output angular acceleration the coupler a_2 and fixed base link a_4 are parallel and the angle $\angle ADC = 90°$, see Fig. 3b. Together, these two conditions further mean that $\angle DCB = 90°$. We will verify both theorems with examples, but also demonstrate that they are incomplete because of the link length conditions, and only partially correct. In fact, the length conditions can impose a folding singularity on each linkage type for certain integer values of link lengths, thereby meaning that the linkage will pass through a singularity as it folds along the base link a_4. Moreover, it is important to note that Eqs. (29) and (31) apply only to these two classes of linkage.



(a) Folding crank-rocker: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 5$.



(b) Folding drag-link: $a_1 = 5$, $a_2 = 2$, $a_3 = 4$, $a_4 = 1$.

Fig. 3. Configurations where $\ddot{\theta}_{4_{\max}}$ occur.

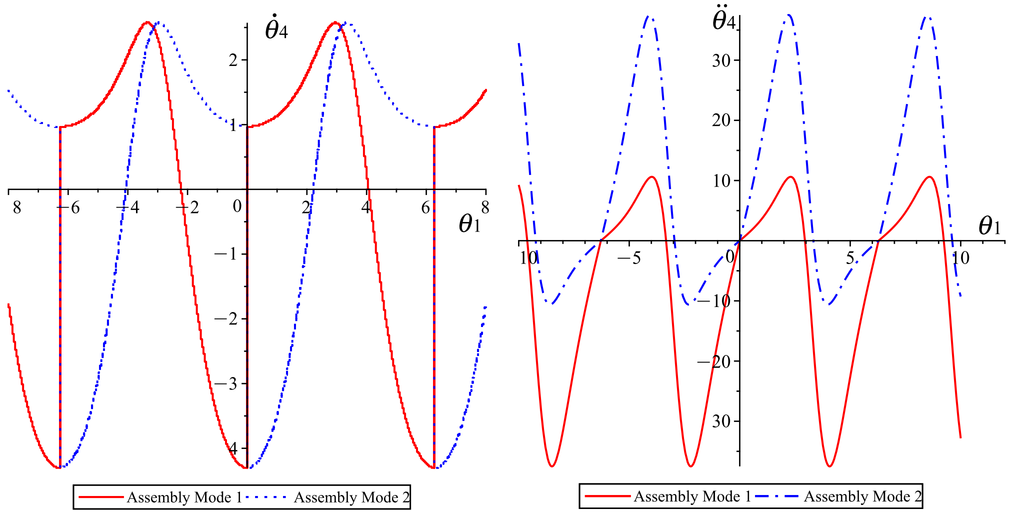
Table 2. Folding crank-rocker and drag-link mechanism link lengths.

Link	Crank-rocker lengths	Drag-link lengths
a_1	1	5
a_2	2	2
a_3	4	4
a_4	5	1

Let a crank-rocker, where a_1 is the crank and a_4 is the rocker, be defined by the link lengths that satisfy Eq. (30), which are listed in Table 2. It turns out that the linkage is a folding drag-link for these lengths. The angular velocity and acceleration profiles for a constant input angular velocity of $\dot{\theta}_1 = 10$ rad/s are illustrated in Fig. 4, where the folding singularity appears as discontinuities in the profile curves. Substituting the link lengths and constant input angular velocity $\dot{\theta}_1 = 10$ into Eq. (29) predicts that a maximum output angular acceleration is $\ddot{\theta}_4 = 37.5 \ \text{rad}/\text{s}^2$. Freudenstein predicts that this extreme angular acceleration will occur when links a_1 and a_2 are parallel and a_2 is perpendicular to a_3.

Furthermore, let a folding drag-link mechanism, where a_1 is the input crank and a_4 is the output crank, be defined by the link lengths that satisfy Eq. (32), which are also listed in Table 2. Substituting the link lengths and constant input angular velocity $\dot{\theta}_1 = 10$ rad/s into Eq. (31) predicts that a maximum output angular acceleration is $\ddot{\theta}_4 = 75 \ \text{rad}/\text{s}^2$. Freudenstein predicts that the extreme angular acceleration for the drag-link will occur when links a_2 and a_4 are parallel and a_3 is perpendicular to a_4. The two linkages were sketched in AutoCAD, and the input angle where an extreme angular acceleration occurs was measured to be precisely $\pm 126.86989765°$ for the crank-rocker/drag-link mechanisms, see Fig. 3.

In order to confirm the results for both the crank-rocker and drag-link mechanisms, a method to compute the extreme output angular accelerations and associated configurations is needed. Hence, the following algorithm is proposed.



(a) The $\dot{\theta}_4 = f(\theta_1)$ angular velocity profile. (b) The $\ddot{\theta}_4 = f(\theta_1)$ angular acceleration profile.

Fig. 4. The angular velocity and acceleration profiles for the Freudenstein crank-rocker.

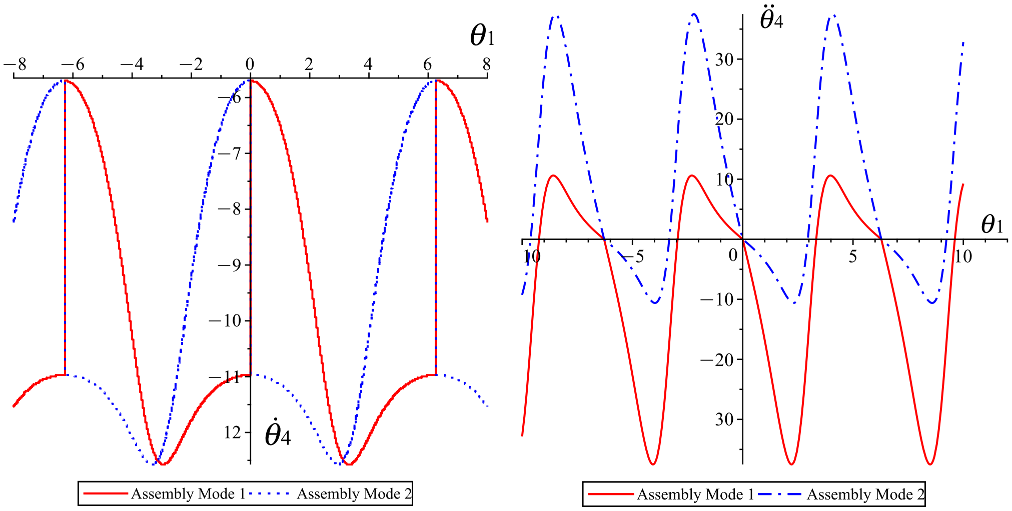
Extreme planar 4R angular acceleration algorithm.

If values for a_1, a_2, a_3, and a_4 are given and the input angular velocity is a constant specified value, we wish to determine the critical values $\theta_{i_{\text{crit}}}$ that result in $\ddot{\theta}_{j_{\text{min/max}}}$, so both θ_j and $\dot{\theta}_j$ must be eliminated from the position, angular velocity, and acceleration IO equations.

1. Convert v_i and v_j in the IO equation to angles as $v = \tan(\theta/2)$ and solve for θ_j.
2. Substitute the expression for θ_j from Step 1 into the $\dot{\theta}_i$-$\dot{\theta}_j$ equation and solve for $\dot{\theta}_j$, which gives $\dot{\theta}_j = f(\theta_i)$, since $\dot{\theta}_i$ is a specified constant.
3. Substitute the expressions for θ_j and $\dot{\theta}_j$ into the $\ddot{\theta}_i$-$\ddot{\theta}_j$ equation.
4. Solve the resulting equation for $\ddot{\theta}_j$, which gives $\ddot{\theta}_j = f(\theta_i)$, since $\ddot{\theta}_i = 0$.
5. Solve $\dfrac{d\ddot{\theta}_j}{d\theta_i} = 0$ for $\theta_{i_{\text{crit}}}$ and determine the values of $\ddot{\theta}_{j_{\text{min/max}}}$ corresponding to each distinct value of $\theta_{i_{\text{crit}}}$.

If the associated points of inflection are required, then the second derivative, $\dfrac{d^2\ddot{\theta}_j}{d\theta_i^2} = 0$, must be solved for the critical input angles.

Following the extreme angular velocity and acceleration algorithms for a constant input angular velocity of $\dot{\theta}_1 = 10$ rad/s, and crank-rocker as well as drag-link link lengths listed in Table 2, we obtain the angular velocity and acceleration profiles for $\dot{\theta}_4 = f(\theta_1)$ and $\ddot{\theta}_4 = g(\theta_1)$ that are illustrated in Figs. 4 and 5. Moreover, the computed values for $\dot{\theta}_{4_{\text{min/max}}}$, $\ddot{\theta}_{4_{\text{min/max}}}$, and the associated $\theta_{1_{crit}}$ for the crank-rocker and drag-link mechanisms are listed in Table 3. It is to be seen that for the crank-rocker, the value for $\ddot{\theta}_{4_{\text{max}}}$ computed with Freudenstein's Eq. (29) is indeed correct, as well as the configuration in which it occurs. See the value for



(a) The $\dot{\theta}_4 = f(\theta_1)$ angular velocity profile. (b) The $\ddot{\theta}_4 = f(\theta_1)$ angular acceleration profile.

Fig. 5. The angular velocity and acceleration profiles for the Freudenstein drag-link.

Table 3. Folding crank-rocker and drag-link mechanism angular velocity and acceleration extrema.

Mechanism	Assembly mode	$\dot{\theta}_{4_{\mathrm{min/max}}}$ rad/s	$\theta_{1_{\mathrm{crit}}}$ rad (deg)	$\ddot{\theta}_{4_{\mathrm{min/max}}}$ rad/s^2	$\theta_{1_{\mathrm{crit}}}$ rad (deg)
Crank-rocker	1	2.573593128	2.951961350 (169.1349266°)	-37.5	-2.214297436 (-126.8698977°)
		-4.301898050	0 (0°)	10.61385310	2.304690338 (132.0490295°)
	2	2.573593128	-2.951961350 (-169.1349266°)	37.5	2.214297436 (126.8698977°)
		1.137464865	0 (0°)	-10.61385310	-2.304690338 (-132.0490295°)
Drag-link	1	-12.57359315	-2.951961350 (-169.1349266°)	-37.5	2.214297436 (126.8698977°)
		-5.698101962	0 (0°)	10.61385310	-2.304690338 (-132.0490295°)
	2	-12.57359315	2.951961350 (169.1349266°)	37.5	-2.214297436 (-126.8698977°)
		-5.698101962	0 (0°)	-10.61385310	2.304690338 (132.0490295°)

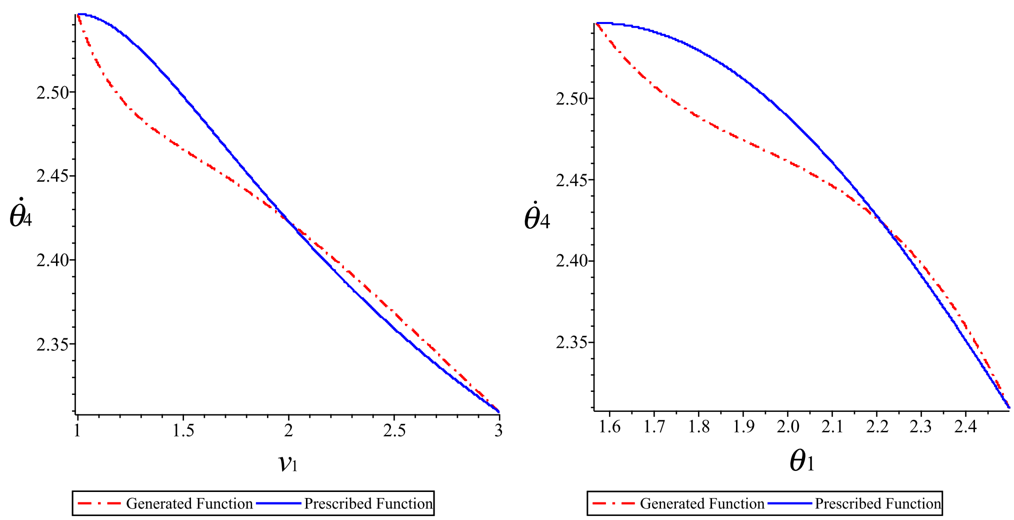
$\theta_{1_{crit}}$ in Assembly Mode 2 in Table 3, which is identical to the empirically measured value in Fig. 3. However, the remaining extreme value for the output angular acceleration is not accounted for. Additionally, it is to be seen for the drag-link that the configuration in which an extreme value for $\ddot{\theta}_{4_{\mathrm{max}}}$ is correct, see the value for $\theta_{1_{crit}}$ in Assembly Mode 2 in Table 3, which is the same as the empirically measured value in Fig. 3. However, the value for the maximum angular acceleration is not correct, it is twice the computed value, see the results listed in Table 3. The values computed with the angular velocity and acceleration extrema algorithms yield results that agree with the plotted angular velocity and acceleration profiles illustrated in Figs. 4 and 5.

3. ANGULAR VELOCITY FUNCTION GENERATOR DIMENSIONAL SYNTHESIS

The previous sections demonstrated that the angular velocity and acceleration extrema in a mechanism may be computed using any of the six v_i-v_j algebraic IO equations and the two algorithms. To do so, the desired output needed to be expressed as a function of only the input angle, e.g. $\dot{\theta}_j = f(\theta_i)$ and $\ddot{\theta}_j = g(\theta_i)$, where f and g represent the functions. This suggests, as Freudenstein mentioned in [8], that one may synthesise a four-bar linkage with prescribed velocity or acceleration characteristics. Let a desired angular velocity profile be expressed as

$$\dot{\theta}_4 \quad = \quad \tan\left(\frac{v_1}{1+v_1^2}\right), \quad 1 \leq v_1 \leq 3. \tag{33}$$

We will demonstrate with an example using exact function generation dimensional synthesis, though the



(a) IO curve in the v_1-$\dot{\theta}_4$ plane.

(b) IO curve in the θ_1-$\dot{\theta}_4$ plane.

Fig. 6. Synthesised angular velocity function IO curves.

continuous approximate dimensional technique [2] may also be used. We select our three input angle parameters to be $v_1 = 1, 2, 3$. The corresponding output angular velocities which satisfy the function are $\dot{\theta}_4 =$ 2.546302490, 2.422793219, 2.309336250 rad/s. We select a constant input angular velocity of $\dot{\theta}_1 = 10$ rad/s and use Eq. (18), the $\dot{\theta}_1$-$\dot{\theta}_4$ IO equation, as our synthesis equation. We identify the v_4 that correspond to our selected values of v_1 by solving Eq. (3), the v_1-v_4 IO equation, for v_4. These values are substituted into Eq. (18) generating three synthesis equations in terms of the unknown link lengths a_1, a_2, a_3, a_4. Since this is a function generation problem the scale of the linkage is arbitrary. Without loss of generality, we set $a_4 = 1$ and obtain the link lengths listed in Table 4. The resulting angular velocity profile is illustrated in Fig. 6. We define the structural error as the common area between the curves, which for Fig. 6a is 0.017595114.

Table 4. Velocity function generator identified link lengths.

$a_1 = 0.1316240266$	$a_2 = 0.5005860676$	$a_3 = 0.5968439840$	$a_4 = 1$

4. CONCLUSIONS

In this paper we presented the six planar 4R four-bar mechanism v_i-v_j algebraic IO equations, one for each distinct pair of the relative angles between the links, and their first two time derivatives. Novel algorithms for computing the extreme values of output angular velocity and acceleration for a specified constant input angular velocity were also proposed. To compute the extrema, the velocities and accelerations must be expressed as functions of only the input angle parameter. Freudenstein's output angular acceleration extrema criteria for crank-rocker and drag-link mechanisms were evaluated and shown to be only partially correct, as well as incomplete. Finally, an example of angular velocity function generation synthesis was presented. The authors believe this is the very first such example to be found in the literature.

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