A GENERAL METHOD FOR DETERMINING ALGEBRAIC INPUT-OUTPUT EQUATIONS FOR THE SLIDER-CRANK AND THE BENNETT LINKAGE

Mirja Rotzoll[1], M. John D. Hayes[1],
[1]*Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, ON K1S 5B6, Canada*
Email: mirjarotzoll@cmail.carleton.ca, john.hayes@carleton.ca

ABSTRACT

In a recent paper we have shown a general method for determining algebraic input-output (IO) equations for planar and spherical 4R linkages. In this paper, we will demonstrate that this method can be similarly applied to the spatial Bennett linkage as well as the planar slider-crank mechanism. The procedure requires describing the linkages with Denavit Hartenberg (DH) parameters, projecting the overall coordinate transformation of the linkage into Study soma coordinates, and solving the resulting system of equations via resultants in the case of the slider-crank, and Gröbner basis in the case of the Bennett mechanism. The procedure is free of trigonometric expressions, and the results of this paper illustrate that the applicability of this method may be extended to other spatial mechanisms.

Keywords: Algebraic input-output equation; RRRP linkage; Bennett; Study soma coordinates.

MÉTHODE GÉNÉRALE POUR DÉTERMINER LES ÉQUATIONS ALGÉBRIQUES D'ENTRÉE-SORTIE POUR LE MÉCANISME BIELLE-MANIVELLE ET LA LIAISON BENNETT

RÉSUMÉ

Dans un article récent, nous avons démontré une méthode générale pour déterminer les équations algébriques d'entrée-sortie (IO) pour les liaisons planaires et sphériques 4R. Dans cet article, nous démontrerons que cette méthode peut être appliquée de la même manière à la liaison spatiale de Bennett ainsi qu'au mécanisme planaire bielle-manivelle. La procédure requiert de décrire les liens avec les paramètres de Denavit Hartenberg (DH), puis de projeter la transformation globale des coordonnées de la liaison en coordonnées Study soma, et enfin de résoudre le système d'équations qui en résulte via les résultants dans le cas du bielle-manivelle, et la base de Gröbner pour le mécanisme de Bennett. Cette procédure a l'avantage de ne pas appeler de fonctions trigonométriques et les résultats de cet article laissent entrevoir que cette méthode pourrait être appliquée à d'autres mécanismes spatiaux.

Mots-clés : Équation algébrique d'entrée-sortie ; liaison RRRP ; Bennett ; coordonnées Study soma.

1. INTRODUCTION

Planar and spatial four-bar mechanisms are commonly used in a wide range of applications, including braking and steering systems in cars, space and aircraft systems, or even laparoscopic surgical tools. In many of these applications, the mechanism's designer is interested in the relation between the input angle of the first link and the output angle of the last link. This relationship is commonly expressed as the input-output (IO) equation. The first analytically derived IO equation for planar four-bar linkages with four revolute (4R) joints was presented by Ferdinand Freudenstein in 1954 [1]. He divided the linkage into two dyads containing a_1/a_2 and a_3/a_4, and used trigonometric relations to obtain the *Freudenstein equation*

$$k_1 + k_2\cos(\theta_4) - k_3\cos(\theta_1) \quad = \quad \cos(\theta_1 - \theta_4), \tag{1}$$

where the linear factors k_i are known as the Freudenstein parameters and are defined by

$$k_1 \equiv \frac{(a_1^2 + a_2^2 + a_4^2 - a_3^2)}{2a_1a_2}; \;\; k_2 \equiv \frac{a_4}{a_1}; \;\; k_3 \equiv \frac{a_4}{a_2}.$$

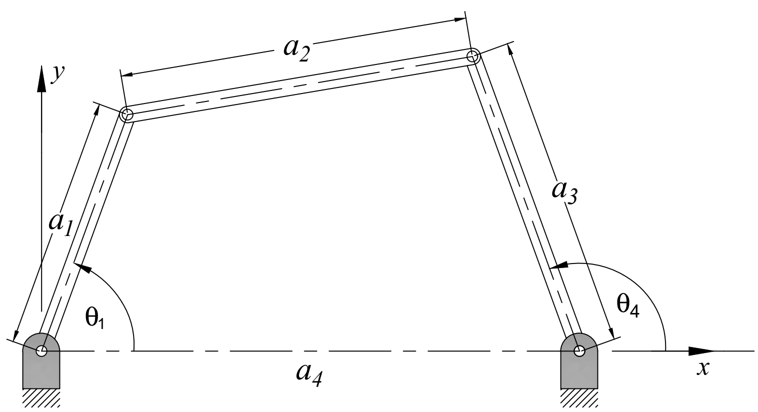

Fig. 1. Planar 4R function generator.

As illustrated in Fig. 1, the parameters a_1, a_2, a_3 and a_4 correspond to the input link, coupler, output link, and the non-moving ground connection; and the variables θ_1 and θ_4 correspond to the input and output angle, respectively.

The first algebraic IO equation for planar 4R linkages was presented in [2] by Bottema and Roth. Their derivation is purely trigonometric. However, by substituting $tan(\theta_1/2) = v_1/w$ and $tan(\theta_4/2) = v_4/w$ the authors map the IO motion in a (v_1, v_4, w)-plane which results in an algebraic version of the equation.

An alternative approach to derive the algebraic version of the 4R linkage was presented by Hayes, Husty and Pfurner [3]. They described the two constraints defined by the circular motion of the input and output links, and mapped these constraints into Study's soma coordinates [2, 4]. With Weierstraß subsitutions [5] they converted the trigonometric expressions of the input and output angle into algebraic ones, and finally eliminated the undesired Study coordinates using resultants to obtain the IO equation.

A slightly different linkage is obtained by exchanging the fourth R joint with a prismatic (P) joint. This linkage is known as the slider-crank, or RRRP linkage, whose most famous application is the piston engine. The linkage allows transforming the reciprocating linear motion of the piston into a rotary motion of the crankshaft. Inversely, in e.g. a hand pump a rotary motion is transformed into a reciprocating motion of the piston in the suction pipe [6]. A general illustration is shown in Fig. 2. In contrast to the 4R IO equation it is easy to see that the coordinate frame in Fig. 2 can be selected in such a way that the slider is perpendicular to the ground connection a_4 without affecting the IO equation.

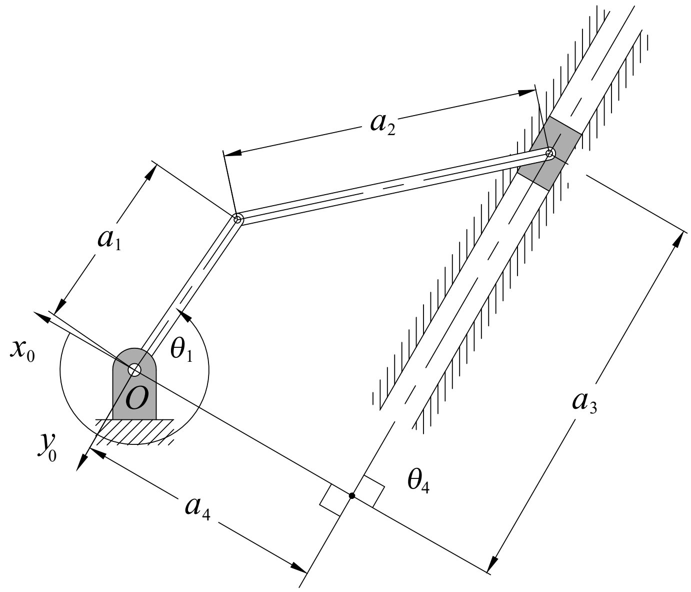

Fig. 2. Planar RRRP function generator.

The literature differentiates between two RRRP linkages, the *in-line* or *central*, and the *offset* or *eccentric* RRRP linkages [7]. It is considered central if the extended line of the slider intersects the rotation centre of the crank. On the other hand, an eccentric RRRP exists if $a_4 \neq 0$ [8]. Thus, a central RRRP is a special case of the eccentric slider crank.

Centric slider-crank IO equations for function generation can be found in every basic mechanics book. Two different trigonometric derivations that also apply to the *eccentric* linkage are given in [2, 9]. They use trigonometric constraints to derive a trigonometric and an algebraic expression of the IO equation, respectively. In a recent publication [10], it was shown that the same IO equation as derived in [3] for planar 4R linkages can be applied to planar RRRP linkages. The only difference between the IO equations is the interpretation of it: while the variables of the 4R are the input and the output angle, the variables of the RRRP are the input angle and the slider distance. To demonstrate the general applicability of the method presented in this paper, we will show that it can also be applied to a particular spatial mechanism, the Bennett linkage which is shown in Fig. 3.

The Bennett linkage is, as the planar 4R, composed of four rigid links that are connected by four R joints. According to the Chebychev–Grübler–Kutzbach criterion the Bennett linkage has a mobility of -2, which in theory prevents it from moving. However, its actual mobility is 1, and thereby, it is the only known mobile spatial 4R linkage. The linkage is able to move if it satisfies the following conditions which were discovered by Bennett and for that reason known as the Bennett conditions [11]

$$\begin{gathered} a_1 = a_3 \qquad\qquad a_2 = a_4 \\ \tau_1 = \tau_3 \qquad\qquad \tau_2 = \tau_4 \\ \frac{\sin(\tau_1)}{a_1} = \frac{\sin(\tau_2)}{a_2}. \end{gathered} \tag{2}$$

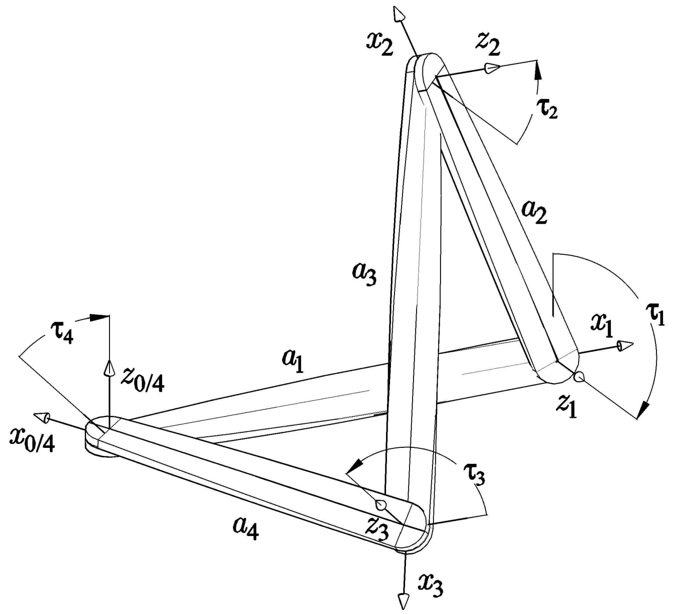

Fig. 3. Spatial 4R function generator: the Bennett linkage.

The IO equation is well known, for example [11–15] show different derivations. While the former authors favoured a geometric approach, Denavit used Cayley-Klein parameters, and Pfurner et al. used an algebraic method to first derive the Bennett conditions and second, obtain an IO equation in terms of the tangent half input and output angles.

In [16], we have presented a general method for determining IO equations for the planar 4R and the spherical 4R linkage. It required describing the open-chain of the linkages with DH parameters, and projecting the overall transformation of the linkage into Study soma coordinates. The open-chain linkage is conceptually closed by equating the Study vector to its identity vector, and to solve the system of equations and simultaneously to eliminate the intermediate link angles we use Gröbner basis. The IO equation for the planar 4R linkage results in

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{3}$$

where

$$\begin{aligned}
A &= (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4) \;=\; A_1A_2;\\
B &= (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4) \;=\; B_1B_2;\\
C &= (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4) \;=\; C_1C_2;\\
D &= (a_1 + a_2 + a_3 + a_4)(a_1 - a_2 + a_3 + a_4) \;=\; D_1D_2;\\
v_1 &= \tan\frac{\theta_1}{2};\\
v_4 &= \tan\frac{\theta_4}{2}.
\end{aligned}$$

In the following sections, we will show how this method can successfully be applied to planar RRRP linkages and the Bennett linkage in order to compute their respective IO equations.

2. DENAVIT HARTENBERG PARAMETER

Displacements of kinematic chains are often parametrised using the Denavit-Hartenberg (DH) convention [17]. They were first introduced by Denavit and Hartenberg in 1955, and are still widely used in the field of robotics. The four associated DH parameters for a single link in the kinematic chain are the link lengths a_i, link twist angles τ_i, joint angles θ_i, and link offsets d_i. Traditionally, the coordinate frames are assigned in the following way:

- the z_i-axis points along the directions of joint $i+1$;
- the x_i-axis is parallel to $z_i \times z_{i-1}$, and it is directed towards z_i;
- the y_i-axis is complementing a right-handed coordinate frame.

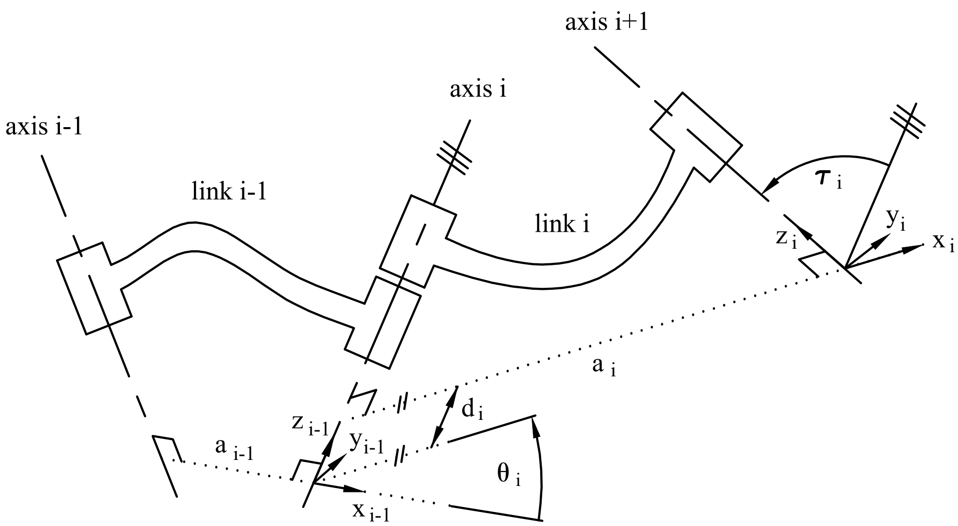

Fig. 4. DH parameter frame assignment and parameter.

Subsequently, the DH parameters are assigned as illustrated in Fig. 4, following the convention

- d_i: distance from the origin of the coordinate system $i-1$ to the intersection of z_{i-1} to x_i, measured along z_{i-1};
- θ_i: rotation angle from x_{i-1} to x_i, measured about z_{i-1};
- a_i: distance from the intersection of z_{i-1} and x_i to the origin of the coordinate system i, measured along x_i;
- τ_i: rotation angle from z_{i-1} to z_i, measured about x_i.

According to this convention the coordinate transformation from the coordinate system for joint i relative to the coordinate system of the previous joint $i-1$ can be divided into two screw displacements, i.e., two pure rotations and two pure translations in terms of the DH parameters

$$\mathbf{T}(d_i) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]; \quad \mathbf{T}(\theta_i) = \left[\begin{array}{ccc|c} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right];$$

$$\mathbf{T}(a_i) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]; \quad \mathbf{T}(\tau_i) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\tau_i) & -\sin(\tau_i) & 0 \\ 0 & \sin(\tau_i) & \cos(\tau_i) & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

Multiplying the rotations and translations following

$$\mathbf{T}(\theta_i) \cdot \mathbf{T}(d_i) \cdot \mathbf{T}(a_i) \cdot \mathbf{T}(\tau_i) \tag{4}$$

yields the transformation between two coordinate frames which is given by

$${}^{i-1}_{i}\mathbf{T} = \left[\begin{array}{ccc|c} \cos\theta_i & -\sin\theta_i\cos\tau_i & \sin\theta_i\sin\tau_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\tau_i & -\cos\theta_i\sin\tau_i & a_i\sin\theta_i \\ 0 & \sin\tau_i & \cos\tau_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & \mathbf{A} & & \mathbf{t} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{5}$$

Hence, to describe the end-effector coordinate frame of a kinematic chain with respect to the base frame, the overall transformation matrix becomes

$${}^{0}_{i}\mathbf{T} = {}^{0}_{1}\mathbf{T}\,{}^{1}_{2}\mathbf{T}\,{}^{2}_{3}\mathbf{T} \dots {}^{i-1}_{i}\mathbf{T}. \tag{6}$$

Applying this algebraic representation to linkages requires that the end-effector coordinate frame coincides with the coordinate frame of the base. Therefore, the overall transformation equates to the identity matrix [18].

3. STUDY'S KINEMATIC MAPPING

Using matrices is one possibility of representing Euclidean displacements where orientation and distances are preserved. Another possibility was introduced by Eduard Study in 1903 [4]. He demonstrated that displacements can be represented as points on a hyper-surface in a seven-dimensional space, known as *kinematic mapping*. These points contain eight coordinates and are known as *Study parameters* or *soma coordinates*, $x = [x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3]^\mathsf{T} \in P^7$.

Representing displacement as points in a higher dimensional space result in algebraic varieties that can be illustrated geometrically and that allow for detailed analysis using algebraic tools. Moreover, in contrast to other displacement representations, such as Euler angles, the kinematic mapping introduced by Study is not subjected to representational singularities. The first four entries of the Study vector x_i are defined as one of the following combinations of the rotation matrix elements a_{ij} of Eq. (5) which excludes $x_i = (0 : 0 : 0 : 0)$

$$x_0 : x_1 : x_2 : x_3 = \begin{cases} 1 + a_{11} + a_{22} + a_{33} : a_{32} - a_{23} : a_{13} - a_{31} : a_{21} - a_{12}, \\ a_{32} - a_{23} : 1 + a_{11} - a_{22} - a_{33} : a_{12} + a_{21} : a_{31} + a_{13}, \\ a_{13} - a_{31} : a_{12} + a_{21} : 1 - a_{11} + a_{22} - a_{33} : a_{23} + a_{32}, \\ a_{21} - a_{12} : a_{31} + a_{13} : a_{23} + a_{32} : 1 - a_{11} - a_{22} + a_{33}. \end{cases} \tag{7}$$

The remaining four entries of the Study vector y_i are defined as linear combination of the first four entries x_i and the elements of the translation vector $\mathbf{t}$ of Eq. (5) following

$$\begin{aligned} y_0 &= \tfrac{1}{2}(t_3x_3 + t_2x_2 + t_1x_1), & y_1 &= \tfrac{1}{2}(t_3x_2 - t_2x_3 - t_1x_0), \\ y_2 &= \tfrac{1}{2}(-t_3x_1 + t_1x_3 - t_2x_0), & y_3 &= \tfrac{1}{2}(-t_3x_0 + t_2x_1 - t_1x_2). \end{aligned} \tag{8}$$

4. SLIDER-CRANK LINKAGE

The first step to derive the IO equation of the RRRP linkage is to identify the associated DH parameters of the linkage. Consider the RRRP linkage as an open-chain which is closed by aligning the base coordinate frame with the end-effector coordinate frame. As a result we can assign the coordinate frames as shown in Fig. 5, leading to the DH parameters as displayed in Table 1.

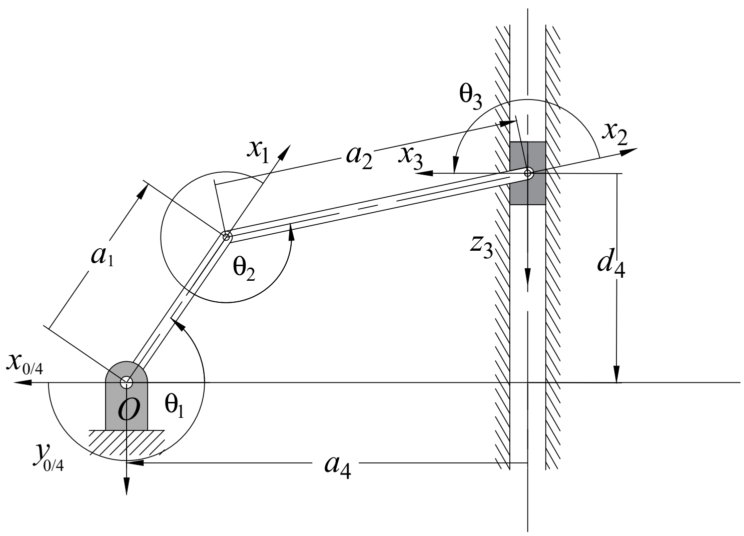

Fig. 5. DH parameter assignment for the RRRP linkage.

Note that the variables in this linkage are the input angle θ_1 and the output slider distance d_4. Moreover, note

Table 1. DH parameters for RRRP linkage.

joint axis i	link angle θ_i	link offset d_i	link length a_i	link twist τ_i
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0
3	θ_3	0	0	$-\pi/2$
4	0	d_4	a_4	$+\pi/2$

that the link lengths and the link offset are directed distances with the directions as indicated in the Fig. 5.

In a second step, the DH parameters are substituted into Eq. (5) for every $i = 1 \ldots 4$. The four transformation matrices are multiplied according to Eq. (6), and the resulting overall transformation ${}^0_4\mathbf{T}$ can be mapped into

Study's coordinates using Eq. (7) and Eq. (8). The vector yields

$$\begin{aligned}
x_0 &= (-2v_2-2v_3)v_1-2v_2v_3+2, \\
x_1 &= 0, \\
x_2 &= 0, \\
x_3 &= -2v_1v_2v_3+2v_1+2v_2+2v_3, \\
y_0 &= 0, \\
y_1 &= ((-d_4v_3-a_1+a_2+a_4)v_2+(-a_1-a_2+a_4)v_3+d_4)v_1+((a_1-a_2+a_4)v_3+d_4)v_2+d_4v_3-a_1-a_2-a_4, \\
y_2 &= (((a_1-a_2+a_4)v_3+d_4)v_2+d_4v_3-a_1-a_2-a_4)v_1+(d_4v_3+a_1-a_2-a_4)v_2+(a_1+a_2-a_4)v_3-d_4, \\
y_3 &= 0,
\end{aligned} \tag{9}$$

where $v_i = \tan(\theta_i/2)$.

The Study array is equated to its identity array, $x = [1:0:0:0:0:0:0:0]^\mathsf{T} \in P^7$, allowing the first and the last coordinate frame in the RRRP linkage to align. Since the Study coordinates are homogeneous, this leaves a system of three equations, i.e., I. $x_3 = 0$; II. $y_1 = 0$; III. $y_2 = 0$ which can easily be solved using resultants. Eliminating the intermediate link angle v_2 from I. and II. yields

$$a_1v_1^2v_3^2+a_2v_1^2v_3^2-a_4v_1^2v_3^2+a_1v_1^2-a_1v_3^2-a_2v_1^2+a_2v_3^2-a_4v_1^2-a_4v_3^2-a_1-a_2-a_4=0. \tag{10}$$

And eliminating the same intermediate angle v_2 from I. and III. yields

$$-d_4v_1^2v_3^2-2a_1v_1v_3^2+2a_2v_1^2v_3-d_4v_1^2-d_4v_3^2-2a_1v_1+2a_2v_3-d_4=0. \tag{11}$$

Finally, eliminating the intermediate angle v_3 from Eq. (10) and Eq. (11) reveals the IO equation for RRRP linkages

$$v_1^2d_4^2+d_4^2+4a_1d_4v_1+(a_1+a_2+a_4)(a_1-a_2+a_4)+(a_1-a_2-a_4)(a_1+a_2-a_4)v_1^2=0. \tag{12}$$

Eq. (12) can be verified via the IO equation of the planar 4R linkage, Eq. (3), as follows. Since the slider of the RRRP linkage is perpendicular to the fixed ground distance a_4, we can substitute $v_4 = \tan(-90°/2) = -1$ into Eq. (3). In addition, the original link length a_3 of the 4R linkage now becomes the slider distance of the RRRP, i.e., a_3 has to be renamed to d_4. After recollecting the equation in its variables v_1 and d_4 it yields Eq. (12).

5. BENNETT LINKAGE

The IO equation for the Bennett linkage is derived following the same procedure. First, as illustrated in Fig. (3) the coordinate frames are attached to the linkage according to the DH convention. The DH coordinate frame assignment allows to define the DH parameters for the linkage. It turns out that the Bennett linkage does not contain any link offsets d_i. However, it contains the four variable joint angles θ_i of the R joints and the design parameters a_i and τ_i. The DH parameters for the Bennett linkage are given in Table 2. To evaluate the overall transformation of the position and orientation of the last joint with respect to the base coordinate system, the DH parameters from Table 2 are substituted into Eq. (5), and multiplied according to Eq. (6).

Recall that the Bennett linkage is subjected to special conditions. The Bennett conditions from Eq. (2)

Table 2. DH parameters for the Bennett linkage.

joint axis i	link angle θ_i	link offset d_i	link length a_i	link twist τ_i
1	θ_1	0	a_1	τ_1
2	θ_2	0	a_2	τ_2
3	θ_3	0	a_3	τ_3
4	θ_4	0	a_4	τ_4

can be reformulated with Weierstraß substitution and expressed algebraically as

$$\begin{aligned}
a_2 &= \frac{a_1\alpha_2(\alpha_1^2+1)}{\alpha_1(\alpha_2^2+1)};\\
a_4 &= \frac{a_1\alpha_2(\alpha_1^2+1)}{\alpha_1(\alpha_2^2+1)};\\
a_3 &= a_1;\\
\alpha_3 &= \alpha_1;\\
\alpha_4 &= \alpha_2;
\end{aligned} \tag{13}$$

where $\alpha_i = \tan(\tau_i/2)$.

Eq. (13) is substituted into the overall transformation matrix which reduces the number of unknown parameters. After mapping the transformation matrix into Study's parameters, the vector yields

$$\begin{aligned}
x_0 = {} & \alpha_1^3\alpha_2^4v_1v_2v_3v_4+\alpha_1^3\alpha_2^4v_1v_2-\alpha_1^3\alpha_2^4v_1v_3+\alpha_1^3\alpha_2^4v_1v_4+\alpha_1^3\alpha_2^4v_2v_3-\alpha_1^3\alpha_2^4v_2v_4+\alpha_1^3\alpha_2^4v_3v_4\\
&+4\alpha_1^2\alpha_2^3v_1v_2v_3v_4-\alpha_1\alpha_2^4v_1v_2v_3v_4+\alpha_1^3\alpha_2^4-2\alpha_1^3\alpha_2^2v_1v_3+2\alpha_1^3\alpha_2^2v_1v_4+2\alpha_1^3\alpha_2^2v_2v_3-2\alpha_1^3\alpha_2^2v_2v_4\\
&-\alpha_1^3v_1v_2v_3v_4+4\alpha_1^2\alpha_2v_1v_2v_3v_4+\alpha_1\alpha_2^4v_1v_2-\alpha_1\alpha_2^4v_1v_3-\alpha_1\alpha_2^4v_1v_4-\alpha_1\alpha_2^4v_2v_3-\alpha_1\alpha_2^4v_2v_4\\
&+\alpha_1\alpha_2^4v_3v_4-\alpha_1^3v_1v_2-\alpha_1^3v_1v_3+\alpha_1^3v_1v_4+\alpha_1^3v_2v_3-\alpha_1^3v_2v_4-\alpha_1^3v_3v_4-4\alpha_1^2\alpha_2^3-\alpha_1\alpha_2^4\\
&-2\alpha_1\alpha_2^2v_1v_3-2\alpha_1\alpha_2^2v_1v_4-2\alpha_1\alpha_2^2v_2v_3-2\alpha_1\alpha_2^2v_2v_4+\alpha_1v_1v_2v_3v_4-\alpha_1^3-4\alpha_1^2\alpha_2-\alpha_1v_1v_2\\
&-\alpha_1v_1v_3-\alpha_1v_1v_4-\alpha_1v_2v_3-\alpha_1v_2v_4-\alpha_1v_3v_4+\alpha_1,\\
x_1 = {} & -2\alpha_1^3\alpha_2^3v_1v_2v_3v_4+2\alpha_1^2\alpha_2^4v_1v_2v_3v_4-2\alpha_1^3\alpha_2^3v_1v_2-2\alpha_1^3\alpha_2^3v_3v_4-2\alpha_1^3\alpha_2v_1v_2v_3v_4+2\alpha_1^2\alpha_2^4v_1v_4\\
&-2\alpha_1^2\alpha_2^4v_2v_3+2\alpha_1\alpha_2^3v_1v_2v_3v_4-2\alpha_1^3\alpha_2^3-2\alpha_1^3\alpha_2v_1v_2-2\alpha_1^3\alpha_2v_3v_4-2\alpha_1^2\alpha_2^4+4\alpha_1^2\alpha_2^2v_1v_4\\
&-4\alpha_1^2\alpha_2^2v_2v_3-2\alpha_1^2v_1v_2v_3v_4-2\alpha_1\alpha_2^3v_1v_2-2\alpha_1\alpha_2^3v_3v_4+2\alpha_1\alpha_2v_1v_2v_3v_4-2\alpha_1^3\alpha_2+2\alpha_1^2v_1v_4\\
&-2\alpha_1^2v_2v_3+2\alpha_1\alpha_2^3-2\alpha_1\alpha_2v_1v_2-2\alpha_1\alpha_2v_3v_4+2\alpha_1^2+2\alpha_1\alpha_2,\\
x_2 = {} & -2\alpha_1^3\alpha_2^3v_1v_3v_4+2\alpha_1^3\alpha_2^3v_2v_3v_4-2\alpha_1^2\alpha_2^4v_1v_2v_3-2\alpha_1^2\alpha_2^4v_2v_3v_4-2\alpha_1^3\alpha_2^3v_1+2\alpha_1^3\alpha_2^3v_2\\
&-2\alpha_1^3\alpha_2v_1v_3v_4+2\alpha_1^3\alpha_2v_2v_3v_4-2\alpha_1^2\alpha_2^4v_1-2\alpha_1^2\alpha_2^4v_4-4\alpha_1^2\alpha_2^2v_1v_2v_3-2\alpha_1\alpha_2^3v_1v_3v_4\\
&-2\alpha_1\alpha_2^3v_2v_3v_4-2\alpha_1^3\alpha_2v_1+2\alpha_1^3\alpha_2v_2-4\alpha_1^2\alpha_2^2v_4-2\alpha_1^2v_1v_2v_3+2\alpha_1^2v_2v_3v_4+2\alpha_1\alpha_2^3v_1+2\alpha_1\alpha_2^3v_2\\
&-2\alpha_1\alpha_2v_1v_3v_4-2\alpha_1\alpha_2v_2v_3v_4+2\alpha_1^2v_1-2\alpha_1^2v_4+2\alpha_1\alpha_2v_1+2\alpha_1\alpha_2v_2,\\
x_3 = {} & \alpha_1^3\alpha_2^4v_1v_2v_3-\alpha_1^3\alpha_2^4v_1v_2v_4+\alpha_1^3\alpha_2^4v_1v_3v_4-\alpha_1^3\alpha_2^4v_2v_3v_4+\alpha_1^3\alpha_2^4v_1-\alpha_1^3\alpha_2^4v_2+\alpha_1^3\alpha_2^4v_3-\alpha_1^3\alpha_2^4v_4\\
&+2\alpha_1^3\alpha_2^2v_1v_2v_3-2\alpha_1^3\alpha_2^2v_1v_2v_4-4\alpha_1^2\alpha_2^3v_2v_3v_4-\alpha_1\alpha_2^4v_1v_2v_3-\alpha_1\alpha_2^4v_1v_2v_4+\alpha_1\alpha_2^4v_1v_3v_4\\
&+\alpha_1\alpha_2^4v_2v_3v_4+2\alpha_1^3\alpha_2^2v_3-2\alpha_1^3\alpha_2^2v_4+\alpha_1^3v_1v_2v_3-\alpha_1^3v_1v_2v_4-\alpha_1^3v_1v_3v_4+\alpha_1^3v_2v_3v_4-4\alpha_1^2\alpha_2^3v_1\\
&-4\alpha_1^2\alpha_2v_2v_3v_4-\alpha_1\alpha_2^4v_1-\alpha_1\alpha_2^4v_2+\alpha_1\alpha_2^4v_3+\alpha_1\alpha_2^4v_4-2\alpha_1\alpha_2^2v_1v_2v_3-2\alpha_1\alpha_2^2v_1v_2v_4-\alpha_1^3v_1\\
&+\alpha_1^3v_2+\alpha_1^3v_3-\alpha_1^3v_4-4\alpha_1^2\alpha_2v_1+2\alpha_1\alpha_2^2v_3+2\alpha_1\alpha_2^2v_4-\alpha_1v_1v_2v_3-\alpha_1v_1v_2v_4-\alpha_1v_1v_3v_4\\
&-\alpha_1v_2v_3v_4+\alpha_1v_1+\alpha_1v_2+\alpha_1v_3+\alpha_1v_4,
\end{aligned}$$

$$
\begin{aligned}
y_0 \;=\;& -2a_1\alpha_1^4\alpha_2^2v_1v_2v_3v_4+4a_1\alpha_1^3\alpha_2^3v_1v_2v_3v_4-2a_1\alpha_1^2\alpha_2^4v_1v_2v_3v_4-2a_1\alpha_1^4\alpha_2^2v_1v_2-2a_1\alpha_1^4\alpha_2^2v_3v_4\\
&-2a_1\alpha_1^2\alpha_2^4v_1v_4-2a_1\alpha_1^2\alpha_2^4v_2v_3-2a_1\alpha_1^4\alpha_2^2-4a_1\alpha_1^3\alpha_2^3-2a_1\alpha_1^2\alpha_2^4-4a_1\alpha_1^2\alpha_2^2v_1v_2-4a_1\alpha_1^2\alpha_2^2v_1v_4\\
&-4a_1\alpha_1^2\alpha_2^2v_2v_3-4a_1\alpha_1^2\alpha_2^2v_3v_4+2a_1\alpha_1^2v_1v_2v_3v_4-4a_1\alpha_1\alpha_2v_1v_2v_3v_4+2a_1\alpha_2^2v_1v_2v_3v_4-2a_1\alpha_1^2v_1v_4\\
&-2a_1\alpha_1^2v_2v_3-2a_1\alpha_2^2v_1v_2-2a_1\alpha_2^2v_3v_4+2a_1\alpha_1^2+4a_1\alpha_1\alpha_2+2a_1\alpha_2^2,\\
y_1 \;=\;& -a_1\alpha_1^4\alpha_2^3v_1v_2v_3v_4+a_1\alpha_1^3\alpha_2^4v_1v_2v_3v_4-a_1\alpha_1^4\alpha_2^3v_1v_2-a_1\alpha_1^4\alpha_2^3v_3v_4+a_1\alpha_1^4\alpha_2v_1v_2v_3v_4\\
&+a_1\alpha_1^3\alpha_2^4v_1v_4-a_1\alpha_1^3\alpha_2^4v_2v_3-4a_1\alpha_1^3\alpha_2^2v_1v_2v_3v_4+4a_1\alpha_1^2\alpha_2^3v_1v_2v_3v_4-a_1\alpha_1\alpha_2^4v_1v_2v_3v_4-a_1\alpha_1^4\alpha_2^3\\
&+a_1\alpha_1^4\alpha_2v_1v_2+a_1\alpha_1^4\alpha_2v_3v_4-a_1\alpha_1^3\alpha_2^4+2a_1\alpha_1^3\alpha_2^2v_1v_4-2a_1\alpha_1^3\alpha_2^2v_2v_3-a_1\alpha_1^3v_1v_2v_3v_4\\
&-2a_1\alpha_1^2\alpha_2^3v_1v_2-2a_1\alpha_1^2\alpha_2^3v_3v_4+4a_1\alpha_1^2\alpha_2v_1v_2v_3v_4-a_1\alpha_1\alpha_2^4v_1v_4+a_1\alpha_1\alpha_2^4v_2v_3\\
&-4a_1\alpha_1\alpha_2^2v_1v_2v_3v_4+a_1\alpha_2^3v_1v_2v_3v_4+a_1\alpha_1^4\alpha_2+4a_1\alpha_1^3\alpha_2^2+a_1\alpha_1^3v_1v_4-a_1\alpha_1^3v_2v_3+4a_1\alpha_1^2\alpha_2^3\\
&+2a_1\alpha_1^2\alpha_2v_1v_2+2a_1\alpha_1^2\alpha_2v_3v_4+a_1\alpha_1\alpha_2^4-2a_1\alpha_1\alpha_2^2v_1v_4+2a_1\alpha_1\alpha_2^2v_2v_3+a_1\alpha_1v_1v_2v_3v_4\\
&-a_1\alpha_2^3v_1v_2-a_1\alpha_2^3v_3v_4-a_1\alpha_2v_1v_2v_3v_4+a_1\alpha_1^3+4a_1\alpha_1^2\alpha_2+4a_1\alpha_1\alpha_2^2-a_1\alpha_1v_1v_4+a_1\alpha_1v_2v_3\\
&+a_1\alpha_2^3+a_1\alpha_2v_1v_2+a_1\alpha_2v_3v_4-a_1\alpha_1-a_1\alpha_2,\\
y_2 \;=\;& -a_1\alpha_1^4\alpha_2^3v_1v_3v_4+a_1\alpha_1^4\alpha_2^3v_2v_3v_4-a_1\alpha_1^3\alpha_2^4v_1v_2v_3-a_1\alpha_1^3\alpha_2^4v_2v_3v_4-a_1\alpha_1^4\alpha_2^3v_1+a_1\alpha_1^4\alpha_2^3v_2\\
&+a_1\alpha_1^4\alpha_2v_1v_3v_4-a_1\alpha_1^4\alpha_2v_2v_3v_4-a_1\alpha_1^3\alpha_2^4v_1-a_1\alpha_1^3\alpha_2^4v_4-2a_1\alpha_1^3\alpha_2^2v_1v_2v_3+4a_1\alpha_1^3\alpha_2^2v_2v_3v_4\\
&-2a_1\alpha_1^2\alpha_2^3v_1v_3v_4-4a_1\alpha_1^2\alpha_2^3v_2v_3v_4+a_1\alpha_1\alpha_2^4v_1v_2v_3+a_1\alpha_1\alpha_2^4v_2v_3v_4+a_1\alpha_1^4\alpha_2v_1-a_1\alpha_1^4\alpha_2v_2\\
&+4a_1\alpha_1^3\alpha_2^2v_1-2a_1\alpha_1^3\alpha_2^2v_4-a_1\alpha_1^3v_1v_2v_3+a_1\alpha_1^3v_2v_3v_4+4a_1\alpha_1^2\alpha_2^3v_1+2a_1\alpha_1^2\alpha_2^3v_2\\
&+2a_1\alpha_1^2\alpha_2v_1v_3v_4-4a_1\alpha_1^2\alpha_2v_2v_3v_4+a_1\alpha_1\alpha_2^4v_1+a_1\alpha_1\alpha_2^4v_4+2a_1\alpha_1\alpha_2^2v_1v_2v_3+4a_1\alpha_1\alpha_2^2v_2v_3v_4\\
&-a_1\alpha_2^3v_1v_3v_4-a_1\alpha_2^3v_2v_3v_4+a_1\alpha_1^3v_1-a_1\alpha_1^3v_4+4a_1\alpha_1^2\alpha_2v_1-2a_1\alpha_1^2\alpha_2v_2+4a_1\alpha_1\alpha_2^2v_1\\
&+2a_1\alpha_1\alpha_2^2v_4+a_1\alpha_1v_1v_2v_3-a_1\alpha_1v_2v_3v_4+a_1\alpha_2^3v_1+a_1\alpha_2^3v_2+a_1\alpha_2v_1v_3v_4+a_1\alpha_2v_2v_3v_4-a_1\alpha_1v_1\\
&+a_1\alpha_1v_4-a_1\alpha_2v_1-a_1\alpha_2v_2,\\
y_3 \;=\;& -2a_1\alpha_1^4\alpha_2^2v_1v_3v_4+2a_1\alpha_1^4\alpha_2^2v_2v_3v_4-4a_1\alpha_1^3\alpha_2^3v_2v_3v_4-2a_1\alpha_1^2\alpha_2^4v_1v_2v_3+2a_1\alpha_1^2\alpha_2^4v_2v_3v_4\\
&-2a_1\alpha_1^4\alpha_2^2v_1+2a_1\alpha_1^4\alpha_2^2v_2-4a_1\alpha_1^3\alpha_2^3v_1-2a_1\alpha_1^2\alpha_2^4v_1+2a_1\alpha_1^2\alpha_2^4v_4-4a_1\alpha_1^2\alpha_2^2v_1v_2v_3\\
&-4a_1\alpha_1^2\alpha_2^2v_1v_3v_4+4a_1\alpha_1^2\alpha_2^2v_2+4a_1\alpha_1^2\alpha_2^2v_4-2a_1\alpha_1^2v_1v_2v_3-2a_1\alpha_1^2v_2v_3v_4+4a_1\alpha_1\alpha_2v_2v_3v_4\\
&-2a_1\alpha_2^2v_1v_3v_4-2a_1\alpha_2^2v_2v_3v_4+2a_1\alpha_1^2v_1+2a_1\alpha_1^2v_4+4a_1\alpha_1\alpha_2v_1+2a_1\alpha_2^2v_1+2a_1\alpha_2^2v_2.
\end{aligned}
$$

Again, to form a closed-loop chain it requires that the base and the fourth coordinate frames align. Hence, the Study array is equated to the identity array. As the Study coordinates are homogeneous, i.e., a point represented by these coordinates remains unchanged if every entry is multiplied by the same factor, the system of equations which has to be solved consists of seven equations.

One method to eliminate the intermediate link angles is using Gröbner basis. A pure lexicographic ordering of $(x_1 > \ldots > x_n) \rightarrow (v_2 > v_3 > v_4 > v_1)$ reveals one polynomial that no longer contains v_2 and v_3

$$(v_4^2+1)(v_1^2+1)((\alpha_1-\alpha_2)v_1v_4-\alpha_1-\alpha_2)=0. \tag{14}$$

Since the expressions (v_4^2+1) and (v_1^2+1) can never be zero, we can safely divide Eq. (14) by these two terms. This yields the IO equation for the Bennett linkage

$$(\alpha_1-\alpha_2)v_1v_4-\alpha_1-\alpha_2=0, \tag{15}$$

where $\alpha_i=\tan(\tau_i/2)$ and $v_i=\tan(\theta_i/2)$. Eq. (15) is identical to the IO relation of the Bennett linkage obtained in [13] after algebraisation with tangent half-angle substitutions.

6. CONCLUSIONS

This paper applied a novel method to derive the IO equation of the planar RRRP and the Bennett linkage. This method has shown previous success in the derivation of the IO equation for planar and spherical 4R linkages. With this paper, however, its applicability has been extended to all known types of 4R linkages: planar; spherical; and spatial.

In general, the procedure requires defining the DH parameters of the examined linkage which are used to calculate the overall change in orientation and position of the last coordinate frame with respect to its base coordinate frame. The transformation is projected into Study coordinates, and the open linkage is conceptually closed by equating the Study array to its identity. Depending on the system of equations that has to be solved to obtain the respective IO equation, an appropriate method to eliminate the intermediate angle parameters has to be chosen. In the case of the planar RRRP linkage using resultants is sufficient, and in the case of the Bennett linkage Gröbner basis leads to the desired result. We believe the presented method can streamline the way IO equations are derived for any type of mechanism which could help leveraging the design process of arbitrary linkages.

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