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MAXIMUM AREA ELLIPSES INSCRIBING SPECIFIC QUADRILATERALS

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Abstract— In this paper synthetic and analytic projective and Euclidean geometry are used to map the unit circle inscribing the unit square to an ellipse possessing the largest area inscribing special prescribed convex quadrilaterals; namely parallelograms and trapezoids. The transformation of the unit circle to the inscribing ellipse is the inverse transformation that maps the vertices of the convex quadrilateral to the vertices of the unit square. We describe the computational geometric details of determining the transformation matrix that is used in the mapping. Additionally, employing synthetic geometric arguments, we prove that the resulting inscribing ellipses are the ones possessing the maximum area in the corresponding one parameter pencil of ellipses inscribing the prescribed convex quadrilaterals.

Keywords- projective, affine, Euclidean geometry; collineations; parallelograms; trapezoids; inscribing ellipses.

I. INTRODUCTION

In this paper classical analytic projective and Euclidean geometry [1-4] are used to determine the largest area inscribing ellipse subject to specific linear constraints, namely convex quadrilaterals, that are either parallelograms or trapezoids. This topic has generated only scant interest in the literature despite a relatively large range of applications. For example, error ellipses subject to linear constraints are important in statistical analysis. Error ellipses arise when considering the covariance between two statistical variables. The lengths of the semi-major and semi-minor error ellipse axes are the square roots of the absolute values of the eigenvalues of the associated covariance matrix [1]. Given additional constant linear constraints on the ellipse, the maximum area ellipse inscribing those constraints is a measure of the maximum covariance between the variables.

Determining the maximum area inscribing ellipse given linear constraints that form convex quadrilaterals provides an alternate approach to characterizing the velocity performance of parallel mechanisms in the presence of actuation redundancy, as reported in [2]. Therein the aim is to determine the ellipse with the largest area that inscribes an arbitrary polygon. In this context, the area of the ellipse is proportional to the kinematic isotropy of the mechanism, while the polygon is defined by the reachable workspace of the mechanism [2]. There, the approach is a numerical maximization problem, essentially fitting the largest area inscribing ellipse starting with a unit circle.

There are only a handful of papers that report investigations into determining maximum area ellipses inscribing arbitrary polygons, to the best of the author's knowledge. The dual problem, that is the problem of determining the polygons of greatest area inscribed in an ellipse is reported in [3]. While interesting, this dual problem is not germane to determining the maximum area ellipse inscribing a polygon. Three papers by the same author [8-10] appear to lead to a solution to the general problem of finding the largest area ellipse inscribing an n-sided convex polygon, however the papers focus on the proof of the existence of a solution rather than an explicit algorithm for computing the ellipse equation, or shape coefficients.

In this paper, we build on the approach presented in [4] to identify the parametric equation of the maximum area ellipse inscribing an arbitrary parallelogram, or trapezoid. A projective collineation is a transformation that maps collinear points onto collinear points in the projective plane. We propose to determine the general planar projective collineation that maps the unit circle inscribing a unit square onto an ellipse that inscribes the prescribed convex polygon. Since the coordinates of the vertices of both the unit square and the given polygon are known, it is a simple matter to compute the transformation that maps the vertices of the given polygon onto the unit square. The inverse of the same transformation is then used to map the homogeneous parametric equation of the inscribing unit circle onto the corresponding ellipse that inscribes the prescribed polygon. The unit circle that inscribes the unit square is clearly its largest inscribing ellipse. The transformed ellipse that inscribes an arbitrary convex quadrilateral will generally not be the one possessing the largest area. However, if the prescribed convex quadrilateral is either a parallelogram or a trapezoid, then the resulting inscribing ellipse will indeed possess the maximum area of the entire pencil of inscribing ellipses. Herein we describe a simple construction for maximum area inscribing ellipses in parallelograms and trapezoids. That is, we show how to identify the projective coordinate transformation that maps a unit circle inscribing a unit square onto an ellipse inscribing a given arbitrary convex parallelogram or trapezoid. By virtue of the properties of parallelograms and trapezoids, we prove, using synthetic arguments, that the resulting inscribing ellipses mapped from the unit circle are indeed those possessing the maximum area.

II. MATHEMATICAL BACKGROUND

It is well known that the closed second order curve possessing the largest area that inscribes the unit square is the unit circle. This is seen to be true when one considers that for an arbitrary convex quadrilateral there is a one parameter pencil of inscribing ellipses whose centres all lie along the open line segment joining the midpoints of the two internal diagonals, or all ellipses in the pencil possess the same centre when the diagonals intersect at their midpoints [5]. The unit square is a special symmetric parallelogram. It is well known that the area maximising ellipse inscribed in a parallelogram is the one whose tangent points are the midpoints of the parallelogram edges [5]. For the unit square, the inscribing unit circle tangent points are the midpoints of the edges of the square, and hence the unit circle possess the largest area of the pencil of inscribing ellipses. The area maximising ellipse that inscribes a trapezoid is the one whose centre is the midpoint of the join of the diagonal midpoints [5].

A. *Projective Transformations*

Two distinct sets of four points in the projective plane P_2 uniquely determine a projective collineation if the points in the two sets are distinct, and if no three are on the same line. Let the first set of four points have the coordinates $W(W_0 : W_1 : W_2)$, $X(X_0 : X_1 : X_2)$, $Y(Y_0 : Y_1 : Y_2)$, and $Z(Z_0 : Z_1 : Z_2)$. Let the second set of four points have the coordinates $w(w_0 : w_1 : w_2)$, $x(x_0 : x_1 : x_2)$, $y(y_0 : y_1 : y_2)$, and $z(z_0 : z_1 : z_2)$.

When expressed as a vector, the ratios implied by the homogeneous coordinates can be expressed by an arbitrary scaling factor:

$$[w_0 : w_1 : w_2]^T = \mu[w_0 : w_1 : w_2]^T. \quad (1)$$

The corresponding Euclidean coordinates are

$$x_w = \frac{\mu w_1}{\mu w_0}; \qquad y_w = \frac{\mu w_2}{\mu w_0}. \quad (2)$$

This is why different scalar multiples of a set of homogeneous coordinates represent the same point in the affine or projective plane.

The projective collineation may be viewed as a linear transformation that maps the coordinates of a point described in a particular coordinate system onto the coordinates of a different point in the same coordinate system. The geometry can be represented by the vector-algebraic relationship

$$\lambda \begin{bmatrix} W_0 \\ W_1 \\ W_2 \end{bmatrix} = \mu \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}. \quad (3)$$

Without loss in generality, we can set $\rho = \lambda/\mu$ and express Equation (3) more compactly as

$$\rho\mathbf{W} = \mathbf{T}\mathbf{w}, \quad \text{or} \quad \mathbf{T}\mathbf{w} - \rho\mathbf{W} = \mathbf{0}. \quad (4)$$

The elements of the linear transformation matrix depend on the details of the mapping. As it represents a general projective collineation there are no orthogonality conditions on the rows or columns of $\mathbf{T}$. This means that the elements can take on any numerical value. Thus the mapping between two points in an arbitrary collineation consists of nine variables. If we wish to determine the mapping given a point and its image then $\mathbf{T}$ represents nine unknowns, but, because of the use of homogenous coordinates, at most eight are independent. Still, to remain general the scaling factor ρ must be counted among the unknowns because the given points come from a Cartesian coordinate system imposed on the Euclidean plane while the mapping is projective. The result is that the coordinates of four points, along with those of their images, are enough to uniquely define the eight independent elements of the transformation matrix and the four independent scaling factors, ρ_i, $i \in \{1, 2, 3, 4\}$.

The vertices of an arbitrary quadrilateral represent four points W, X, Y, and Z. We consider the image of these four points w, x, y, and z, to be the vertices of the square containing the unit circle centred on the origin of the Cartesian coordinate system in which the quadrilateral is defined. Now a set of equations must be written so that the elements of $\mathbf{T}$ can be computed in terms of the point and image coordinates:

$$\begin{gathered} t_{11}w_0 + t_{12}w_1 + t_{13}w_2 - \rho_1 W_0 = 0, \\ t_{21}w_0 + t_{22}w_1 + t_{23}w_2 - \rho_1 W_1 = 0, \\ t_{31}w_0 + t_{32}w_1 + t_{33}w_2 - \rho_1 W_2 = 0, \\ t_{11}x_0 + t_{12}x_1 + t_{13}x_2 - \rho_2 X_0 = 0, \\ \vdots \\ t_{31}z_0 + t_{32}z_1 + t_{33}z_2 - \rho_4 Z_2 = 0. \end{gathered} \quad (5)$$

Equations (5) represent 12 equations in 13 unknowns, 12 of which are independent, hence we can arbitrarily scale the elements of $\mathbf{T}$ by $1/t_{11}$, thereby setting $t_{11} = 1$. It is a simple matter to solve for the 12 unknowns, however we only require the eight independent elements of $\mathbf{T}$.

B. *Ellipses in the Euclidean Plane*

A general conic in the Euclidean plane is a projection of a unique projective curve. That is, a projective curve has many different affine views: ellipses, hyperbolae, and parabolae are all different affine views of the same projective curve projected into the Euclidean plane using different points of perspective of the curve in the projective plane [6]. Using homogenous coordinates, any conic c in the affine and Euclidean planes can be described by a general second order implicit polynomial equation of the form

$$a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 = 0, \quad (6)$$

or in matrix form

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = 0, \tag{7}$$

where $\mathbf{x}$ is a vector representing the homogeneous coordinates of a point on the conic, and $\mathbf{A}$ is a symmetric 3×3 matrix with at least one non-zero element which represents the shape coefficients of the conic. In component form the arrays are:

$$\begin{bmatrix} x_0 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = 0. \tag{8}$$

For a conic to be one that is regular and non-degenerate it must be that $\det(\mathbf{A}) \neq 0$. An affine classification for regular non-degenerate conics is given by evaluating the *discriminant* of the coefficient matrix $\mathbf{A}$. The discriminant of $\mathbf{A}$ is defined to be the 2×2 subdeterminant, denoted by Δ, of its quadratic form, denoted by $\mathbf{Q}$, such that

$$\Delta := \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2. \tag{9}$$

There are three possibilities [6]:

$$c \text{ is } \left\{ \begin{array}{c} \text{an ellipse} \\ \text{a parabola} \\ \text{an hyperbola} \end{array} \right\} \text{ iff } \Delta \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0.$$

The centre of a central conic c can be computed using the partial derivatives of the general conic, Equation (6), with respect to x_1 and x_2, then solving the resulting pair of linear equations for the pair of x_1 and x_2 [5]. The solution of this linear system represents the Cartesian coordinates of the centre, if it exists. However, the equations for the centre coordinates are easily generalised.

$$\frac{\partial c}{\partial x_1} = 2a_{01}x_0 + 2a_{11}x_1 + 2a_{12}x_2 = 0, \tag{10}$$

$$\frac{\partial c}{\partial x_2} = 2a_{02}x_0 + 2a_{12}x_1 + 2a_{22}x_2 = 0. \tag{11}$$

Solving Equations (10) and (11) for x_1 and x_2 yields the centre coordinates $(x_{1_{\mathrm{cen}}}, x_{2_{\mathrm{cen}}})$:

$$\begin{bmatrix} x_{1_{\mathrm{cen}}} \\ x_{2_{\mathrm{cen}}} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}^2} \begin{bmatrix} x_0(a_{01}a_{22} - a_{02}a_{12}) \\ x_0(a_{01}a_{12} - a_{02}a_{11}) \end{bmatrix} \tag{12}$$

Note that the denominator on the left hand side of Equation (12) is exactly the discriminant of the general conic coefficient matrix. Hence, Equation (12) will reveal a unique solution for the coordinates of the centre for any central conic (ellipse or hyperbola); infinitely many solutions if the conic is a degenerate pair of parallel lines; and no finite solution if the conic is a parabola.

The matrix $\mathbf{A}$ and its discriminant can be used to evaluate the area of a conic. If c is an ellipse then the area that it encloses can be computed using the formula [7]

$$\mathrm{area}(c) = \left(\frac{\det(\mathbf{A})}{\Delta^{3/2}} \right) \pi. \tag{13}$$

Note that for hyperbolae and parabolae, where $\Delta \leq 0$, the value of this function is either purely imaginary, or infinite.

While we don't require the major and minor axes of the ellipses for the application in this paper, for completeness we describe an algebraic method to determine them. The quadratic form associated with the general planar implicit polynomial equation of the second degree, Equation (6), is

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0, \tag{14}$$

and the associated symmetric matrix is

$$\mathbf{Q} := \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}. \tag{15}$$

The eigenvalues of the characteristic equation of the quadratic form are defined as the determinant

$$|\lambda\mathbf{I} - \mathbf{Q}| = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{12} & \lambda - a_{22} \end{vmatrix} = 0, \tag{16}$$

where the λ are the eigenvalues of $\mathbf{Q}$ and $\mathbf{I}$ is the 2×2 identity matrix. The resulting equation is a quadratic in terms of the eigenvalues:

$$\lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}^2. \tag{17}$$

The solutions to Equation (17) reveal two real eigenvalues for every non-degenerate conic, which in general are:

$$\lambda = \frac{1}{2}\left(a_{11} + a_{12} \pm \sqrt{a_{11}^2 + 2a_{11}a_{12} - 4a_{11}a_{22} + 5a_{12}^2} \right). \tag{18}$$

If the conic is an ellipse, then the eigenvectors associated with each of the eigenvalues are parallel to the directions of the major and minor ellipse axes: the major axis is parallel to the eigenvector corresponding to the eigenvalue possessing the smallest absolute value. Given the eigenvalues, the eigenvectors are the solutions to

$$|\lambda\mathbf{I} - \mathbf{Q}| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{19}$$

Now the major and minor axes of an ellipse are obtained by establishing the parametric equations of the lines through the centre, established with Equation (12), which are parallel to the eigenvectors.

C. *Polar Lines and Pole Points With Respect to a Conic*

The pole and polar are respectively a point and a line that have a unique reciprocal relationship with respect to a given conic section. If the point lies on the conic section, its polar is the tangent line to the conic section at that point. Hence, given a conic section and a line tangent to the conic, the corresponding pole point is the tangent point between the polar line and conic. For an ellipse that inscribes a convex quadrilateral, the edges of the quadrilateral are polar lines to the ellipse, and the pole points are the tangent points between the edges and the ellipse.

Given the homogeneous coordinates of a pole point, denoted by the vector $\mathbf{p}$, on a general non-degenerate conic, the vector of line coordinates, $\mathbf{l}_p$, for the corresponding polar line are obtained with the product

$$\mathbf{l}_p \quad = \quad \mathbf{A}\mathbf{p}. \tag{20}$$

Hence, it is a simple matter to determine the pole point coordinate vector given a polar line and a conic, which is:

$$\mathbf{p} \quad = \quad \mathbf{A}^{-1}\mathbf{l}_p. \tag{21}$$

Equation (21) will be used to determine the pole points for our generated ellipses to prove that they are the inscribing ellipses possessing the maximum area.

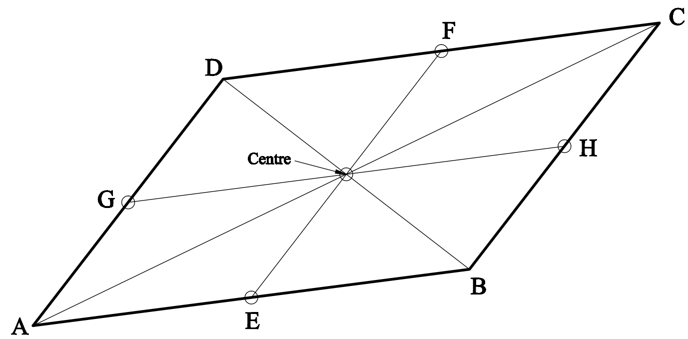

Figure 2. Pole points and centre of largest area Ellipse inscribing a parallelogram.

D. Relevant Properties of Squares, Parallelograms, and Trapezoids

The projective transformation method employed in this paper to transform a unit circle inscribing a unit square to an ellipse inscribing a parallelogram or trapezoid exploits the fact that similar properties are preserved by the transformation. Consider the square illustrated in Figure 1. There is a pencil of ellipses inscribing the square, all centred at the midpoint of the two diagonals, AC and BD, which is where they intersect at the centre square. The one possessing the largest area is the circle, whose pole points are the midpoints of the four edges, E, F, G, and H.

Consider the parallelogram illustrated in Figure 2. The similarity transformation that maps a square to a parallelogram preserves the property that the largest inscribing ellipse, among the pencil of inscribing ellipses centred at the intersection of the diagonals AC and BD, has pole points located at the mid points of the edges E, F, G, and H [5, 6].

Trapezoids are a geometric departure from parallelograms. However, two opposite edges in a trapezoid are always parallel, see Figure 3. The lines containing the two non parallel edges always intersect in a finite point, G in Figure 3. While the two diagonals, AC and BD, intersect on the interior of the trapezoid, they never both intersect at their midpoints, t and s. Similar to any convex quadrilateral, any inscribing ellipse has its centre located on the open line segment joining Points t and s. The bounding ellipses in the one parameter pencil inscribing the trapezoid, or any convex quadrilateral, are the two diagonals (doubly mapped) whose centres are at points t and s, respectively. The inscribing ellipse possessing the greatest area is the one whose centre is located at the midpoint of the open line segment joining the midpoints of the two diagonals [5, 6].

The similarity property that is maintained in the projective correspondence between the square and trapezoid vertices in the Euclidean plane is that the maximum area inscribing ellipse has its centre at the midpoint of the open line segment joining the midpoints of the two diagonals, and has pole points at the midpoints of the two parallel edges [7]. Additionally, the line joining the midpoints of the two diagonals is parallel to the

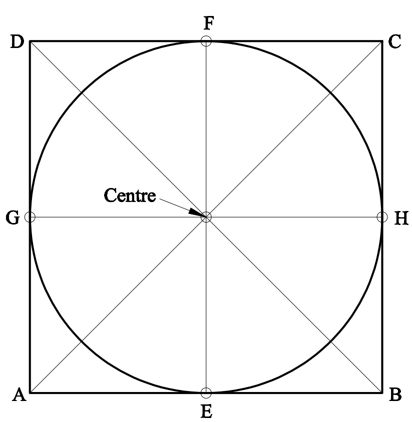

Figure 1. Pole points and centre of largest area Ellipse inscribing a square.

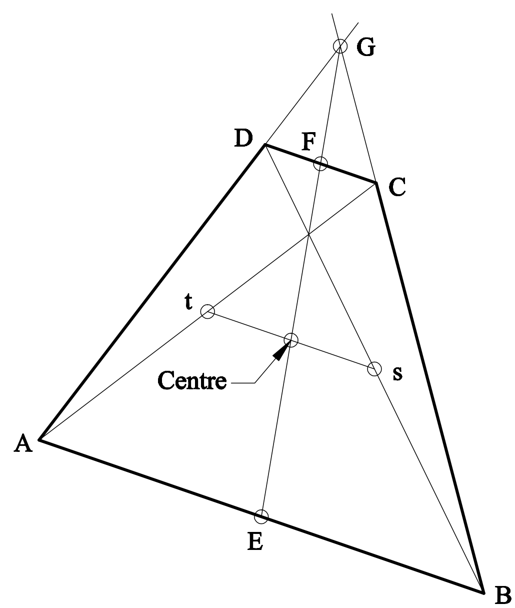

Figure 3. Pole points and centre of largest area Ellipse inscribing a trapezoid.

two parallel edges, and the point where the extension of the two non parallel edges intersect (G), the midpoints of the two parallel edges (E and F), the intersection point of the two interior diagonals, and the midpoint of the line segment joining the midpoints of the two diagonals are all collinear.

III. Examples

We now consider two examples where the bounding quadrilaterals are a parallelogram and a trapezoid, respectively.

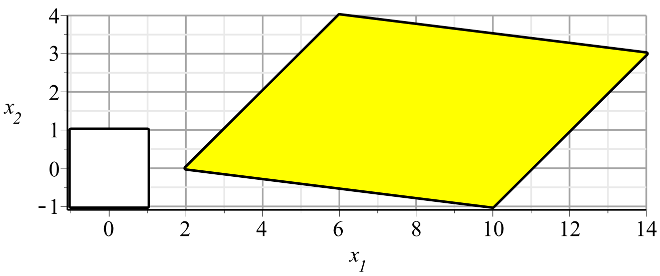

Figure 4. Unit square and a boundary parallelogram.

A. *Parallelogram*

Consider the unit square and parallelogram shown in Figure 4. The homogeneous coordinates of the vertices of the boundary quadrilateral are $W(1:2:0)$, $X(1:10:-1)$, $Y(1:14:3)$, and $Z(1:6:4)$, while the image points, the square vertices, have homogeneous coordinates $w(1:1:1)$, $x(1:1:-1)$, $y(1:-1:-1)$, and $z(1:-1:1)$. The projective collineation defined by the vertices of the two quadrilaterals is

$$T = \frac{1}{18}\begin{bmatrix} 18 & 0 & 0 \\ -26 & 4 & -4 \\ -20 & 1 & 8 \end{bmatrix}. \tag{22}$$

The unit circle, centred on the origin, inscribing the unit square is the largest area inscribing ellipse. The matrix $\mathbf{T}$ is used to transform its parametric equation. But, because of how the problem has been posed, the circle represents the image of the ellipse that inscribes the boundary quadrilateral. To obtain the parametric equation of the desired ellipse, the inverse of $\mathbf{T}$ premultiplies the unit circle parametric equation:

$$\mathbf{e} = \mathbf{T}^{-1}\mathbf{c}, \tag{23}$$

where $\mathbf{c} = [1 : \cos(\theta) : \sin(\theta)]$, and the resulting normalised parametric ellipse equation is

$$\mathbf{e} = \begin{bmatrix} 1 \\ 8 + 4\cos(\theta) + 2\sin(\theta) \\ 1/2(3 - \cos(\theta) + 4\sin(\theta)) \end{bmatrix}. \tag{24}$$

The resulting ellipse is the affine image of the unit circle, see Figure 5. Using Equation (24) it is a simple matter to compute five points on the ellipse, and use these to obtain the implicit form, i.e. Equation (6), of the ellipse, which is computed to be:

$$-1075.71000x_0^2 - 24.31791x_1^2 - 114.43723x_2^2 + 354.75541x_0x_1 + 160.21212x_0x_2 + 22.88745x_1x_2. \tag{25}$$

Now, using Equation (12), the Cartesian coordinates of the ellipse centre are computed to be $(8, 3/2)$, which are the same as those of the mutual midpoints of the diagonals. Using Equation (21), the Cartesian coordinates of the pole points of the ellipse are $(6, 1/2)$, $(12, 1)$, $(10, 7/2)$, and $(4, 2)$, which are the corresponding midpoints of the four parallelogram edges illustrated in Figure 5. By virtue of these facts, the computed ellipse is the one in the one parameter pencil of inscribing ellipses possessing the greatest area. Using Equation (13), that area is 28.27 square generic drawing units.

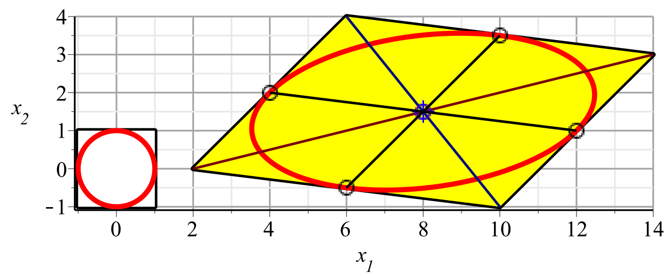

Figure 5. Maximum area ellipse inscribing a parallelogram.

B. *Trapezoid*

Consider the unit square and trapezoid illustrated in Figure 6. The homogeneous coordinates of the vertices of the boundary quadrilateral are $W(1:2:-1)$, $X(1:6:1)$, $Y(1:5:4)$, and $Z(1:3:3)$, while the image points, the square vertices, have the same coordinates as before. The projective collineation defined by the vertices of the two quadrilaterals is

$$T = \frac{1}{10}\begin{bmatrix} 10 & 1 & -2 \\ -28 & 7 & 0 \\ -2 & -3 & 6 \end{bmatrix}. \tag{26}$$

Using Equation (23) we obtain the normalised parametric equation for the ellipse:

$$\mathbf{e} = \frac{1}{3 + \sin(\theta)}\begin{bmatrix} 3 + \sin(\theta) \\ 4(3 + \cos(\theta) + \sin(\theta)) \\ 7 + 2\cos(\theta) + 7\sin(\theta)) \end{bmatrix}. \tag{27}$$

The resulting ellipse is the affine image of the unit circle, see Figure 7. The implicit form of the ellipse is computed to be:

$$-2.794252x_0^2 - 0.2315005x_1^2 - 0.1299652x_2^2 + 1.6245650x_0x_1 - 0.0649826x_0x_2 + 0.1299652x_1x_2. \tag{28}$$

Using Equation (12), the Cartesian coordinates of the ellipse centre are $(4, 7/4)$, which are the same as those of the midpoint

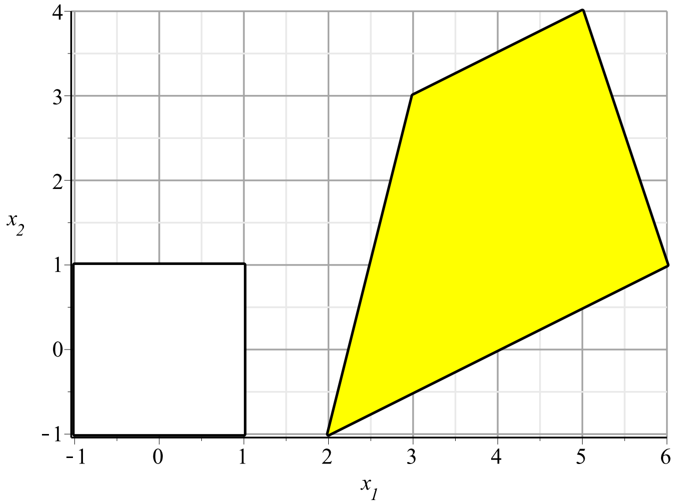

Figure 6. Unit square and a boundary trapezoid.

of the open line segment joining the midpoints of the diagonals. Using Equation (21), the Cartesian coordinates of the pole points of the ellipse are $(4,0)$, $(16/3,3)$, $(4,7/2)$, and $(8/3,5/3)$, the first and third corresponding to the midpoints of the two parallel trapezoid edges, see Figure 7. By virtue of these facts, the computed ellipse is the one in the one parameter pencil of inscribing ellipses possessing the greatest area. Using Equation (13), that area is computed to be 7.78 square generic drawing units.

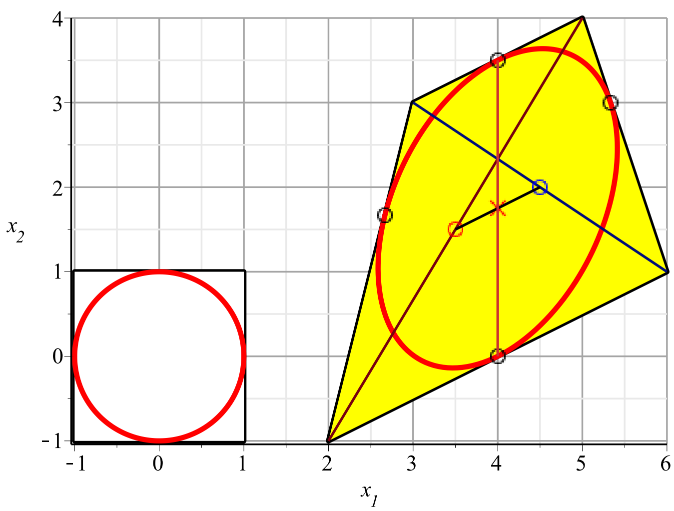

Figure 7. Maximum area ellipse inscribing a trapezoid.

IV. Conclusions and Future Work

In this paper projective transformations were used to map the unit circle inscribing the unit square to ellipses inscribing arbitrary parallelograms and trapezoids possessing the maximum area among the one parameter pencil of all inscribing conics. Synthetic geometric arguments were used to prove the claim that the derived projective transformations preserve the property of the unit circle inscribing the unit square being the inscribing ellipse possessing the largest area. This work has applications to determining the upper bound on error ellipses given specific linear constraints, and for determining the maximum area inscribing ellipse given linear constraints that form convex quadrilaterals which characterize the velocity performance of parallel mechanisms in the presence of actuation redundancy.

Given an arbitrary convex quadrilateral, there is a one parameter pencil of inscribing ellipses all centred on the open line segment joining the midpoints of the diagonals. Only one possesses maximum area. Future work will aim to generalise this approach to determine the maximum area ellipse inscribing arbitrary convex quadrilaterals. After that, we will extend the approach to n-sided arbitrary convex polygons.

References

[1] W. Press, S. Teukolsky, W. Vetterling, and B. Flannery, Numerical Recipes in C, 2$^{\text{nd}}$ Edition, Cambridge University Press, Cambridge, England, 1992.

[2] S. Krut, O. Company, and F. Pierrot, "Velocity Performance Indexes for Parallel Mechanisms with Actuaion Redundancy," Proceedings of the First Workshop on Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators, Eds. C.M. Gosselin & I. Ebert-Uphoff, Québec City, QC., Canada., Oct 3-4, 2002, pp. 46–56.

[3] V. W. Parker and J. E. Pryor, "Polygons of Greatest Area Inscribed in an Ellipse," The American Mathematical Monthly, vol. 51 (4), April, 1944, pp. 205–209.

[4] M. J. D. Hayes, P. J. Zsombor-Murray, and A. Gfrerrer, "Largest Ellipse Inscribing an Arbitrary Polygon," Proceedings of the 19th Canadian Congress of Applied Mechanics (CANCAM 2003), University of Calgary, Calgary, AB, Canada., June 2003.

[5] G. Salmon, A Treatise on Conic Sections, 6$^{\text{th}}$ edition, Longmans, Green, and Co., London, England, 1879.

[6] J. W. Rutter, Geometry of Curves, Chapman Hall/CRC Mathematics Series, 2000.

[7] A. Gfrerrer, "The Area Maximizing Inellipse of a Convex Quadrangle," private communication, December 19, 2002.