The Chebyshev–Grübler–Kutzbach Mobility Criterion Revisited

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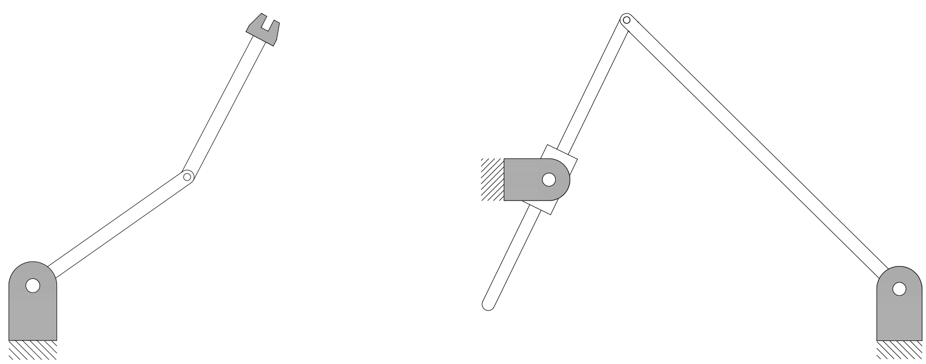
Abstract. The Chebyshev–Grübler–Kutzbach (CGK) mobility criterion is a very well known means of computing the relative degree of mobility possessed by a kinematic chain consisting of jointed rigid links. The relative degree of mobility indicates the number of independent parameters, such as revolute joint angles or prismatic joint translation distances, that are required to define a geometric configuration of the kinematic chain. In the 1800's Pafnuty Chebyshev, Martin Grübler, and later in the early 1900's Karl Kutzbach proposed mobility criteria that sum the unconstrained degree of freedom of the rigid links in the mechanical system and subtract the constraints imposed by the joints. Sometimes the CGK criterion yields an incorrect result. In this paper, the idea of accounting for the symmetry operations in the finite symmetry groups of the paradoxical kinematic chains that cause the criterion to predict degrees of mobility that contradict physical reality is proposed for the first time.

Keywords: degree of mobility, configuration space, finite symmetry groups

1 Introduction

An unconstrained rigid body in the Euclidean plane, $\mathbb{E}^2$, has three degrees of freedom (DOF). It can translate linearly independently in two mutually orthogonal directions in the plane and it can rotate independently about any axis perpendicular to the plane. This is a special case of general three DOF motions in Euclidean three dimensional space, $\mathbb{E}^3$, where the three freedoms can be any of the $\binom{6}{3}$, i.e., 6 choose $3 = 20$ combinations of translations and rotations. For the general case in $\mathbb{E}^3$, an unconstrained rigid body has six DOF: it can translate in three mutually orthogonal and linearly independent directions; and it can rotate about three mutually orthogonal and linearly independent basis vector axes directions. In $\mathbb{E}^3$ there is but one combination of the six translations and rotations.

A kinematic chain consists of more than one rigid body constrained so as to allow relative motion between them. These rigid bodies are termed the links in the chain. Individual links are connected to one, or more, other links with kinematic pairs [1, 2], also known as joints, which impose motion constraints on the links and maintain a fixed geometric relationship between the rigid bodies. The degree of connectivity (DOC) [1] of a link is the number of other links to which it is connected by joints, which is necessarily an integer. A kinematic chain is said to be simple if each of its links is coupled to at least one or at most two other links, $0 < \text{DOC} \leq 2$. If one link has a $\text{DOC} > 2$ then the chain is said to be complex. If all its links are coupled to two other links, $\text{DOC} = 2$, then the chain is closed, as in Figure 1b; otherwise, it is open, see Figure 1a. In Figure 1, R indicates *revolute* joint, or R-pair, whereas P indicates *prismatic* joint, or P-pair. Note that in an open kinematic chain, all links are coupled to two other links, except for the first and last ones, which are coupled to only one other link. Closed kinematic chains are termed linkages (or mechanisms), whereas open kinematic chains are called manipulators [2, 3].



(a) Open RR kinematic chain. (b) Closed $PRRR$ kinematic chain.

Fig. 1: Open and closed simple kinematic chains.

Any joint connecting two neighboring rigid bodies removes at least one relative DOF. If the joint removes no DOF then the bodies are not connected. If the joint removes 3 DOF in the plane $\mathbb{E}^2$, or 6 DOF in $\mathbb{E}^3$, the two bodies are a rigid structure. The joints are classified as *lower* or *higher* kinematic pairs [1]. Lower kinematic pairs impose surface constraints, such as a pair of mating cylinders. Higher kinematic pairs impose point, or line constraints; a cam and knife-edge follower for example. Moreover, constraints can be classified as *holonomic* and *non-holonomic* [4]. The term holonomic is derived from the Greek word *holos* meaning *integer*. It describes constraints that may be expressed in *integral form*: in terms of displacements, as opposed to *differential form*, i.e., in terms of linear and angular velocities. Differential form kinematic constraints involving link angular velocities are generally non-holonomic unless the motion is planar and occurs without slip [4]. In this paper, we will restrict ourselves to kinematic chains jointed with lower kinematic pairs imposing holonomic constraints.

The geometric constraints of the linkage, or manipulator, allow for the calculation of all of the configuration parameters in terms of a minimum set, which are the number of input parameters required to specify a configuration of the mechanical system. The cardinality of the minimum set of the required input parameters is called the mobility. It is also known as the dimension of the configuration space [3]. The generic mobility of an arbitrary mechanical system has traditionally been taken as the sum of unconstrained DOF for the links in the chain less the constraints imposed by the joints [3].

The earliest recorded investigations into mobility of a mechanical system date to the middle of the 19^{th} century by Pafnuty Chebyshev [5], Martin Grübler [6], and many others. Later in the early part of the 20^{th} century the well known work by Karl Kutzbach [7] was published. A comprehensive review of the development of mobility formulations can be found in a review paper by Grigore Gogu from 2005 [8] where most of the contributions are discussed. Over the years, the Theory of Machines and Mechanisms community has mostly embraced variants of one mobility criterion, namely that broadly known as the Chebyshev–Grübler–Kutzbach (CGK) mobility criterion. It is generally acknowledged that the criterion is incomplete because for some kinematic chains, the application of the CGK formula leads to erroneous results. The linkages that have mobility when the CGK formula predicts no mobility are known as *paradoxical linkages* [9]. The vast body of literature over the last century contains many investigations into paradoxical linkages, but only considers the nature of the mobility [9–12].

The recent work of Josef Schicho [13] investigates reasons for, and a method to predict the paradoxical chains, but there is no obvious means presented for compensating the CGK formula to resolve the paradox. In this paper, the novel idea of accounting for the number of symmetry operations contained in the finite symmetry groups of the paradoxical kinematic chains that cause the criterion to predict degrees of mobility that contradict physical reality is proposed for the first time, thereby resolving the paradox.

2 Groups

A group is a collection of elements of any sort contained in a set, together with a binary operation that combines any pair of set elements into another element contained in the set. The group must also contain the identity and inverse elements. An isometry is a congruent rigid body transformation that preserves distance. A set of isometries constitutes a group if it contains the inverse of each and the product of any two. The number of distinct congruent transformations contained in the set is the order of the group, which is infinite for the group of isometries. Whereas the set of symmetry operations of a polygon or a polyhedron form a finite group, called the symmetry group of the figure, whose cardinality is a finite integer.

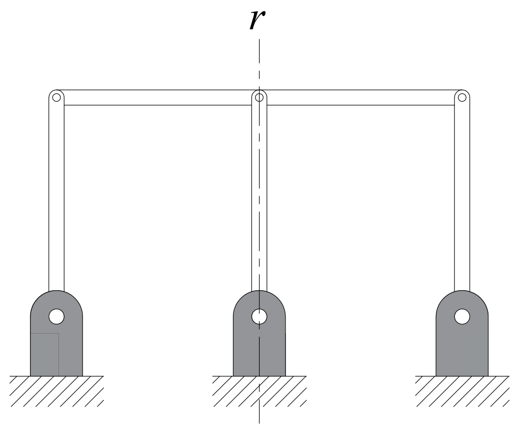


Fig. 2: Double parallelogram linkage with bilateral symmetry.

2.1 Finite Symmetry Groups

Symmetry can be described in terms of a figure, a schematic diagram of a mechanical system for example, to which certain isometries (rigid body transformations) can be applied, called *symmetry operations*, which leave the whole figure unchanged while permuting its parts [14]. The double parallelogram linkage illustrated in Figure 2 has bilateral symmetry of a particular sort, meaning that when it is in the configuration illustrated in the figure, it contains the identity in addition to one symmetry operation consisting of the reflection in the vertical mirror line r. The identity operation can be thought of as the product of a reflection in r with its inverse. Thus the corresponding finite symmetry group of the double parallelogram linkage has order 2. The finite symmetry group is therefore indicated by S_2: a single reflection and the identity. It is important to note that this one reflection represents the groups only symmetry operation leading to a single permutation, and hence S_2 is a finite group.

It is well known that a closed spatial kinematic chain consisting of $n = 4$ rigid links, each connected to its two neighbours by an R-pair, has a degree of mobility of $m = -2$ and is therefore doubly over-constrained [15]. There are only three known cases where $m = 1$: (i) when the four R-pair axes are parallel; (ii) when the four R-pair axes intersect in a point; (iii) when the R-pair axes, taken in order, have common normals which are consecutively concurrent in a finite point, and alternately equal in length [15]. The first two cases are the well known planar and spherical $4R$ linkages (mechanisms). The third case was discovered by G.T. Bennett and first published in 1903 [16].

The term *isogram* was first used by Bennett [15] to describe any quadrilateral, whether planar, spherical, or spatial, which has alternate edges equal in length. A generic Bennett mechanism, or *isogram* as he called it, is illustrated in Figure 3. The Bennett linkage is, as the planar 4R, composed of four rigid links that are connected by four R-pairs. According to the CGK criterion the Bennett linkage has a mobility of $m = -2$, which in theory prevents it from moving. Indeed, the mobility of $m = -2$ implies that the linkage is over-constrained. However, its actual mobility is $m = 1$, and is the only known mobile spatial 4R mechanism. The linkage is able to move if it satisfies the following symmetry conditions which were discovered by Bennett and for that reason known as the Bennett conditions [16]:

$$\begin{aligned} a_1 &= a_3 & a_2 &= a_4 \\ \tau_1 &= \tau_3 & \tau_2 &= \tau_4 \\ &\frac{\sin(\tau_1)}{a_1} = \frac{\sin(\tau_2)}{a_2}. \end{aligned} \tag{1}$$

It can be shown that the kinematic geometry of every Bennett linkage is defined by a deformable tetrahedron [10]. It is easy to see that the intersections of the four R-pair axes and link centre-lines are the four vertices of a generally irregular tetrahedron. Owing to the Bennett conditions, there is one case where the link lengths can all be the same, i.e. $a_1 = a_2 = a_3 = a_4$. For this case, there is at least one configuration where the vertices are those of a regular tetrahedron. Therefore, according to H.S.M. Coxeter [14], pp. 270-272, the finite symmetry group of the Bennett mechanism has order S_4. The regular tetrahedron is symmetrical by reflection in the plane that joins any edge to the midpoint of the opposite edge, resulting in the three permutations of one of its four faces as well as the identity.

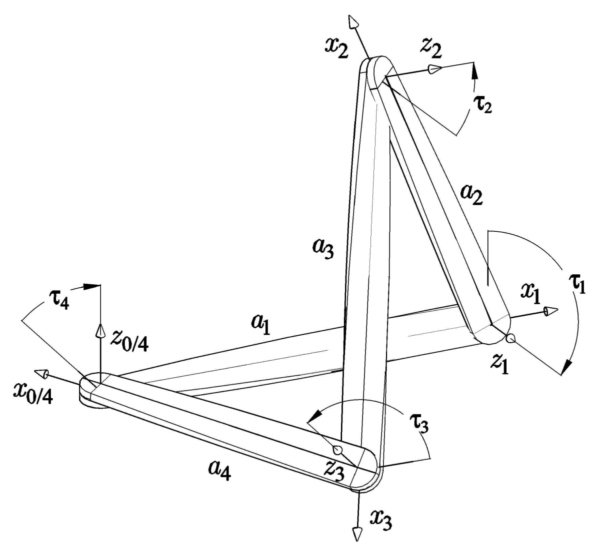


Fig. 3: A Bennett mechanism (isogram).

Table 1: Lower Pair Sub-groups and Their Dimension

Lower Pair	G_S	$\dim(G_S)$
E	$\mathcal{E}$	3
S	$\mathcal{S}$	3
C	$\mathcal{C}$	2
H	$\mathcal{H}$	1
P	$\mathcal{P}$	1
R	$\mathcal{R}$	1

2.2 Computing DOF Using Group Concepts

The relative motion associated with each of the lower kinematic pairs illustrated in Figure 4 constitute a sub-group of the group of Euclidean displacements G_6 under the binary product operator (i.e. the composition of two displacements). Thus, the motions of all R-pairs are a sub-group of G_6, the motions of all E-pairs are a sub-group of G_6, etc.. The dimension of these sub-groups is defined to be the DOF of the relative motion permitted by the lower pair. The dimension is indicated by $\dim(G_S)$, where $G_S \subset G_6$. These sub-groups, together with their corresponding dimensions are listed in Table 1.

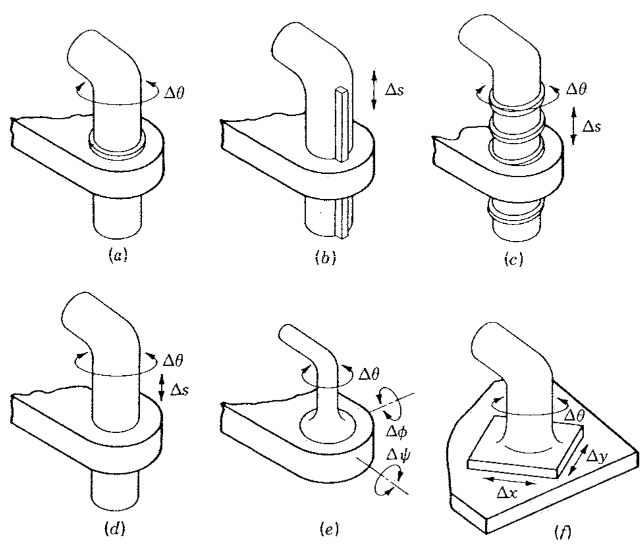


Fig. 4: The six lower pairs: (a) R, revolute; (b) P, prismatic; (c) H, helical; (d) C, cylindrical; (e) S, spherical; (f) E, planar.

Let the product of two sub-groups indicated by

$$G' = G_1 * G_2$$

be the composition of the displacements they represent. Let $\dim(G') = d$. In the mathematical model which follows, d represents the maximum possible motion group dimension. In Euclidean space, $\mathbb{E}^3$, the absolute maximum is $d = 6$; in the Euclidean plane, $\mathbb{E}^2$, the absolute maximum is $d = 3$. This is because in $\mathbb{E}^2$ the origin of the moving reference frame E, see Figure 5, can translate independently in both the X and Y basis vector directions in the relatively non-moving reference frame, Σ, in any linear combination, and E can change it's orientation about any axis normal to the relatively fixed and moving coordinate systems. For example, consider the RPR linkage shown in Figure 5, which implies a serial chain of four links connected to ground by a sequence of revolute-prismatic-revolute joints. In $\mathbb{E}^3$, $d = 6$ because there are three linearly independent translation directions and three linearly independent rotations about the basis vector axes directions. The i^{th} kinematic pair imposes μ_i constraints on the two links it couples. For

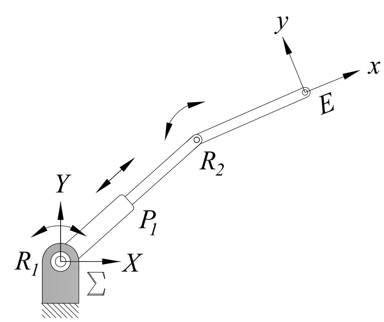


Fig. 5: Reference frame E has 3 DOF relative to $\Sigma \;\Rightarrow\; d = 3$.

example, in $\mathbb{E}^3$ R-, P-, and H-pairs impose five constraints, a C-pair imposes four constraints, while S- and E-pairs each impose three. However, in $\mathbb{E}^2$ S-, H-, and C-pairs are undefined, while E-pairs introduce no constraints but R- and P-pairs each introduce two.

2.3 The Chebyshev-Grübler-Kutzbach (CGK) Formula

Clearly, n unconstrained rigid links have $d(n-1)$ relative DOF, given that one of the links is designated as a non-moving reference link. As noted earlier, any joint connecting two neighboring rigid bodies removes at least one relative DOF. If the joint removes no DOF then the bodies are not connected. If the joint removes 3 DOF in $\mathbb{E}^2$, or 6 DOF in $\mathbb{E}^3$ the two bodies are a rigid structure. Summarising this discussion, the mobility m of a kinematic chain, relative to one fixed link in the chain, can be expressed as what is known as the Chebyshev-Grübler-Kutzbach (CGK) formula:

$$m = d(n-1) - \sum_{i=1}^{j} \mu_i - \zeta, \tag{2}$$

where $d = \dim(G')$, n is the number of links including the fixed one, μ_i is the number of constraints imposed by the i^{th} joint, j is the number of joints, and ζ represents the number of idle DOF of the chain. The idle DOF of a chain are the number of independent single DOF motions that do not affect the transmission of motion from the input to the output links of the chain.

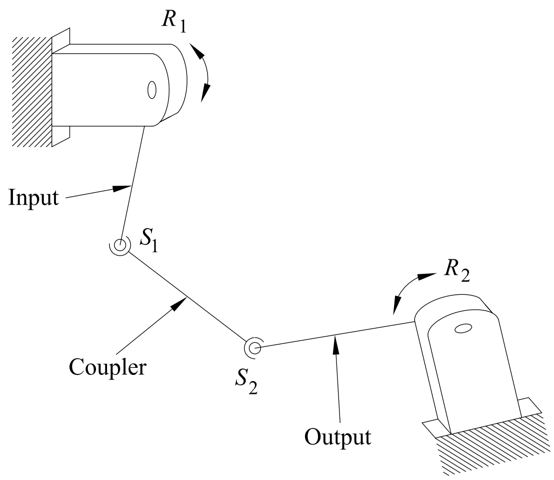


Fig. 6: An $RSSR$ spatial linkage has 1 degree of mobility.

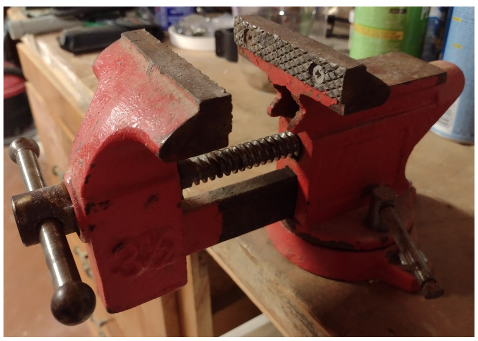
Consider the $RSSR$ linkage shown in Figure 6. The coupler is free to spin about its own longitudinal axis between S_1 and S_2. The input link drives the linkage by rotating about axis R_1. The output link rotates about axis R_2. Clearly, the free spinning of the coupler does not contribute to the transmission of motion from the input link to the output link of the chain. Hence, in this case, $\zeta = 1$. This is an example in $\mathbb{E}^3$ so $d = 6$. There are four links, including the rigid frame to which R_1 and R_2 are fixed, thus $n = 4$. Each R-pair imposes $\mu = 5$ constraints, while each S-pair imposes $\mu = 3$. Thus:

$$\begin{aligned} m &= 6(4-1) - (2(5) + 2(3)) - 1, \\ 18 - 16 - 1 &= 1. \end{aligned}$$

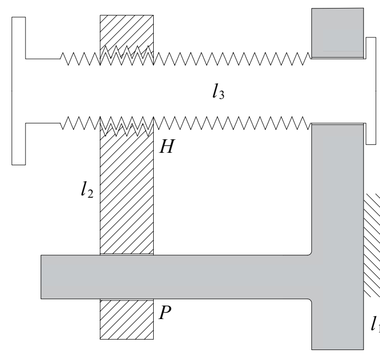
The dimension d of the motion space must be carefully evaluated for some linkages. Consider a bench vice, illustrated in Figure 7, which presents some interesting diversion. At first glance, it is planar, so we immediately set $d = 3$ without hesitation:

$$d = 3,\ n = 3,\ \mu_i = 2,\ j = 3,\ \zeta = 0,$$
$$\Rightarrow m = d(n-1) - \sum_{i=1}^{j} \mu_i - \zeta = \ 3(3-1) - 3(2) - 0 = 6 - 6 = \ 0.$$

Experience insists that a bench vice has mobility $m = 1$! This seeming anomaly



(a) Bench vice.



(b) Bench vice schematic.

Fig. 7: Typical bench vice.

is an artifact of representation. The R- and H-axes are parallel while the translation direction of the P-pair is also parallel, see Figures 7a and 7b.

The P- and R-pairs are kinematically equivalent to a single C-pair. Moreover, the axis of the H-pair is parallel to the axis of the C-pair. Therefore, the dimension of the motion sub-group represented by the common bench vice is $d = 2$, **not** $d = 3$. Thus:

$$d = 2,\ n = 3,\ \mu_i = 1,\ j = 3,\ \zeta = 0,$$
$$m \ = \ d(n-1) - \sum_{i=1}^{j} \mu_i - \zeta \ = \ 2(3-1) - 3(1) - 0 = 4 - 3 = \ 1.$$

One must be careful blindly applying the CGK formula. Still, note that we had to treat the C-pair as a P-pair and an R-pair separately.

The geometric model of the CGK formula is incomplete. Certain geometric mechanism properties are not modeled by it. As a result, there are many mechanisms with mobility $m > 0$ that are not identified by the CGK formula. Such seemingly paradoxical linkages are said to be *over-constrained* [11]. The Bennett $4R$ Linkage is one of the best known examples of an over-constrained mechanism.

3 Mobility Revisited

Evidently the CGK mobility formula is sensitive to, among other things, correct assessment of the dimension of the motion space, d. However, even after ensuring that the correct value for d has been assigned, the group of paradoxical linkages that lead the CGK formula to err in its mobility prediction still exists, but reasons for the error in the prediction of mobility is not immediately obvious [13]. Given the observations of the symmetry groups of the planar double parallelogram and the Bennett mechanisms, the order of the finite symmetry group of a mechanism possessing symmetry operations should somehow be included in the CGK formula.

Let us restrict the definition of "paradoxical linkage" to mean one possessing mobility for which the CGK formula predicts that the mechanism is rigid: there are sufficiently many constraints so that there should be no freedom left for motion, except moving the mechanism itself as a whole like a single rigid body. The observations on the orders of the symmetry groups of the double parallelogram and the Bennett mechanisms in Section 2.1 suggest modifying the CGK formula in the following way to resolve the mobility paradox:

$$m = d(n-1) + (S_G - 1) - \sum_{i=1}^{j} \mu_i - \zeta, \tag{3}$$

where the additional term $S_G - 1$ is the order of the corresponding finite symmetry group less the identity symmetry operation. If the mechanism possesses no symmetry then S_G=1, consisting of only the identity and Equation (3) will still yield correct results. We will now revisit the double parallelogram and the Bennett.

3.1 Double Parallelogram Mechanism Mobility

As determined in Section 2.1, the order of the symmetry group of the double parallelogram mechanism is

$$S_G = 2,$$

consisting of a single reflection and the identity. Now application of Equation (3) yields the correct result:

$$d = 3,\ n = 5,\ S_G = 2,\ \mu_i = 2,\ j = 6,\ \zeta = 0,$$

$$m \ = \ d(n-1) + (S_G - 1) - \sum_{i=1}^{j} \mu_i - \zeta \ = \ 3(5-1) + (2-1) - 6(2) - 0 = \ 1.$$

Even if the number of closed loops in the double parallelogram linkage is increased, the linkage is still kinematically equivalent to a single parallelogram mechanism with $m = 1$. Alternately, one could scale the $(S_G - 1)$ term in Equation (3) with the number of additional loops making the term $l(S_G - 1)$, where l is the number of loops of double parallelogram components.

3.2 Bennett Mechanism Mobility

The Bennett linkage is, at present, the only known spatial over-constrained $4R$ mechanism possessing mobility $m = 1$. For this mechanical system there are no kinematic equivalences, so modification of the CGK criterion is required to resolve the predicted mobility from $m = -2$ to its actual mobility of $m = 1$. Here, again the symmetry group order term resolves the paradox. As determined in Section 2.1, the order of the symmetry group of the Bennett mechanism is

$$S_G = 4,$$

consisting of three reflections and the identity. Now application of Equation (3) yields the correct result:

$$d = 6,\ n = 4,\ S_G = 4,\ \mu_i = 5,\ j = 4,\ \zeta = 0,$$

$$m \;=\; d(n-1) + (S_G - 1) - \sum_{i=1}^{j} \mu_i - \zeta \;=\; 6(4-1) + (4-1) - 4(5) - 0 = \;1.$$

4 Conclusions

In this paper we have carefully re-examined the Chebyshev–Grübler–Kutzbach (CGK) mobility criterion for computing the relative degree of mobility possessed by a kinematic chain consisting of jointed rigid links. The relative degree of mobility indicates the number of independent parameters, such as revolute joint angles or prismatic joint translation distances, that are required to define a geometric configuration of the kinematic chain. Sometimes the CGK criterion predicts an incorrect result and the corresponding linkages are considered paradoxical. There is a tremendous volume of archival literature examining the existence of paradoxical linkages, but efforts to resolve the paradoxes seem to be lacking. In this paper, the idea of including the dimension of the finite symmetry groups of the paradoxical kinematic chains that cause the criterion to predict degrees of mobility that contradict physical reality was examined for the first time. A modification to the CGK formula was proposed based on the cardinality of the finite groups of symmetry operations of some paradoxical linkages and was shown to be consistent both in the presence and absence of topological symmetry.

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