KINEMATIC ANALYSIS AND SYNTHESIS

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Chapter 1

Introduction

This book is intended to be a comprehensive look at the kinematics of planar, spherical, and spatial mechanical systems. Some traditional and some novel approaches to analysis and synthesis of mechanisms are introduced and described in sufficient detail so that the reader can apply them. Many examples are examined in rigorous detail, and many end of chapter exercises are stated. The intended audience is students at the senior undergraduate or graduate level in mechanical engineering, but practising engineers, mathematicians, computer scientists, etc., may find some of the novel approaches to be of utility for their current research in analysis and design of mechanisms for function generation and rigid body guidance using four, five, or six-bar mechanisms.

Chapter 1, this chapter, summarises the elementary concepts and definitions needed to understand some of the material presented in subsequent chapters. Chapter 2 summarises the Euclidean and non-Euclidean geometries, and associated algebras, used in the chapters on kinematic analysis and synthesis (design). Chapter 3 is a brief outline of analytic projective geometry that is heavily relied upon in subsequent chapters. Chapter 4 is an introduction to the theory and application of Gröbner bases which are used in the derivation of the equations used in Chapters 6 and 7. Chapter 5 examines rigid body displacements as elements in the isometry group, also summarising the relevant details of group and finite group theory. Chapter 6 is a broad summary of kinematic analysis and presents new methods using algebraic polynomials with the trigonometric details embedded in the equations, where the displacement equations for any joint pairing in planar, spherical, and spatial four-bar mechanisms requires only the solution of a quadratic equation. Finally, Chapter 7 presents traditional and novel algebraic approaches to kinematic synthesis of mechanisms for function generation and for rigid body guidance.

1.1 Mechanisms

A mechanism is a device that transforms one motion into another. Many subclasses of mechanisms exist. For instance, machines and engines are but two.

Machines: mechanical systems which transmit substantial forces, applying power, or changing its direction. Terms such as *force, torque, work* and *power* describe the predominant concepts. Consider for example differentials, clutches, and brakes.

Engines: mechanical systems where generated forces are associated with the conversion of energy of high temperature fluids, electric current, gravitational or elastic potential energy, etc., to shaft power.

While a mechanism can be used to transmit force and power, the predominant associated concepts involve attaining a desired motion. Thus it may be defined as an assemblage of links coupled by mechanical constraints such that there can be relative motion between the links.

There is a direct analogy between the terms *structure, mechanism,* and *machine* to the branches of the science of mechanics, see Figure 1.1.

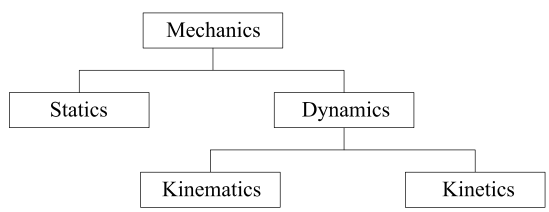


Figure 1.1: Branches of Mechanics.

Mechanics has two main constituents: statics and dynamics.

Statics: analysis of constrained systems of mechanical elements of whose configuration is invariant with time.

Dynamics: analysis of constrained systems of mechanical elements whose relative configuration changes with time.

Dynamics, in turn, has two main components: kinematics and kinetics.

Kinematics: the study of motion, without considering the forces causing the motion: position, orientation, displacement, velocity, acceleration, etc..

Kinetics: Equates motions to the forces that cause them.

Statics concerns structures: systems of mechanical elements that cannot move relative to each other. Kinematics concerns mechanisms. Kinetics concerns machines and engines.

Because kinematics is independent from the inertial properties of the mechanical elements comprising a mechanism, it can be considered as purely geometric. Geometric configurations can always be represented and manipulated algebraically. The secret to understanding kinematics is to understand geometry, and to understand that if a particular geometric attribute is difficult to visualize, it can always be transformed, using algebra, to a different geometric representation that is easier to see. It is also important to understand that algebra is rationalised geometry: that is, geometry without the pictures.

The *rigid body* assumption (i.e. the distance between every pair of points on a mechanical element remains constant during all states of motion) is the key that allows one to consider kinematics and kinetics separately. For flexible bodies, the shape of the bodies themselves, and therefore their motions, are force dependent. Kinetics and kinematics must be considered simultaneously, significantly increasing the complexity of analysis and synthesis. Even though there are no absolutely rigid bodies in reality, one can, when applied loads are known, design the components to behave as rigid relative to the loads. Thus the rigid body assumption can usually always be justified.

1.2 Analysis and Synthesis

There are two fundamentally opposite aspects of the study of mechanisms: analysis and synthesis.

Analysis: techniques allowing for quantifiable evaluation of an existing, or proposed mechanism.

Synthesis: a process whose output prescribes the sizes, shapes, material (when considering *kinetic* synthesis, but only *kinematic* synthesis will be considered here), and arrangements of components so that the resulting mechanism will perform the desired task.

In mechanical engineering design effort is generally directed towards analysis, but the goal is generally always synthesis. Moreover, analysis is a tool that must be used to evaluate a synthesised mechanism. *Circularity*, as will soon become clear, is vitally important to the theory of kinematics.

1.3 Kinematic Chains

A more general, yet more precise, term for *mechanism* is *kinematic chain*. A kinematic chain is defined as a collection of rigid bodies coupled by mechanical constraints such that there can be relative motion between the rigid bodies. The individual rigid bodies are called *links* in the chain. The kinematic chains are classified according to how the links are connected.

1.3.1 Simple Kinematic Chains

A kinematic chain is *simple* if each link in the chain is coupled to *at most* two other links. The degree-of-connectivity (DOC) of a link indicates the number of rigid bodies coupled to it. If all links are binary ($DOC = 2$) the simple chain is *closed*. For example, a four-bar mechanism. Alternately the simple chain is

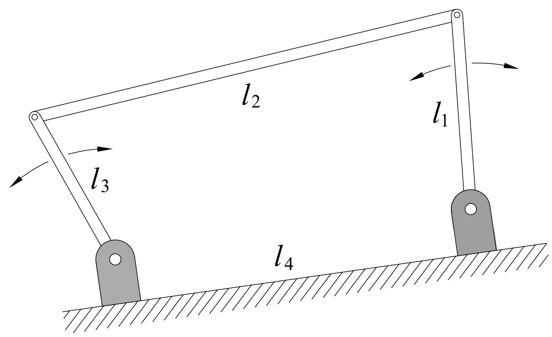


Figure 1.2: A typical planar four-bar mechanism is a simple closed chain.

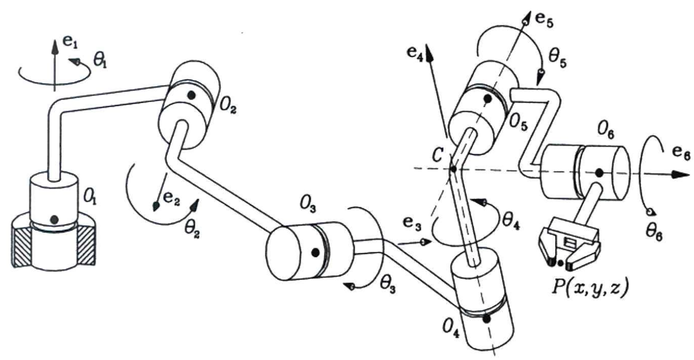
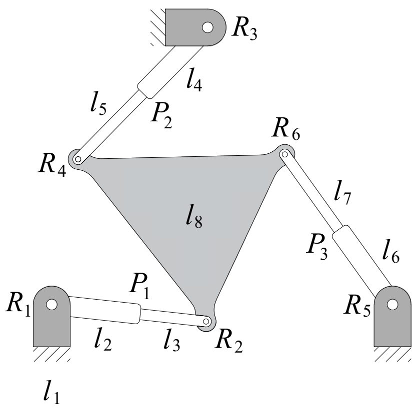


Figure 1.3: A typical *serially-connected* 6-axis robot is a simple open chain.

open with the first and last links having $DOC = 1$. Figures 1.2 and 1.3 illustrate closed and open simple kinematic chains, respectively.

1.3.2 Complex Kinematic Chains

A chain is complex if *at least* one of its links has a $DOC \geq 3$. A complex kinematic chain may always be decomposed into simple kinematic sub-chains. Planar 3-legged platforms and Stewart-Gough type 6-legged spatial platforms (flight simulators) are typical examples of complex kinematic chains.



(a) Central triangular-shaped link and fixed base link have $DOC = 3$.

(b) A CAE 6-legged flight simulator where the moving platform and fixed base link have $DOC = 6$.

Figure 1.4: Planar and spatial complex kinematic chains.

A somewhat less typical example of a complex kinematic chain is the Peaucellier inversor [1]. In the configuration shown in Figure 1.5, the linkage converts circular motion to straight line motion, without sliding.

When lengths $AD = DC$, the motion of point C on the circle centered at D is mapped to the motion of point P along a straight line. When link CD is removed, the motion of points C and P are algebraic inverses. Other famous *approximate* straight line linkages were developed by James Watt, Richard Roberts, and Pafuntlij Chebyshev in the 1700's and 1800's [2].

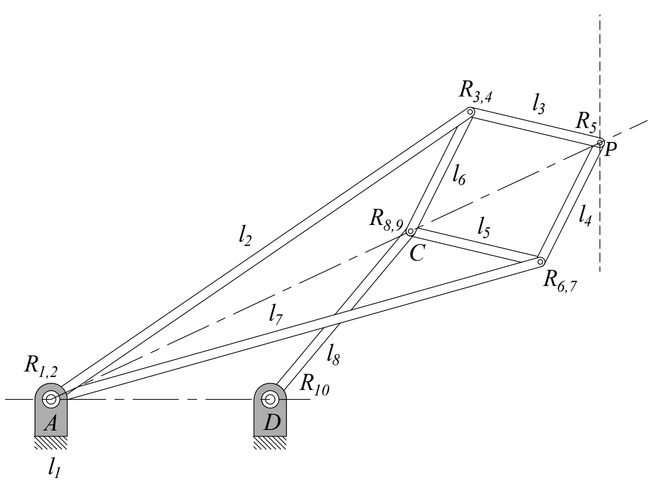


Figure 1.5: Peaucellier inversor: eight-bar linkage from 1864.

1.4 Degree-of-Freedom (DOF)

The degree-of-freedom (DOF) of a kinematic chain is defined to be an integer value corresponding to the minimum number of independent parameters required to fully describe an arbitrary configuration of every link in the kinematic chain [3, 4, 5]. There is usually one parameter defined on the field of real numbers associated with each DOF. Because any one of these parameters may be changed without necessitating a change in any of the others, they are all independent. Such parameters are historically called *generalized coordinates* [5]. For the study of kinematics, generalized coordinates usually represent measures of distance and angle.

An unconstrained rigid body free to move in 3-dimensional Euclidean space, $\mathbb{E}^3$, has 6 DOF. The DOF are generally taken to be three translations parallel to three linearly independent basis directions and three rotations about three linearly independent axes, although any system of six *generalized coordinates* is sufficient. That is, the six numbers need not be three distances and three angles, nor do the coordinate axes need to be orthogonal [6]. While it is not necessary that the rotation axes be parallel to the translation directions, it is usually convenient to use a 3D orthogonal reference frame to describe the space of a motion, the rotation axes respectively parallel to the coordinate axes.

An unconstrained rigid body free to move in the Euclidean plane $\mathbb{E}^2$ has 3 DOF, generally taken to be two linearly independent translations and a rotation about an axis perpendicular to the plane. Mechanical constraints are imposed on rigid bodies to limit their motion as required. In this sense, constraints are the complements of DOF. For instance, if a rigid-body has two DOF in Euclidean space $\mathbb{E}^3$, four constraints must be imposed.

A closed kinematic chain constitutes a mechanism if both its DOF is greater than zero, and if a relatively non-moving reference link is specified [2]. It is called a *constrained mechanism* if the chain is closed and its DOF is identically equal to 1 (e.g. a four-bar mechanism) [4]. A general mechanism can have more than one DOF, and need not be a closed chain (e.g. a serially connected 6-axis robot). The kinematic chain is a statically determinate structure if its DOF is identically zero. The kinematic chain is a hyper-static structure if its DOF is less than zero. Such structures are termed statically under-determined, or over-constrained. We see that, philosophically, the upper bound on DOF is 6, while there is no lower bound.

1.5 Mobility: Chebyshev-Grübler-Kutzbach (CGK) Formula

The mobility, m, and the DOF of a single rigid body are terms that can be used interchangeably. But this is not the case for a kinematic chain. The important philosophical difference concerns the upper bound of DOF ≤ 6 in $\mathbb{E}^3$. The mobility of the corresponding kinematic chain can be any integer value, positive or negative. Consider a spatial serial kinematic chain with 7 rigid links connected with 7 motorised joints. The distal link in the chain has 6 DOF, but the 7 motorised joints connecting the distal link to the fixed base link mean that the mobility of the kinematic chain is $m = 7$. When $m >$ DOF the kinematic chain is said to be redundantly actuated.

The mobility of a kinematic chain can be computed using the Chebyshev-Grübler-Kutzbach (CGK) formula [2, 4, 7, 8, 9], among many others [10]. Clearly, n unconstrained rigid links have a relative mobility of $d(n-1)$, given that one of the links is designated as a non-moving reference link, where d can be described as the dimension of the space of the motion. In general for planar mechanisms $d = 3$ and for spatial mechanisms $d = 6$, but as will be seen this mobility model leads sometimes to inconsistencies mostly because certain important elements of the geometry of the kinematic chain is not considered in Equation (1.1) [11], see Chapter 5. Any joint connecting two neighboring rigid bodies removes at least one relative DOF. If the joint removes no DOF then the bodies are not connected. If the joint removes 3 DOF in the plane, or 6 DOF in $\mathbb{E}^3$ the two bodies are a rigid structure.

Summarising this discussion, the mobility m of a kinematic chain, relative to one fixed link in the chain, can be expressed as the equation known as the CGK formula:

$$m = d(n-1) - \sum_{i=1}^{p} \mu_i - \zeta \qquad (1.1)$$

where,

n = number of links (including the ground link),
d = dimension of the motion space,
μ_i = the number of constraints imposed by the i^{th} joint (pair),
p = is the number of joints (pairs),
ζ = represents the number of idle DOF of the kinematic chain.[1]

If

$m > 0 \Rightarrow$ the number of joint values required to attain a configuration,
$m = 0 \Rightarrow$ a determinate structure with no rigid body motion,
$m < 0 \Rightarrow$ structure is over-constrained and statically indeterminate.

The Kutzbach criteria [2, 9] for mobility in 3-dimensional space provides a slightly different formula:

$$m = 6(n-1) - 5j_1 - 4j_2 - 3j_3 - 2j_4 - j_5 - \zeta \qquad (1.2)$$

[1] An idle DOF may exist in a kinematic chain, but it makes no contribution to the transfer of motion from the input link to the output link. Consider a rigid link connected to two spherical joints, see Section 1.6. The rigid link can spin freely about its longitudinal centreline without changing the relative locations of the spherical joints. The ability to spin about its centreline is an idle DOF.

where,

n	=	number of links (including the ground link),
$6n$	=	DOF of mechanism (before connecting links),
-1	=	the relatively non-moving base link,
j_i	=	number of joints with i DOF,
ζ	=	the number of idle DOF of the kinematic chain.

In planar or spherical kinematics, the Kutzbach criterion reduces to

$$m = 3(n-1) - 2j_1 - j_2. \tag{1.3}$$

See Chapter 5 for examples applying these equations, and for discussion on *paradoxical* linkages where these mobility formulas fail to predict the true mobility, and for resolution of the paradoxes, as in [11].

1.6 Kinematic Pairs

The term *kinematic pair* indicates a joint between *two* links, hence the use of the word *pair*. Joints are mechanical constraints imposed on the links. Those involving surface contact are called *lower pairs*. Those normally involving point, line, or curve contact are *higher pairs*. Lower pairs enjoy innate practical advantages over higher pairs.

1. Applied loads are spread continuously over the contacting surfaces.

2. They can, in general, be more easily and accurately manufactured.

1.6.1 Lower Pairs

There are six types of lower pair (see Figure 1.6) classified in the following way.

1. **R-Pair.** The revolute R-pair is made up of two congruent mating surfaces of revolution. It has one rotational DOF about its axis.

2. **P-Pair.** The prismatic P-pair comprises two congruent non-circular cylinders, or prisms. It has one translational DOF.

3. **H-Pair.** The helical H-pair, or screw, consists of two congruent helicoidal surfaces whose elements are a convex screw and a concave nut. For an angle θ of relative rotation about the screw axis there is a coupled translation of distance S in a direction parallel to the screw axis. The sense of translation depends on the *hand* of the screw threads and on the sense of rotation. The distance S is the thread *pitch* for a rotation of $\theta = 360°$. When $S = 0$ it becomes an R-pair; when $S = \infty$ it becomes a P-pair. The H-pair has one DOF specified as a translation or a rotation, coupled by the pitch S.

4. **C-Pair.** The cylindrical C-pair consists of mating convex and concave circular cylinders. They can rotate relative to each other about their common axis, and translate relative to each other in a direction parallel to the axis. Hence the C-pair has two DOF: one rotational, the other translational.
5. **S-Pair.** The spherical S-pair consists of a solid sphere which exactly conforms with a spherical shell. They are also called *ball-joints.* S-pairs permit three rotational DOF about intersecting orthogonal axes.
6. **E-Pair.** The planar E-pair, for the German word "ebene", meaning "level" or "plane", is a special S-pair comprising two concentric spheres of infinite radius. They permit two orthogonal translations and one rotational DOF about an axis orthogonal to the plane of translation. They provide 3 DOF in total.

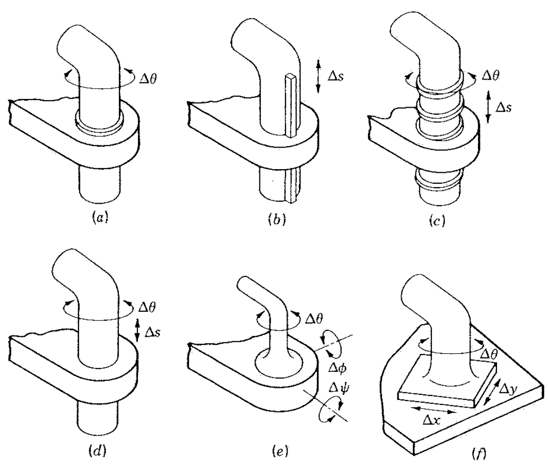


Figure 1.6: The six lower pairs: (a) revolute or pin; (b) prismatic; (c) helical; (d) cylindrical; (e) spherical; (f) planar.

Table 1.1: Summary of the lower pairs and their respective DOF.

Pair	DOF
R	1
P	1
H	1
C	2
S	3
E	3

1.6.2 Higher Pairs

Higher pairs are important because they often offer the most direct means of achieving complicated motions. The main drawback is that they are typically more complicated, implying that they are more expensive to design and manufacture. A few examples are mating spur gears, rack and pinion, cam and follower. The higher pairs may be classified according to the nature of the relative motion between the jointed links.

1. **Pure Sliding.** The relative motion is pure translation as in a reciprocating cam activating a knife-edge or mushroom head follower, or the finger tip of a robot hand sliding on a flat surface. See Figures 1.7 (a) and (b), as well as Figure 1.8 (a), for example.

2. **Pure Rolling.** The relative motion involves rolling without slip, such as the tangential pitch circles or mating sets of spur gears, or rack and pinion system, see Figure 1.7 (c) and Figure 1.8 (b).

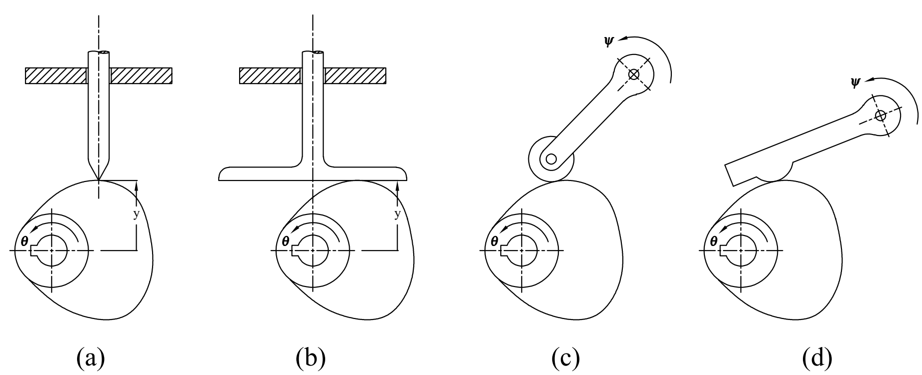


Figure 1.7: Plate cams with (a) an offset reciprocating knife-edge follower; (b) a reciprocating flat-face follower; (c) an oscillating roller follower; and (d) an oscillating curved-shoe follower.

3. **Combination of Sliding and Rolling.** In rotating cam and follower systems the tip of the follower slides along any constant radius of curvature portions of the cam surface. As the cam rotates and, relative to the follower, its radius of curvature changes, the follower rotates about the same axis. As this occurs, the follower tip will also roll on the cam surface. This can be observed in Figure 1.7 (d).

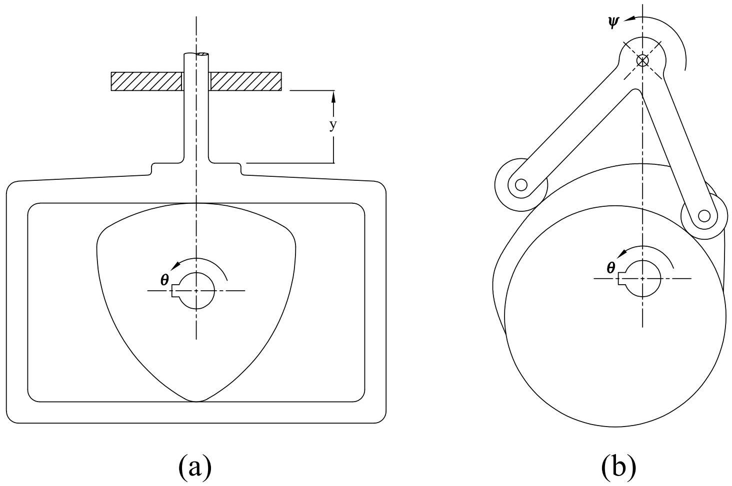


Figure 1.8: (a) A constant-width Reuleaux triangle cam with a reciprocating flat-face follower. (b) Conjugate cams with an oscillating roller follower.

1.7 Holonomic and Non-Holonomic Constraints

The term *holonomic* is derived from the Greek word *holos* meaning *integer*. It describes constraints that may be expressed in *integral form*, i.e., in terms of displacements, as opposed to *differential form* in terms of linear and angular velocities. Differential form kinematic constraints involving link angular velocities are generally non-holonomic unless the motion is planar and occurs without slip, see any text on advanced dynamics [12].

The i^{th} of n constraint equations confining the motion of a rigid body can be written in the form of a function:

$$f_i(q_1, q_2, ..., q_m, t) \quad = \quad 0, \tag{1.4}$$

where the m q's are constrained generalized coordinates, t stands for time, and the subscript i indicates a particular constraint equation. Any limitation placed on the generalized coordinates restricts the position of the rigid body, and hence are called *position constraints*.

Position constraints impose restrictions on the velocity as well. The velocity constraints are obtained by differentiating Equation (1.4) with respect to time:

$$\dot{f}_i = \frac{df_i}{dt} = \sum_{j=1}^{m} \left[\frac{\partial}{\partial q_j} f_i(q_1, q_2, ..., q_m, t) \right] \dot{q}_j + \left[\frac{\partial}{\partial t} f_i(q_1, q_2, ..., q_m, t) \right] = 0, \tag{1.5}$$

where the $\dot{q}$'s are called *generalized velocities*, dq_i/dt. Equations (1.4) and (1.5) are equivalent in the limitations they impose, as long as the initial position and orientation are specified.

A more general form for the velocity constraint equations is obtained by replacing the derivatives by arbitrary coefficients that are functions of only the generalized coordinates and time:

$$\sum_{j=1}^{m} a_{ij}(q_1, q_2, ..., q_m, t)\dot{q}_j + b_i(q_1, q_2, ..., q_m, t) \quad = \quad 0. \tag{1.6}$$

Equations (1.5) and (1.6) represent equivalent constraints if the corresponding coefficients of each generalized velocity and of the velocity independent term are the same up to a multiplicative factor, which may itself be a function of the generalized coordinates and time, $g_i = g_i(q_1, q_2, ..., q_m, t)$. A velocity constraint is derivable from a position constraint, and vice-versa, if and only if:

$$a_{ij} g_i \quad = \quad \frac{\partial f_i}{\partial q_j}, \;\; b_i g_i \quad = \quad \frac{\partial f_i}{\partial t}. \tag{1.7}$$

The velocity constraint equations are holonomic, meaning integrable, if they satisfy Equation (1.7), otherwise they are non-holonomic.

This terminology refers to a differential form of degree 1 equation called the *Pfaffian form* [13, 14]. The Pfaffian form is obtained by multiplying Equation (1.6) through by dt, giving:

$$\sum_{j=1}^{m} a_{ij}(q_1, q_2, ..., q_m, t)dq_j + b_i(q_1, q_2, ..., q_m, t)dt \quad = \quad 0. \tag{1.8}$$

When Equation (1.7) is satisfied, multiplying Equations (1.8) by the g_i functions transform the Pfaffian forms to perfect differentials of each function f_i. This leads to the following definition:

> **Definition:** A velocity constraint is holonomic if there exists an integrating factor g_i for which the Pfaffian form of the constraint equation becomes a perfect differential. In this case, it may be integrated yielding the position constraint on the generalized coordinates.

The concept of a holonomic constraint may be viewed from a geometric perspective. The generalised coordinates q_i may be taken to be the basis $(q_1, q_2, ..., q_m)$ of an m-dimensional (mD) constraint space. The constrained motion in the constraint space is a locus of points as the motion evolves over time. Consider a holonomic constraint $f_i(q_1, q_2, ..., q_m, t) = 0$. At any instant in time t the position of the rigid body is confined to some surface in the constraint space. The corresponding Pfaffian form of the velocity constraints states that infinitesimal displacements *must* be in the corresponding tangent plane to the constraint surface at that point.

When the constraint is non-holonomic, the constraint surface is *not* defined. Hence, the velocity constraint cannot be integrated. In this case the Pfaffian form of the constraint equation restricts infinitesimal displacements to lie in a plane that can only be defined by the current state of motion and the position level kinematics cannot be directly obtained from the velocities by integration.

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