Chapter 5

Linear Geometry

The kinematic analysis and synthesis of mechanisms, or any type of linkage, is greatly facilitated by suitable geometric representation, or algebraic formulation in a geometry that results in the simplest complete solution. Most engineers are only acquainted with 2D and 3D variants of Euclidean geometry. What other geometries are there? This of course begs at least three questions. What is geometry? How can "geometry" be defined? How can one geometry be differentiated from another?

The word *geometry*, which is originally a Greek word, means *earth-measure*. Its first applications were to determine the area of farms so taxes on the land could be levied. The 13 books of *The Elements* [1], compiled by Euclid in about 300 BC, summarized the state of the art of geometry 2300 years ago. The Elements also contains a great amount of number theory. We tend to equate *synthetic geometry* with the propositions and axioms set down in The Elements and using them to derive and prove theorems. The term *analytic geometry* shifts to the cartesian representation of Euclidean geometry developed by Rene Descartes, 1596-1650, so we can use coordinates and develop algebraic equations relating the coordinates. It is widely believed that the geometry contained in Euclid's Elements is perfect and complete: there are no flaws in the text, and all of geometry is to be found there. It turns out that Euclidean geometry is but one in an infinite series. Let's take a quick look at how this came to be known, and in so doing, come to understand what geometry really implies.

5.1 Euclid's Basic Assumptions

In Book I of The Elements, Euclid states ten assumptions as his basis for proving all theorems using logic and only a collapsible compass, like a piece of string, and a straight edge without a scale. All proofs can be traced back to the assumptions which are taken to be self-evident truths. They consist of five *postulates* and five *axioms*, or *common notions*. The postulates are of a geometric nature, whereas the axioms are more general. Today however, we tend to use the words postulate and axiom interchangeably.

Common Notions (Axioms)

1. Things which are equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things which coincide are also equal.
5. The whole is greater than the part.

Postulates

1. A straight line may be drawn between two distinct points.
2. A finite straight line may be produced to any length in a straight line.
3. A circle may be described with any center and any distance from the center.
4. All right angles are equal.
5. If a straight line meets two other straight lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.

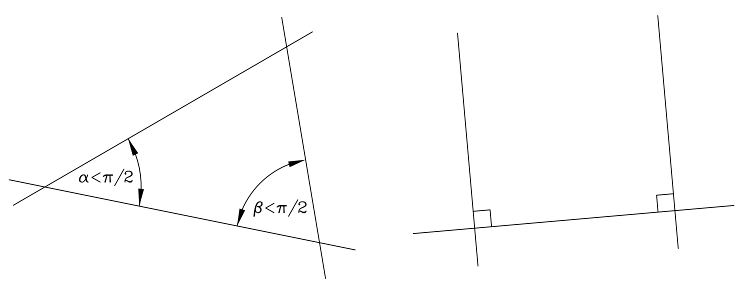


Figure 5.1: Euclid's parallel postulate.

The fifth postulate, or *parallel postulate* as it has come to be known, which is illustrated in Figure 5.1, has been the subject of study since the time it was published 2300 years ago. In fact, attempts to "prove" the parallel postulate led to the discovery of the *non-Euclidean* geometries: elliptic and hyperbolic, where the fifth postulate is pushed to opposite extremes. Before briefly examining the validity of the parallel postulate in elliptic and hyperbolic geometry, let us restate it in a more convenient form as:

> for each line l and each point P not on l, there is exactly one, i.e. one and only one, line through P parallel to l.

5.2 Riemannian Elliptic Geometry

G.F Bernhard Riemann, 1826-1866, was a German mathematician who made significant contributions to analysis, number theory, and geometry [2, 3]. He observed that in the elliptic plane, the parallel postulate is inconsistent. That is, given a line l and a point P not on l, there are no lines containing P parallel to l. This observation is based on the model he devised in 1854 for the elliptic plane, which is described next.

Analogous to the *straight* line in the plane is the *geodesic* line on a curved surface. The geodesic is the shortest curve on the surface connecting two points. The elliptic plane is modeled by central projection of the points in E_2 onto the surface of a hemisphere, see Figure 5.2. Each point P in the plane σ yields a line

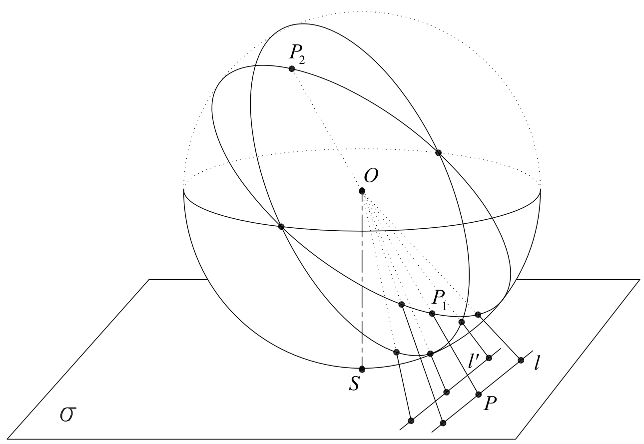


Figure 5.2: Central projection model of the elliptic plane.

OP, joining it to O, the centre of the sphere. This diameter intersects the sphere in two antipodal points, similar to north and south poles, P_1 and P_2 which are both images of the the same point P under the central projection. Each line l in σ yields a plane Ol, joining it to O. This diametral plane intersects the sphere in a great circle: a circle whose center is coincident with the sphere center O. Great circles on a sphere are its only geodesics. Hence, all great circles are *straight* lines on the sphere. If we allow the plane σ to be bounded by the line at infinity, L_∞, then the equator are the points at infinity, doubly mapped. We abstractly define the antipodal points to be one and the same point.

All parallel lines in σ intersect the sphere as arcs of great circles that all meet at the same points on the equator. This is because every pair of great circles on a sphere intersect in a pair of antipodal points. Since the antipodal points are defined to be the same, all parallel lines in the elliptic plane, modelled by great circles on the sphere, intersect in the same point at infinity. But, lines in different directions have different points at infinity, all on the same line, L_∞. When L_∞ is treated like any other line, the elliptic plane becomes a model for the real projective plane, as will soon seem obvious, but more needs to be discussed.

The main conclusion is that in the elliptic plane the fifth postulate of Euclid must be replaced with:

> given line l and point P not on l, there are *no lines* containing P that are parallel to l. In other words, all lines in the plane which contain P intersect l.

5.3 Hyperbolic Geometry

Hyperbolic geometry was discovered independently in about 1826 [2] by Nikolai Lobachevsky (1782-1856), Janos Bolyai (1802-1860), and Carl Friedrich Gauss (1777-1855). This was the first truly non-Euclidean geometry compared to Riemann's elliptic geometry which dates to about 1854. The model of the

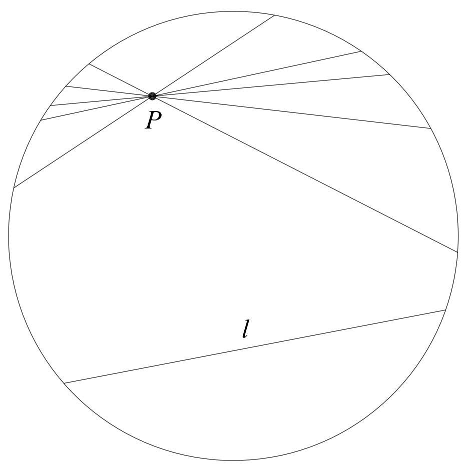


Figure 5.3: Model of the hyperbolic plane.

hyperbolic plane is a subset of the Euclidean plane. The points of the hyperbolic plane are those on the interior of a circle in the Euclidean plane, excluding those points on the circumference. Thus lines are finite, but unbounded, chords of the given circle. The geometry on this surface is hyperbolic geometry. Distance and angles are defined in a different way compared to Euclidean and elliptic geometry. But it is easy to see in Figure 5.3 that for a given line l and point P not on it there are infinite lines that contain P that do not intersect l, and hence are parallel to l. Thus we must replace Euclid's fifth postulate with:

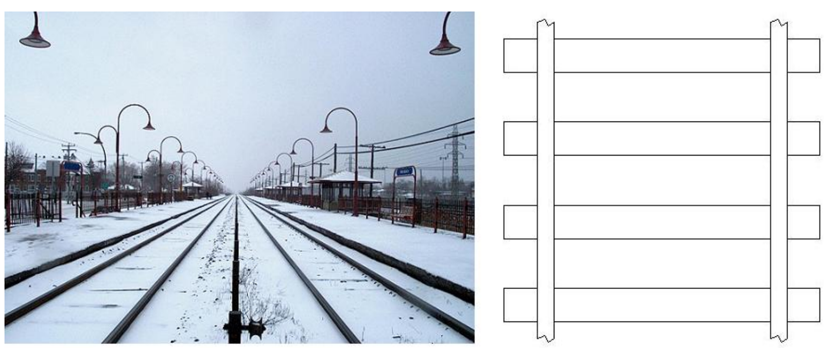
> given line l and point P not on l, there are *an infinite number* of lines containing P that are *parallel* to l.

Many of Euclid's axioms and postulates are valid in elliptic and hyperbolic geometry, but many, such as the fifth postulate, are not. The point to emphasize is that for 2200 years there was only Euclidean geometry. Then in the space of 30 years suddenly there were two new, axiomatically consistent, non-Euclidean geometries. Suddenly, geometry was realized to be far from a *closed* subject. The 19^{th} century brought about many great advances. Those who made significant discoveries in the subjects we will examine were Julius Plücker (1801-1868), his Ph.D. student Felix Klein (1849-1925), and Arthur Cayley (1821-1895), among many others.

Before looking at Klein's Erlangen Programme, an analytical way to systematically derive different geometries, we'll examine synthetic *projective geometry*. It turns out that projective geometry is the most general *linear* geometry, from which all other *linear* geometries, including elliptic, hyperbolic, and Euclidean geometry, also called parabolic, are derived [4], these conic monikers are discussed further in Section 5.7.5. In fact, Cayley's collected mathematical papers [5] from 1889, on page 592 contains the following famous quote.

> "The more systematic course in the present introductory memoir would have been to ignore altogether the notions of distance and metrical geometry ... Metrical geometry is a part of descriptive geometry, and descriptive geometry is all geometry."

Cayley used the word *descriptive* where today we would say *projective*. But the observation is clear: projective geometry is the foundation of all linear geometries.



(A) (B)

Figure 5.4: (A) What we see. (B) What is really there.

5.4 Synthetic Projective Geometry

We can think of projective geometry as Euclidean geometry with some axioms "left out", or changed. For instance, there is no parallelism, and no way to measure angles, or the distance between points. In fact, with our vision we see a 2D stereo projection of a 3D Euclidean world. Our view of E_3 changes every time we move our eyes. We see a projection of E_3 onto the projective plane, P_2, of our vision. What we see with our eyes is physically vastly different from the things we are looking at, train tracks for example, see Figure 5.4.

It was exactly this problem of reconciling the geometry of our local macro physical existence with the very different geometry of our vision that led people like Albrecht Durer (1471-1526), Johann Kepler (1571-1630), Gerard Desargues (1593-1662), and Blaise Pascal (1623-1662) to create a set of axioms that would be consistent with what we see. Their work ultimately led to the discovery of the most general geometry, proved by Klein in 1872 [6], in which colinear points map to colinear points. It is now known as *projective geometry.* The development of projective geometry was inspired by the troubling observation that lines which are known to be parallel appear to intersect when viewing them from a specific vantage point. The illustration in Figure 5.5 depicts a generalisation of this problem. In the figure, Durer's assistant plots the locations of points on the lute that are projected onto the plane that the assistant measures in, and then transfers the locations to the paper. Note that the projector is a piece of string attached to a pointer at one end which passes through an eye attached to the wall. The string is attached to a weight at the opposite end to keep the projectors approximately straight lines. All projectors pass through the eye attached to the wall so that it functions as a vantage point for observing the lute, thereby approximating what one would see if the focal point of their eyes was located at the eye in the wall.

For another example, the "parallel" lines of a pair of train tracks appear to converge at a point on a third line, the horizon L_∞, which is illustrated in Figure 5.6. In that figure points A and B lie on Line r while points C and D lie on line s. Similar to the eye attached to the wall in Figure 5.5, the human eye in Figure 5.6 is located at the vantage point, while the conceptual plane of vision is pierced by lines r and s at points A and C, respectively. In the plane of vision the projected points B' and D' appear to be closer together than the

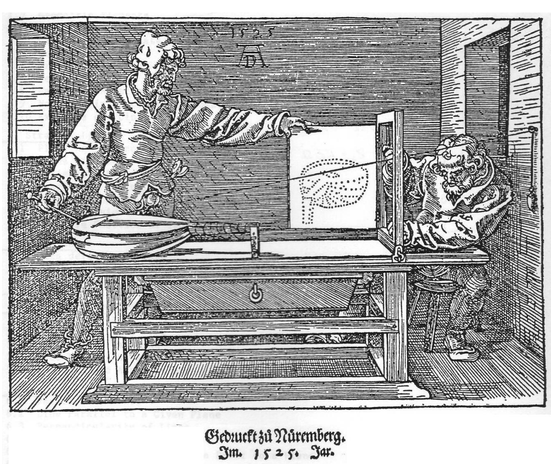


Figure 5.5: Albrecht Durer projections.

actual points. Moreover, the projections of lines r and s appear to intersect in a point on a line orthogonal to their directions, labelled L_∞.

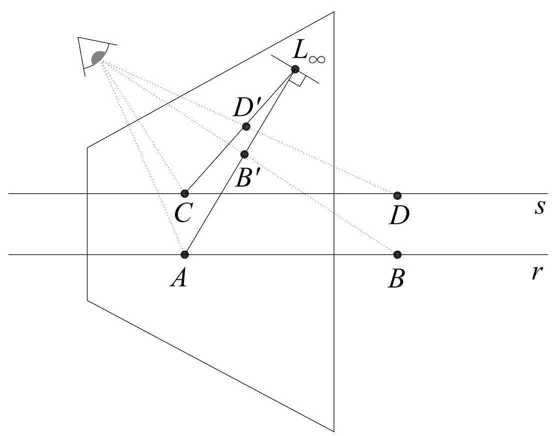


Figure 5.6: The parallel lines of a pair of train tracks appear to converge to a point on a third line, the *horizon*, or the *line at infinity*, L_∞.

In general, in projective space P_3, any two parallel lines from E_3 meet at a point on a line which is perpendicular to the two parallel lines. In fact, all lines parallel to r and s will appear to converge to exactly the same point. This point is called the *point at infinity* of the class of parallel lines to which r and s belong. So, for every unique direction there is a unique point at infinity, also called an *ideal point*. We can extend E_3 by adding a point at infinity for each direction. The totality of all the points at infinity in a plane lie on the *line at infinity*, L_∞. The totality of all lines at infinity lie on the *plane at infinity*, π_∞.

To synthesize projective geometry let's take five non-metric theorems from E_3 and remove from them the idea of parallelism.

Euclidean Theorems

$E1$**:** Two distinct points determine one and only one line.

$E2$**:** Three distinct non-colinear points, also any line and a point not on the line, determine one and only one plane.

$E3$**:** Two distinct coplanar lines either intersect in one point, or are parallel.

$E4$**:** A line not in a given plane either intersects the plane in a point or is parallel to the plane.

$E5$**:** Two distinct planes either intersect in a line or are parallel.

Projective Theorems

$P1$: Two distinct points determine one and only one line.

$P2$: Three distinct non-colinear points, also any line and a point not on the line, determine one and only one plane.

$P3$: Two distinct coplanar lines determine one and only one point.

$P4$: A line not in a given plane intersects the plane in one point.

$P5$: Two distinct planes determine one and only one line.

Comparing the Euclidean (E) and projective (P) theorems we see the P theorems are shorter and free from "either/or" constructions. But a far more important gain is the concept of *duality*. For each theorem in the projective plane P_2 another is obtained by simply exchanging key words. In the projective plane P_2 the dual elements are *line* and *point*: compare $P1$ and $P3$; $P1$ is obtained from $P3$ by changing the words *point* and *line*. The dual elements of projective space are *point* and *plane*, compare P_1 and P_5.

5.4.1 Theorem of Pappus

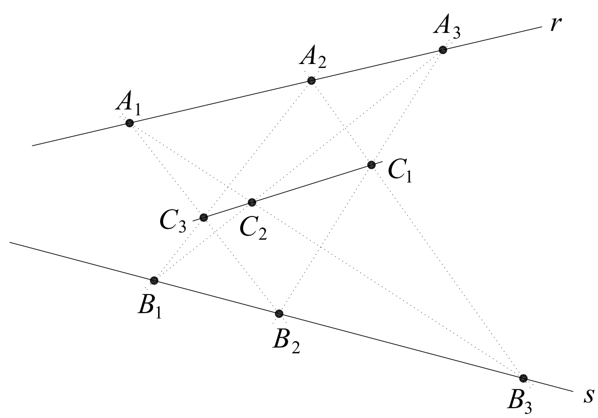


Figure 5.7: Theorem of Pappus: hexagon theorem.

Pappus of Alexandria was one of the last great Greek mathematicians of Antiquity. He is known for his *Collection*, circa 340 AD [7], which is a compendium of mathematics in eight volumes, the bulk of which still survives! In the book Pappus uses the terms *analysis* and *synthesis* in the way they are defined in modern mathematics and kinematic geometry. The Collection also contains his famous *hexagon* theorem, also called simply the *theorem of Pappus*. Nothing is known of his life other than from his own writings where he identifies himself as a teacher in Alexandria. The theorem of Pappus is a nice example of projective

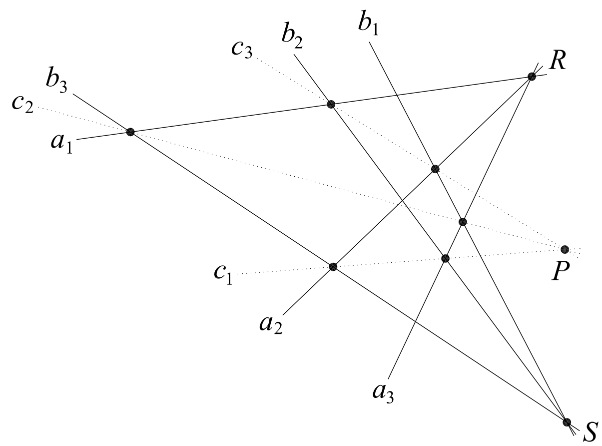


Figure 5.8: Dual Theorem of Pappus.

duality in P_2. Because it is a theorem that is independent of measurements of lengths and angles it is equally valid in E_2 as well as P_2. However, Pappus did not know that, as all he ever knew was the Euclidean plane, E_2. To begin, consider the following definitions:

1. $AB \cdot CD$ is the point of intersection of line segments joining point pairs AB and CD.

2. $ab \cdot cd$ is the line on the points of intersection of line pairs ab and cd, where a, b, c, and d are lines.

Theorem: In the projective plane P_2, let A_1, A_2, A_3 be any distinct *points* on any *line* r and B_1, B_2, B_3 be any other three distinct *points* on any other *line* s; then the *points* $C_1 = A_2B_3 \cdot A_3B_2$, $C_2 = A_1B_3 \cdot A_3B_1$, $C_3 = A_1B_2 \cdot A_2B_1$ are collinear, see Figure 5.7.

The reason this theorem is called the hexagon theorem is because it was originally stated as: *if the six vertices of a hexagon lie alternately on two lines, the three points of intersection of pairs of opposite sides are collinear*. Of course, the edges of this hexagon self-intersect and it is therefore not convex, but it is still a six sided planar figure with six vertices and six edges.

Dual: Let a_1, a_2, a_3 be any three distinct *lines* on any *point* R and b_1, b_2, b_3 be any other three distinct *lines* on any other *point* S; then the lines $c_1 = a_2b_3 \cdot a_3b_2$, $c_2 = a_1b_3 \cdot a_3b_1$, $c_3 = a_1b_2 \cdot a_2b_1$ are concurrent, see Figure 5.8.

The vertices of the hexagon to which Pappus refers are more obviously visible in Figure 5.8 illustrating the dual theorem as the points of intersection of the three lines on each of points R and S where the line pairs with different indices in each set intersect. Pappus' theorem and its dual involve nine points and nine lines which can be drawn in an infinite number of ways; although they are apparently different, they are *projectively equivalent* [7].

5.4.2 Pascal's Theorem

Blaise Pascal, 1623-1662, was a French mathematician, physicist, and inventor. He was an important mathematician, helping create two major new areas of research: he wrote a significant treatise on the subject of projective geometry at the age of 16 and later corresponded with Pierre de Fermat on probability theory, strongly influencing the development of modern economics and social science. Pascal's earliest work was in the natural and applied sciences where he made important contributions to the study of fluids, which is the work he is best known for among engineers. Moreover, the SI unit for pressure is the Pascal. In a treatise on geometry that Pascal published as a 16 year old in 1640, he proved an important theorem, illustrated with two examples in Figure 5.9.

Theorem: In the projective plane P_2, if a simple hexagon $A_1A_2A_3A_4A_5A_6$, either concave or convex, is inscribed in a conic, the intersections

$$R = A_1A_2 \cdot A_4A_5, \;\; S = A_2A_3 \cdot A_5A_6, \;\; T = A_3A_4 \cdot A_6A_1,$$

of the three pairs of opposite sides are collinear.

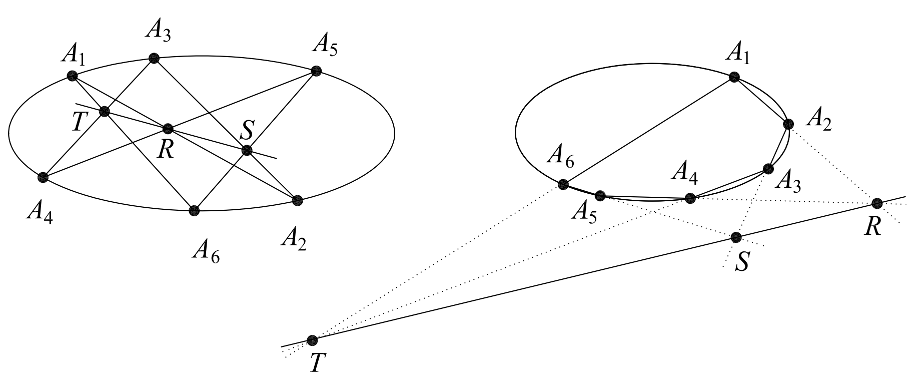


Figure 5.9: Pascal's theorem.

The line dual of Pascal's theorem was proved by Charles Julien Brianchon, 1783-1864, a French mathematician and chemist, in 1806, 166 years after Pascal proved his. However, Brianchon did not use the principle of duality and proved his theorem using synthetic geometric reasoning, so this dual theorem properly carries Brianchon's name. If a mathematician from today was transported back to 1640 to meet with Pascal and describe the point-line duality of the projective plane they would have been able to show Pascal that he actually had proved two theorems by simply exchanging the words *point* and *line*. Brianchon could have then directed his attention to other work.

Both the theorems of Pascal and Brianchon fail in E_2 for regular hexagons and hexagons with different edge lengths but whose opposite edges are still parallel and hence do not intersect. But, in P_2 the extensions of the pairs of opposite sides meet on the line at infinity, see Figure 5.10. For a degenerate conic consisting of two lines, Pascal's theorem and the theorem of Pappus are identical. In fact, in P_2 Pascal's and Pappus' theorems are abstractly isomorphic, in other words identical, period! In E_2 Pappus' theorem is always true, but not Pascal's theorem.

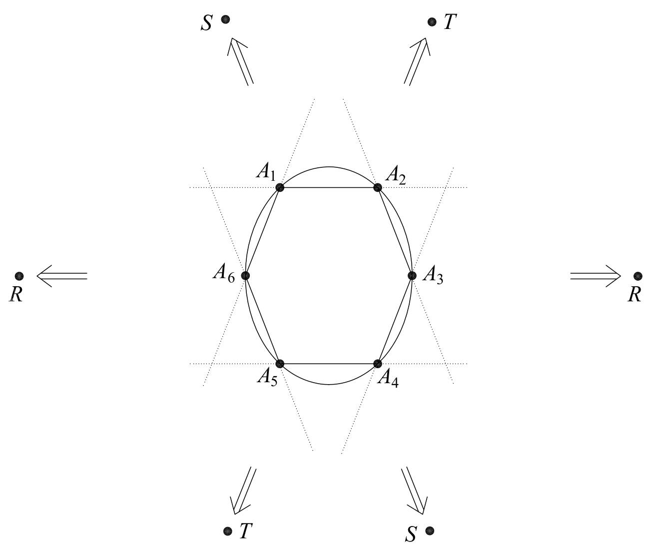


Figure 5.10: Pascal's theorem is always true in P_2, but not in E_2.

Moral of the Story

One must use the appropriate geometry depending on the goal. For dimensional synthesis the goal is to identify sizes and relative locations of links in a mechanism. If the location of the axis of a prismatic joint is needed then E_2 is not sufficient. You must identify a geometry that is axiomatically consistent with the needs of the design problem. This requires geometric thinking. No software package can help with that.

5.4.3 Losses and Gains

The extension of E_2 and E_3 with the ideal elements of points, lines, and planes at infinity results in a much more elegant geometry because of duality. However, we lose the metric, similarity, and betweeness.

Every line in $P_{2,3}$ has a unique point at infinity; you get to this point no matter which direction you travel on the line. This means we can model a projective line as a closed curve, see Figure 5.11.

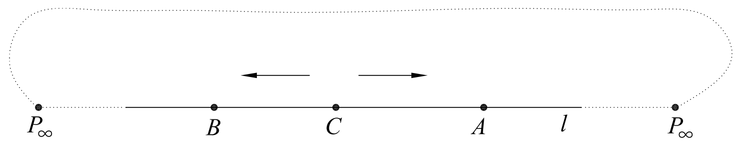


Figure 5.11

Direction To get to B from A you can go in either direction. Hence, we lose the concept of direction.

Betweeness Is C between B and A? Or, is A between C and B? We can only have *separation* between sets of 4 points. For example, in Figure 5.12 points B and D separate points A and C.

Figure 5.12: Points B and D separate Points A and C.

5.5 Homogeneous Coordinates

Let O be the origin of the Cartesian coordinate system, shown in Figure 5.13. Let S be a distinct point in the plane. The ray passing through O and S is described by the coordinate pair (x, y). Another distinct point $Q \neq O$, on ray OS is described by the pair $(\mu x, \mu y)$, where $\mu \in \mathbb{R}$ (ie., a real number). As $\mu \to \pm\infty$ the seemingly meaningless pair (∞, ∞) is obtained [8].

To remedy this representational problem, the point pairs may be represented by two ratios, given by ordered triples (x_0, x_1, x_2). If $x_0 \neq 0$, then the point S can be uniquely described as:

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0}. \tag{5.1}$$

Then any triple of the form $(\lambda x_0, \lambda x_1, \lambda x_2)$ (for $\lambda \neq 0$) describes exactly the same point S. In other words, two real points are equal if the triples representing them are proportional. This is because

$$\frac{\lambda x_1}{\lambda x_0} = \frac{x_1}{x_0} = x, \quad \text{and} \quad \frac{\lambda x_2}{\lambda x_0} = y.$$

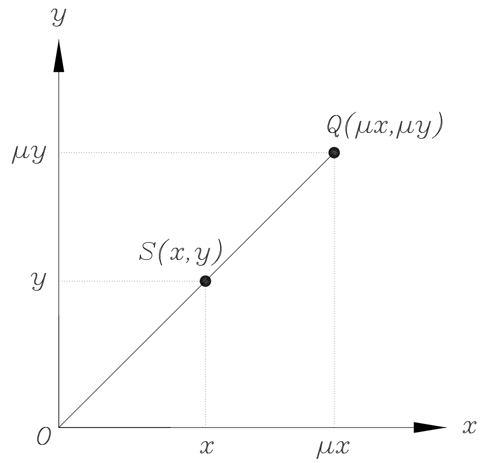


Figure 5.13: Cartesian coordinates in E_2.

The corresponding coordinates $(x_0 : x_1 : x_2)$ are called *homogeneous coordinates*. When $x_0 = 1$ the Cartesian coordinate pair (x, y) is recovered.

The Cartesian coordinates $(\mu x, \mu y)$, $\mu \neq 0$, of the family of points on the ray through Q in Figure 5.13 can be expressed in homogeneous coordinates as ratios:

$$(\mu x, \mu y) = (x_0 : \mu x_1 : \mu x_2) = (\frac{x_0}{\mu} : x_1 : x_2).$$

In E_2 as $\mu \to \pm\infty$ the homogeneous coordinates $(0 : x_1 : x_2)$ are obtained. There is no point on the line OS to which this triple can correspond because E_2 is unbounded. However, in the projective extension of the Euclidean plane[1] the triple $(0 : x_1 : x_2)$, P_2 describes the *point at infinity (ideal point)* on the line OS. Since the same triple is obtained regardless if $\mu \to +\infty$ or $\mu \to -\infty$, a unique point at infinity is associated with the line OS in P_2. Hence, an ordinary line adjoined by its point at infinity is a closed curve [10].

For $x_0 = 0$ the triple $(0 : 0 : 0)$ describes neither an ideal point nor a real point on OS. $(0 : x : y) = (0 : 0 : 0)$ seems to imply that $S = O$, which is a contradiction in the construction of the ray OS. The trivial triple $(0 : 0 : 0)$ is therefore not included in the point set comprising the projective extension of E_2.

All lines in E_2 which are extended to their points at infinity have the homogenising coordinate $x_0 = 0$. The totality of all the existing points at infinity, with the exception of $(0 : 0 : 0)$, are described by $x_0 = 0$. The extended Euclidean plane which includes all the points at infinity is called the *projective*

[1]The *projective plane*, P_2, can be thought of as the Euclidean plane, E_2, to which the *line at infinity* has been added. The generalisation of this concept of *extension* is attributed to Herman Grassmann [9].

plane P_2. Since $x_0 = 0$ is a linear equation, it represents the *line at infinity*, L_∞.

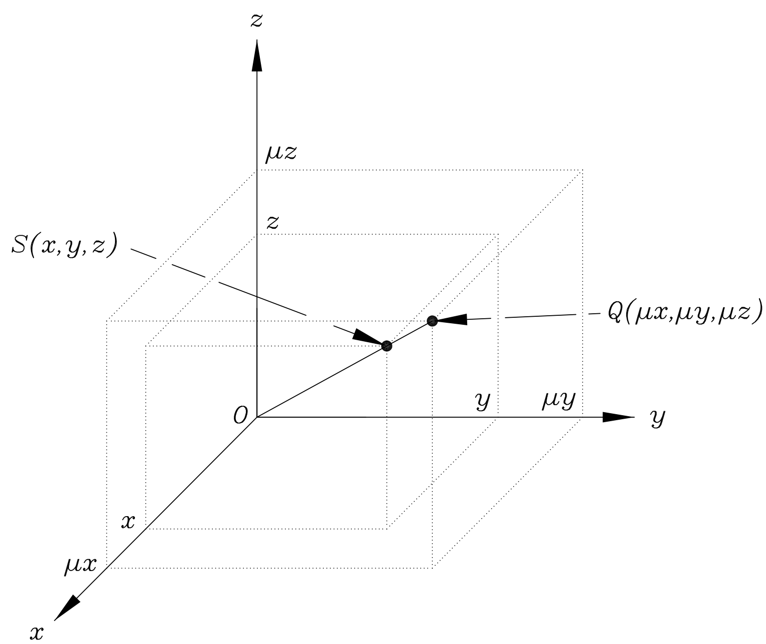


Figure 5.14: Cartesian coordinates in E_3.

Entirely analogous statements can be made for 3D Euclidean space, E_3. This space is covered by a Cartesian coordinate system with origin O and axes x, y, z. The axes are usually defined as orthogonal. Such an orthogonal Cartesian system is illustrated in Figure 5.14. The homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ of the point $S \in E_3$ are defined as:

$$x = \frac{x_1}{x_0},\ y = \frac{x_2}{x_0},\ z = \frac{x_3}{x_0},\ x_0 \neq 0. \tag{5.2}$$

As in two dimensional projective space, when $x_0 = 1$ the Cartesian coordinate triple (x, y, z) is recovered. It should be noted that in general the choice of homogenising coordinate is arbitrary. Over the course of time the following conventions have developed.

1. In North America and the British Commonwealth the homogenising coordinate is taken to be the last one. The coordinate indices begin with the number 1. In the plane, $(x_1 : x_2 : x_3)$ represent the coordinates of a point, with x_3 the homogenising coordinate. In space, a point is described with $(x_1 : x_2 : x_3 : x_4)$, x_4 being the homogenising coordinate. In general, the homogenising coordinate in an nD space has the index $n + 1$.

2. In Europe the first coordinate, given the index 0, is taken to be the homogenising one. Thus, x_0 represents the homogenising coordinate regardless of the dimension of the coordinate space.

Both conventions shall be employed henceforth. This is to underscore the idea that such a restriction is arbitrary and unnecessary in the context of projective geometry, discussed in Section 5.7. However, where required the homogenising coordinate shall be explicitly identified.

5.6 Duality: Point, Line, and Plane Coordinates

In the Euclidean plane a general line has the equation

$$Ax + By + C = 0, \tag{5.3}$$

where A, B and C are arbitrary constants defining the slope and intercepts with the coordinate axes. The x and y that satisfy the equation are points on the line. Using homogeneous coordinates this linear equation becomes

$$X_0 x_0 + X_1 x_1 + X_2 x_2 = 0, \tag{5.4}$$

where the X_i characterise lines (*i.e.,* $X_0 = C,\ X_1 = A,\ X_2 = B$) and the x_i characterise points. Now Equation (5.4) represents Equation (5.3) as an equation that is linear in the X_i as well as the x_i. Every term in Equation (5.4) is *bilinear*, or *homogeneously linear*. This should explain the etymology of the term *homogeneous coordinates*. The X_i are substituted for the A, B and C to underscore the bilinearity and symmetry.

Equation (5.4) may be viewed as a locus of variable points on a fixed line, or as a pencil of variable lines on a fixed point. The X_i define the line and are hence termed *line coordinates*, indicated by the ratios $[X_0 : X_1 : X_2]$; whereas the x_i define the point and bear the name *point coordinates*, indicated by the ratios $(x_0 : x_1 : x_2)$. Note the distinction that line coordinates are contained in square brackets, [ ], while point coordinates have parentheses for delimiters, ( ). Equation (5.4) is a *bilinear* equation describing the mutual incidence of point and line in the plane. Thus, point and line are considered as *dual* elements in the *projective plane* P_2. The importance of this concept is that any valid theorem concerning points and lines yields another valid theorem by simply exchanging these two words [11]. For example, the proposition

any two distinct points determine one and only one line

is dualised by exchanging the words point and line giving a different proposition,

any two distinct lines determine one and only one point.

In space the mutual incidence of point and plane is given by the bilinear equation

$$X_0 x_0 + X_1 x_1 + X_2 x_2 + X_3 x_3 = 0, \tag{5.5}$$

where the x_i remain point coordinates, however the X_i are now *plane coordinates*, the dual elements of 3D *projective space* P_3 being point and plane. Because of the duality, the roles of coefficient and variable are interchangeable. For instance, Equation (5.5) can represent the family points on a fixed plane, or the family planes on a fixed point.

The importance of the principle of duality as a conceptual tool can not be over-emphasised. It shall be employed frequently in the analysis presented in subsequent lectures.

5.6.1 Computing Point, Line, and Plane Coordinates

A necessary and sufficient condition that three distinct points in the plane, represented by the homogeneous coordinates as $(x_0 : x_1 : x_2)$, $(y_0 : y_1 : y_2)$ and $(z_0 : z_1 : z_2)$, be collinear is that [11, 12, 13, 14]

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} = 0.$$

It then follows that the line determined by two distinct points $(y_0 : y_1 : y_2)$ and $(z_0 : z_1 : z_2)$ has an equation that is easily obtained employing *Grassmannian expansion* [4, 9, 15]:

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} x_0 - \begin{vmatrix} y_0 & y_2 \\ z_0 & z_2 \end{vmatrix} x_1 + \begin{vmatrix} y_0 & y_1 \\ z_0 & z_1 \end{vmatrix} x_2 = 0,$$

where a variable point on a fixed line has *point coordinates* $(x_0 : x_1 : x_2)$ and, dually, a variable line on a fixed point has *line coordinates*

$$[X_0 : X_1 : X_2] = \left[\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} : \begin{vmatrix} y_2 & y_0 \\ z_2 & z_0 \end{vmatrix} : \begin{vmatrix} y_0 & y_1 \\ z_0 & z_1 \end{vmatrix} \right], \tag{5.6}$$

note that the columns in the middle determinant have been exchanged to eliminate the negative sign. Comparing the coordinates, it is to be seen that Equation (5.4) represents this exact duality.

A similar relation exists when the equation of a plane is written using homogeneous coordinates. In E_3 a necessary and sufficient condition that four points, whose homogeneous point coordinates are $(x_0 : x_1 : x_2 : x_3)$, $(y_0 : y_1 : y_2 : y_3)$, $(z_0 : z_1 : z_2 : z_3)$ and $(w_0 : w_1 : w_2 : w_3)$, be coplanar is that [10, 11]

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \\ w_0 & w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

It follows that the plane determined by three distinct points has an equation, again obtained with the Grassmannian expansion, given by Equation (5.5). A variable point on a fixed plane has *point coordinates* $(x_0 : x_1 : x_2 : x_3)$, while the principle of duality means that a variable plane on a fixed point has *plane coordinates*

$$[X_0 : X_1 : X_2 : X_3] \;=\; \left[\begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} : \begin{vmatrix} y_0 & y_3 & y_2 \\ z_0 & z_3 & z_2 \\ w_0 & w_3 & w_2 \end{vmatrix} : \begin{vmatrix} y_0 & y_1 & y_3 \\ z_0 & z_1 & z_3 \\ w_0 & w_1 & w_3 \end{vmatrix} : \begin{vmatrix} y_0 & y_2 & y_1 \\ z_0 & z_2 & z_1 \\ w_0 & w_2 & w_1 \end{vmatrix} \right],$$

where again, to eliminate the negative multipliers, two columns in the second and fourth determinants have been exchanged.

5.7 Definition of Linear Geometry

Every geometry of space whose group of transformations are *collineations* which contain the sub-group $\mathcal{G}_7$ can be derived from projective geometry. This geometry has the smallest set of invariants. It is also the most general. This means that not every theorem valid in projective geometry is valid in the sub-geometries defined by less general collineations, recall the discussion on the theorems of Pappus and Pascal in Sections 5.4.1 and 5.4.2. The sub-geometries usually have a larger set of invariants. It was Arthur Cayley who first realised that "projective geometry is all geometry" [16] however, it was Felix Klein who provided the means to systematically derive the sub-geometries [4].

5.7.1 The Erlangen Programme

In 1872 Felix Klein gave his famous inaugural address at the Friedrich-Alexander University in Erlangen, Germany, the text of which is now known as the *Erlangen Programme* [6]. Relying on the earlier work of Arthur Cayley [16], it was intended to show how Euclidean and non-Euclidean geometry could be established from projective geometry. However, Klein's contributions turned out to be more general, leading to a whole series of new geometries. Today, they are known as *Cayley-Klein*[2] *geometries* and the spaces in which they are valid are *Cayley-Klein spaces* [19] (discussed in Section 5.7.5). The following summary of the Erlangen Programme was provided by Klein, himself, in [4]:

> Given any group of transformations[3] in space which includes the principal group, $\mathcal{G}_7$, as a sub-group, then the invariant theory of this group gives a definite kind of geometry, and every possible geometry can be obtained in this way.

According to the Erlangen Programme, the following *dual* propositions are always valid [20]:

[2]This term is attributed to Sommerville[17, 18].

[3]The terms *transformation* and *linear transformation* shall be used interchangeably. This is because all transformations used in this work are linear.

1. A geometry on a space defines a group of linear transformations[4] in that space.
2. A group of linear transformations in a space defines a geometry on that space.

Moreover, the character of a geometry is determined by the relations which remain invariant under the associated group of linear transformations [4, 24].

These linear transformations are of the form

$$\mathbf{A}\mathbf{x} \quad = \quad k\mathbf{b}, \tag{5.7}$$

where $\mathbf{x}$ and $\mathbf{b}$ are the $n+1$ homogeneous coordinates of two points in an n dimensional space, $\mathbf{A}$ is a nonsingular $(n+1)\times(n+1)$ matrix and k is a proportionality constant arising from the use of the homogeneous coordinates.

An *invariant* is defined [4, 22, 24] as a function of the coordinates under the transformation such that

$$\phi(b_0,\ldots,b_n) \quad = \quad \Delta^p\phi(x_0,\ldots,x_n), \tag{5.8}$$

where Δ is the determinant of the matrix $\mathbf{A}$ (which is, by definition, nonsingular) and p is a weighting factor, and the $n+1^{st}$ coordinates are those with the 0 index. If $p=0$ then ϕ is an *absolute invariant*, otherwise it is a *relative invariant* with weight p [22]. Klein's definition of a geometry involves absolute invariants, *i.e.*, functions of the coordinates which remain unchanged by the associated group of transformations [20].

5.7.2 Transformation Groups

Projective Transformations

The *projective transformations* in projective space P_3 may be thought of as 4×4 matrix operators that are collineations. It is important to note that an $(n+1)$D homogeneous coordinate space is required to analytically describe the elements of an nD projective space. These matrices are non-singular by definition. They are sometimes referred to as *structure matrices* [25] since changing the structure of the matrix changes the character of the geometry it represents. A transformation of P_3 may be written as

$$\mathbf{P} \quad = \quad \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{bmatrix}, \tag{5.9}$$

[4] The modern understanding of *linear transformation* is limited to those defined on metric vector spaces. However, in this work the term *linear transformation* refers to any nonsingular injective collineation (*i.e.*, a one-to-one transformation that maps collinear points onto collinear points), in any space. We use the transformations as $n\times n$ matrix operators, but care must be taken because they operate on $n\times 1$ matrices, and not vectors. For instance, a vector space can not be defined on P_3 using 4D vectors, whose elements are composed of homogeneous coordinates, because there is no $\mathbf{0}$ element, which, when added to any other element $\mathbf{v}$ leaves $\mathbf{v}$ unchanged: $\mathbf{v}+\mathbf{0}=\mathbf{v}$. In P_3 the point $(0:0:0:0)$ is not defined. Hence, the more general definition must be used. The interested reader is directed to [21, 22, 23].

where the 16 elements are arbitrary, but all contain a common factor owing to the use of homogeneous coordinates. Hence, the *projective group* of all collineations in P_3 has fifteen parameters, and is termed $\mathcal{G}_{15}$ [4]. Because there are no restrictions on the elements, with the exception that the determinant of the matrix never vanishes, they are the most general linear geometric transformations in 3D space. The fundamental invariant of $\mathcal{G}_{15}$ in particular, and n dimensional projective geometry in general, is the *cross ratio* of four collinear points. The cross ratio is the fundamental invariant of all linear transformation groups, and hence all linear geometries.

Cross Ratio

The concept of cross ratio is one of the oldest now known to be part of projective geometry. It is the only invariant of projective geometry, but is also the fundamental invariant in every linear geometry. It is believed that the theory was known to Pappus of Alexandria, who lived between approximately 290-350 [2, 14, 26]. We can work with the cross ratio using metric concepts from Euclidean geometry and making the required extensions to allow for elements at infinity, but here it will be analytically defined in the plane as follows [14]:

Definition 5.7.1 *If the collinear points A, B, C, and D, at least three of which are distinct, on a projective line have coordinates $(a_0 : a_1)$, $(b_0 : b_1)$, $(c_0 : c_1)$ and $(d_0 : d_1)$, respectively, then the real number*

$$CR(A,B;C,D) \quad = \quad \frac{\begin{vmatrix} a_0 & a_1 \\ c_0 & c_1 \end{vmatrix}\begin{vmatrix} b_0 & b_1 \\ d_0 & d_1 \end{vmatrix}}{\begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}\begin{vmatrix} a_0 & a_1 \\ d_0 & d_1 \end{vmatrix}} \tag{5.10}$$

if it exists is the cross ratio *of the four points in the order A, B, C, D. If the number does not exist, the cross ratio is said to be* infinite*.*

It is important to note that the coordinate with the 0 index is the homogenising coordinate while the coordinate with the index 1 is essentially the location of the point along the line. Evaluating the first determinant in Equation 5.10 yields $a_0c_1 - c_0a_1$ which can be interpreted as the directed distance from A to C. Normalising the projective coordinates of a point in the plane on a line by dividing all the coordinates by the homogenising coordinate means that we can use metric concepts and the cross ratio of the four collinear points is expressed as the ratios of directed distances along the line as

$$CR(A,B;C,D) \quad = \quad \left(\frac{AC}{BC}\right)\left(\frac{BD}{AD}\right). \tag{5.11}$$

Consider the four points on the line illustrated in Figure 5.15. Without loss in generality the points can be spaced at equidistant intervals relative to the coordinate system attached to the line. Considering the coordinates of points

A, B, C, and D to be $(a_0 : a_1) = (1 : 2)$, $(b_0 : b_1) = (1 : 3)$, $(c_0 : c_1) = (1 : 4)$, $(d_0 : d_1) = (1 : 5)$, the cross ratio of the points in the order A, B, C, D is

$$CR(A, B; C, D) = \left(\frac{AC}{BC}\right)\left(\frac{BD}{AD}\right) = \left(\frac{4-2}{4-3}\right)\left(\frac{5-3}{5-2}\right) = \frac{4}{3}.$$

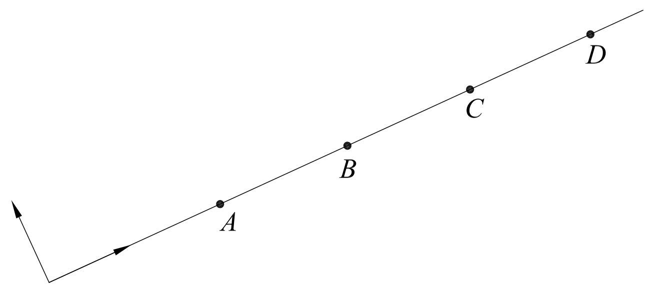


Figure 5.15: Cross ratio of four points on a line.

If one of the points along the line is at infinity, then the ratio containing the homogenising coordinate that is 0 is simply not included in the computation. In general, when C is midway between A and B while D is at infinity then $CR = -1$ and the four points are said to be in a harmonic sequence, however any four finite points on a line whose cross ratio is $CR = -1$ are in a harmonic sequence.

Affine Transformations

The equations of general affine transformations in affine space A_3 contain twelve arbitrary coefficients. Thus, the *affine group* is indicated by $\mathcal{G}_{12}$. It should be apparent that $\mathcal{G}_{12} \subset \mathcal{G}_{15}$. This transformation group of A_3 is typically expressed as:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{bmatrix}. \tag{5.12}$$

Affine geometry can be considered as more *rich* than projective geometry because its set of invariants includes more than just the cross ratio. For example, affine transformations leave the plane at infinity, $x_0 = 0$, invariant, which is generally not the case for projective transformations.

Euclidean Transformations

The group of Euclidean transformations of E_3, also a subgroup of $\mathcal{G}_{15}$, are represented by

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{bmatrix}. \tag{5.13}$$

However, $\mathbf{E}$ contains a 3×3 proper orthogonal sub-matrix *i.e.,* having a determinant of $+1$ representing a change in orientation [27]. The principal group, $\mathcal{G}_7$, represents the most general Euclidean collineations [21]. The Euclidean displacement group $\mathcal{G}_6$ is characterised by the property that both distance and sense are invariant under $\mathcal{G}_6$ [7].

5.7.3 Invariants

Recall that an absolute invariant is defined to be a function of the coordinates of an element in the given geometry which remains invariant under the associated linear transformation group [4, 14]. The Euclidean displacement group $\mathcal{G}_6$ is defined in a metric space, see Section 5.7.4. In addition to the preservation of distance and sense, its set of invariants contains a special imaginary quadratic form. First consider $\mathcal{G}_3 \subset \mathcal{G}_6$. The equation of an arbitrary circle, k, in E_2 with radius r and centre $\mathcal{C}(x_c,\ y_c)$ is:

$$(x - x_c)^2 + (y - y_c)^2 = r^2. \tag{5.14}$$

Expressing Equation (5.14) using homogeneous coordinates $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$ produces

$$(x_1 - x_c x_0)^2 + (x_2 - y_c x_0)^2 = r^2 x_0^2. \tag{5.15}$$

The intersection with the line at infinity $x_0 = 0$ is given by the equations

$$x_1^2 + x_2^2 = 0, \quad x_0 = 0. \tag{5.16}$$

The constants r, x_c and y_c which characterise the circle do not appear in the result. Thus, every circle in the plane intersects the line at infinity in exactly the same two points, namely,

$$I_1(0 : 1 : i), \quad I_2(0 : 1 : -i). \tag{5.17}$$

They are widely called the *imaginary*, or *absolute circle points* [4, 10, 23, 28]. It can be shown, in the same way, that every sphere cuts the plane at infinity in the imaginary conic:

$$x_1^2 + x_2^2 + x_3^2 = 0, \quad x_0 = 0, \tag{5.18}$$

which is called the *imaginary*, or *absolute sphere circle*.

These absolute quantities account for the apparent deficiency of Bezout's theorem [22, 29] for the intersections of algebraic curves and surfaces. That is, two curves of order n and m will intersect in at most nm points; similarly, two surfaces of order n and m will intersect in a curve of, at most, order nm. Clearly, two circles intersect in at most two points, while two spheres intersect in a circle; a second order curve. Since every circle contains I_1 and I_2, two circles intersect in at most four points, and Bezout's theorem is seen to be true. The same applies for spheres; they intersect in a curve which splits into a real and an imaginary conic.

To summarise, the invariants of $\mathcal{G}_3$ include those of the projective and affine planes, but additionally include the line at infinity and two imaginary conjugate points on it, namely I_1 and I_2. The invariants of $\mathcal{G}_6$ include those of projective and affine 3D space, including the plane at infinity and an imaginary conic on it: the imaginary sphere circle.

5.7.4 Metric Spaces

The material on metric spaces presented here is reproduced from Chapter 3 for convenient reference. Metric and non-metric geometries may be looked upon as *special cases* of projective geometry. Before continuing, some definitions are required.

Definition 5.7.2 *The* **Cartesian Product** *of any two sets, $\mathcal{S}$ and $\mathcal{T}$, denoted $\mathcal{S} \times \mathcal{T}$, is the set of all ordered pairs (s, t) such that $s \in \mathcal{S}$ and $t \in \mathcal{T}$.*

Definition 5.7.3 *Let $\mathcal{S}$ be any set. A function d mapping $\mathcal{S} \times \mathcal{S}$ into $\mathbb{R}$ is a* **metric** *on $\mathcal{S}$ iff [30]*

1. $d_{s_1 s_2} = 0$ *iff* $s_1 = s_2$;
2. $d_{s_1 s_2} \geq 0, \ \forall\, s_i \in \mathcal{S}$;
3. $d_{s_1 s_2} = d_{s_2 s_1}, \ \forall\, s_i \in \mathcal{S}$;
4. $d_{s_1 s_2} + d_{s_2 s_3} \geq d_{s_1 s_3}, \ \forall\, s_1, s_2, s_3 \in \mathcal{S}$.

A *metric space* is a set $\mathcal{S}$, together with a metric d defined on $\mathcal{S}$. A *metric geometry* on that space is defined by the group of linear transformations which leave the metric invariant. For example, Euclidean space is a metric space because it contains the set $\mathcal{P}$ of all points. The metric defined on $\mathcal{P}$ is Euclidean distance,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \tag{5.19}$$

which is an invariant of $\mathcal{G}_6$. Thus, Euclidean geometry is a metric geometry. It is important to note that a rule to measure distance in a space is not sufficient to make the space metric. All four conditions in Definition 5.7.3 must be satisfied. An example of a geometry containing a distance rule and distinct points with zero distance between them is *Isotropic Geometry*. The transformations associated with the isotropic plane are [31]

$$\begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}. \tag{5.20}$$

Distance in this geometry is measured by the difference of the x-coordinates of two points: $d = x_2 - x_1$. The distance between two points is clearly invariant under the transformation in Equation 5.20, but it is also clear that there exist an infinite number of distinct points possessing the same x-coordinate and therefore have zero distance between them. The complete enumeration of all such *degenerate geometries* was given by Sommerville in [17].

5.7.5 Cayley-Klein Spaces and Geometries

Projective geometry can be developed from the fundamental elements of point, line, plane and Hilbert's axioms [32] of incidence, order and continuity independently of the concept of metric. Thus, in projective geometry there is no rule to measure and the only absolute invariant is the cross ratio of four points [2]. In defining a *Cayley-Klein* space one could start with projective geometry and define a *rule* to measure distance. Usually this is done by introducing a quadratic form. For instance, Euclidean geometry can be developed from projective geometry by building upon the foundation of *Cayley's principle* [16] that *projective geometry is all geometry* using Klein's Erlangen Programme, *i.e.*, the theory of algebraic invariants. Euclidean geometry can be obtained by adjoining, or constraining, P_3 with the special quadratic form [4]

$$x_1^2 + x_2^2 + x_3^2 = 0, \tag{5.21}$$

which represents the absolute sphere circle, obtained in Equation 5.18. It is an imaginary quadric containing all points with a vanishing norm. This quadratic form is induced by the Euclidean distance function between the homogeneous coordinates of points $(x_0, x_1 : x_2 : x_3)$ and $(y_0 : y_1 : y_2 : y_3)$

$$d = \frac{\sqrt{(x_1y_0 - y_1x_0)^2 + (x_2y_0 - y_2x_0)^2 + (x_3y_0 - y_3x_0)^2}}{x_0y_0}. \tag{5.22}$$

The quadratic form, or *norm*, belonging to this rule is

$$x_1^2 + x_2^2 + x_3^2.$$

Equations (5.21) and (5.22) are fundamental invariants of $\mathcal{G}_6$. However, Equation (5.21) is independent of x_0. An entirely different quadratic form in P_3 can be obtained by adding x_0^2 to Equation (5.21):

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0. \tag{5.23}$$

Changing the quadratic form changes the rule for measuring magnitudes. For instance, the signs could be changed as follows:

$$x_0^2 - x_1^2 + x_2^2 + x_3^2 \quad = \quad 0. \tag{5.24}$$

Each new rule gives a different form of *space*. These are the *Cayley-Klein* spaces. The first quadratic form, Equation (5.21) gives Euclidean, or *parabolic* space. Equation (5.23) gives *Riemann* non-Euclidean, or *elliptic* space, while Equation (5.24) gives *Lobachevskii* non-Euclidean, or *hyperbolic* space [4, 19]. In each of these spaces there is a group of transformations that leaves the norm invariant. These characterise the corresponding geometries [33].

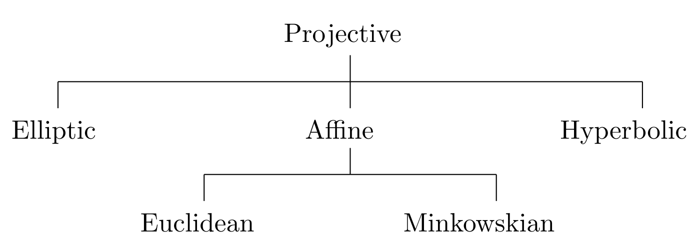
Equation (5.21) may be viewed as sphere with no volume. The distance between two distinct points on this virtual quadric vanishes. The term *virtual* means that only complex points lie on it. Similarly, Equation (5.23) may be viewed as a virtual ellipsoid. Whereas, Equation (5.24) represents a real hyperboloid of two sheets.

The non-Euclidean geometries were serendipitously discovered by efforts to prove Euclid's parallel axiom: *given a line g and a point P, not on g, there is one, and only one line p through P that does not intersect g.* The Euclidean model of Riemann's elliptical plane is a unit sphere, recall Figure 5.2. Straight lines on a sphere are geodesics, *i.e.*, great circles. All great circles intersect in two anti-podal points. If the they are taken to be the same point, then there are no parallel lines in the elliptic plane, because all lines intersect in a point [3].

The Euclidean model for Lobachevskii's hyperbolic plane is the points contained in a unit circle, excluding points on the circumference, recall Figure 5.3. Straight lines are chords of the circle, the end points excluded. Thus, given a line g and a point P not on g in the hyperbolic plane there are an infinite number of lines through P that do not intersect g [3].

Klein was the first to make use of the terms elliptic, parabolic and hyperbolic to classify these geometries [4]. The use of these names implies no direct connection with the corresponding conic sections, rather they mean the following. A central conic is an ellipse or hyperbola according as it has no asymptote or two asymptotes. Analogously, a non-Euclidean plane is elliptic or hyperbolic according as each of its lines contains no point at infinity, or two [7].

However, many other possibilities exist. For instance 4D Minkowskian geometry [34] is well known for its application to Einstein's *Special Theory of Relativity* [2]. It differs from the other geometries in that time differentials are among its set of elements. In the following hierarchy, each geometry can be derived from the one above it by some kind of condition imposed on the transformation group [2].



5.8 Representations of Displacements

It is convenient to think of the relative displacement of two rigid-bodies in E_3 as the displacement of a Cartesian reference coordinate frame E attached to one of the bodies with respect to a Cartesian reference coordinate frame Σ attached to the other [28]. Without loss of generality, Σ may be considered as fixed while E is free to move. Then the position of a point in E in terms of the basis of Σ can be expressed compactly as

$$\mathbf{p}' = \mathbf{A}\mathbf{p} + \mathbf{d}, \tag{5.25}$$

where, $\mathbf{p}$ is the 3×1 position vector of a point in E, $\mathbf{p}'$ is the position vector of the same point in Σ, $\mathbf{d}$ is the position vector of the origin of frame E expressed in Σ, $O_{E/\Sigma}$, and $\mathbf{A}$ is a 3×3 proper orthogonal rotation matrix, *i.e.*, its determinant is $+1$.

It is clear from Equation (5.25) that a general displacement can be decomposed into a pure rotation and a pure translation. The representation of the translation is straightforward: it is given by the position vector in Σ of 0_E. However, there are many ways to represent the orientation. For example *fixed angle* or *Euler angle* representations may be used. There are twelve distinct ways to specify an orientation in each representation. This is because the rotation is decomposed into the product of three rotations about the coordinate axes in a certain order, with twelve distinct permutations. The axes of the fixed frame are used in the fixed angle representation, also called *roll, pitch, yaw* angles [27], while the axes of the moving frame are used for the Euler angle representation.

5.8.1 Orientation: Euler-Rodrigues Parameters

An invariant representation for rotations is given by the *Euler-Rodrigues parameters* [35]. Using *Cayley's formula for proper orthogonal matrices* [27, 28], matrix $\mathbf{A}$ in Equation (5.25) can be rewritten in the following form [28]:

$$\mathbf{A} = \Delta^{-1}\begin{bmatrix} c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ 2(c_1c_2+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ 2(c_1c_3-c_0c_2) & 2(c_2c_3+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 \end{bmatrix}, \tag{5.26}$$

where

$$\Delta = c_0^2+c_1^2+c_2^2+c_3^2,$$

and the c_i, called *Euler-Rodrigues parameters* [28, 36], are defined as

$$\begin{aligned} c_0 &= \cos\tfrac{\varphi}{2}, \\ c_1 &= s_x \sin\tfrac{\varphi}{2}, \\ c_2 &= s_y \sin\tfrac{\varphi}{2}, \\ c_3 &= s_z \sin\tfrac{\varphi}{2}. \end{aligned}$$

The c_i may be *normalised* such that $\Delta = 1$, in which case $\mathbf{s} = [s_x, s_y, s_z]^T$ is a unit direction vector parallel to the axis and φ is the angular measure of a given rotation about that axis. The Euler-Rodrigues parameters are *quadratic invariants* of any given rotation [36].

Since $\mathbf{s}$ is a unit vector, it is immediately apparent that the c_i are not independent, but related by

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 \quad = \quad 1.$$

The geometric interpretation of the four Euler-Rodrigues parameters is that an orientation may be viewed as a point on a unit hyper-sphere in a four-dimensional space. Assembled into a 4×1 array, the Euler-Rodrigues parameters are the *unit quaternions* invented by Sir William Hamilton [35]. The group of spherical displacements, $SO(3)$, are elegantly represented with unit quaternions.

5.8.2 Displacements as Points in Study's Soma Space

In 1903 Eduard Study showed [37] that Euclidean displacements may be represented by eight parameters, or coordinates in a seven dimensional homogeneous projective space. Thus, displacements can be represented as points; fundamental elements in this space. His work was undoubtedly inspired by that of Julius Plücker and Felix Klein. Klein's Erlangen Programme gave rise to a systematic method for constructing new geometries based on the algebraic invariants of the associated transformation groups. However, it was Plücker who first suggested the idea of taking the line as the fundamental element of space [38]. Various types of line coordinates were introduced by Cayley and Grassmann [39]; Plücker adopted a coordinate system which is a special form of these. The success of Plücker's work was hindered by the Cartesian analysis that he employed [38, 40, 41]. Klein, Plücker's student, introduced the system of coordinates determined by six linear complexes in mutual involution: on any line common to two linear complexes a one-to-one correspondence of points is determined by the planes through the line by taking the poles of each plane for the complexes. If a certain condition is satisfied connecting the coefficients of the two complexes, then these pairs of points form an involution [39]. Moreover, Klein's observation that the line geometry of Plücker is point geometry on a quadric contained in a five dimensional space was of critical importance in the conceptualisation of the soma space [18].

Plücker and soma coordinates are analogous in that the set of all lines, in the case of Plücker coordinates, and the set of all displacements, in the case of soma coordinates both exist as the set of points covering special quadric surfaces in higher dimensional spaces. Points not on the respective quadrics represent neither lines nor displacements. Since both quadrics have identical forms, it is instructive to first examine how Plücker coordinates come about, and the nature of their constraint surface, before moving on to Study's soma.

5.8.3 Plücker Coordinates

Plücker developed line coordinates [40, 41] to address the need of describing lines as the fundamental elements of his *neue Geometrie* [38]. Line coordinates may be obtained from Cartesian coordinates by considering the following: a line on the intersection of two planes, or dually the ray on two points. In the former case, the Plücker coordinates specify the linear pencil of planes and are generally called *axial Plücker coordinates*. In the latter case, they are called *ray Plücker coordinates*. If $X(x_0 : x_1 : x_2 : x_3)$ and $Y(y_0 : y_1 : y_2 : y_3)$ are the homogeneous coordinates of two different points on a line, the *Grassmannian sub-determinants* [4] of the associated 2×4 matrix composed of the point coordinates, comprise the homogeneous Plücker coordinates of the line [42]:

$$p_{ik} = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix} \qquad i, k \in \{0, \ldots, 3\},\ i \neq k.$$

Of the twelve possible Grassmannians, only six are independent, since $p_{ik} = -p_{ki}$. Traditionally, the following six are used

$$(p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}). \tag{5.27}$$

These six coordinates collected in a 6×1 matrix are called the *Plücker array*.

The six Plücker coordinates in the sequence given in Equation (5.27) can be interpreted as consisting of two sets of three parameters which are each a vector in E_3, called *Plücker vectors*. Assuming that the first three Plücker coordinates are not all zero, then both vectors can be normalised thus:

$$\mathbf{p} = \frac{(p_{01} : p_{02} : p_{03})}{\sqrt{p_{01}^2 + p_{02}^2 + p_{03}^2}}, \tag{5.28}$$

$$\overline{\mathbf{p}} = \frac{(p_{23} : p_{31} : p_{12})}{\sqrt{p_{01}^2 + p_{02}^2 + p_{03}^2}}. \tag{5.29}$$

The two vectors, $\mathbf{p}$ and $\overline{\mathbf{p}}$ are duals of each other, and the space in which they exist can be considered a dual vector space. The first vector, consisting of the elements of $\mathbf{p}$, is proportional to the direction of the distance between points x and y on the line in E_3, while the dual three, consisting of the elements of $\overline{\mathbf{p}}$, represent the moment of the line segment with respect to the origin of the coordinate system in which x and y are defined. Considering the Plücker array as two dual vectors leads to some elegant analytic methods for robot analysis, where lines can represent the R-pair axes and P-pair directions in a robot mechanical system. Some of these methods are described in Chapter 5, *Analytic Projective Geometry*.

A line, however, is uniquely determined by a point and three direction cosines. The Plücker coordinates are super-abundant by two, hence there are two constraints on the six parameters. First, because the coordinates are homogeneous, there are only five independent ratios. It necessarily follows that

$$(p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}) \neq (0 : 0 : 0 : 0 : 0 : 0). \tag{5.30}$$

Second, the six numbers must obey the following quadratic condition:

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} \quad = \quad 0. \tag{5.31}$$

The first condition, Equation (5.30), is known as the *non-zero condition*. The quadric condition, represented by Equation (5.31) is called the *Plücker identity* [39], also known as the *Plücker condition*. Geometrically, it represents a four-dimensional quadric hyper-surface in a five-dimensional projective homogeneous space, called *Plücker's quadric*, $\mathcal{P}_4^2$ [2, 26]. Distinct lines in Euclidean space are distinct points on $\mathcal{P}_4^2$, but an array of six numbers that doesn't satisfy the Plücker condition does not represent a line.

The Plücker quadric can be derived in the following way [42]. Consider the determinant of a matrix composed of the homogeneous coordinates of two points $X(x_i)$ and $Y(y_i)$, $i \in \{0,1,2,3\}$, counted twice. Obviously, the determinant vanishes because of the linear dependence between rows 1 and 3, and between rows 2 and 4. This determinant can be expanded using 2×2 sub-determinants, known as minors, along the first two rows, according to the *Laplacian expansion theorem* [21]. That is, multiply the product of the minor with its complement, the determinant of the matrix of the rows and columns not in the minor, by $(-1)^h$, where h is the sum of the numbers denoting the rows and columns in which the minor appears. This gives

$$\begin{aligned}
0 = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{vmatrix} &= (-1)^{3+(1+2)} \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \\
&(-1)^{3+(1+3)} \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix} \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + (-1)^{3+(1+4)} \begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \\
&(-1)^{3+(2+3)} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix} + (-1)^{3+(2+4)} \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix} + \\
&(-1)^{3+(3+4)} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} = 2(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}).
\end{aligned} \tag{5.32}$$

Since $p_{13} = -p_{31}$, Equation 5.32 simplifies to Equation 5.31.

Now attention is turned towards determining the structure of the quadric hyper-surface $\mathcal{P}_4^2$. The important observation is that Equation (5.31) contains only bilinear cross-terms. This implies that the quadric has been rotated out of its *standard position*, or *normal form* [43]. The quadratic form associated with $\mathcal{P}_4^2$, can be represented using a 6×6 symmetric matrix, $\mathbf{M}$ [44]:

$$\mathbf{p}^T\mathbf{M}\mathbf{p} \quad = \quad [p_{01}, \cdots, p_{12}] \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{01} \\ \vdots \\ p_{12} \end{bmatrix}.$$

This quadratic form can be orthogonally diagonalised with another 6×6 matrix $\mathbf{P}$, constructed with the eigenvectors of $\mathbf{M}$. The matrix $\mathbf{P}$ is easily found to be

$$\mathbf{P} \;=\; \frac{\sqrt{(2)}}{2}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Now, pre-multiplying $\mathbf{M}$ with the transpose of $\mathbf{P}$ and post-multiplying with $\mathbf{P}$ itself gives the diagonalised matrix, $\mathbf{D}$, *i.e.*, $\mathbf{P}^T\mathbf{M}\mathbf{P} = \mathbf{D}$:

$$\mathbf{D} \;=\; \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Matrix $\mathbf{D}$ reveals the normal form of $\mathcal{P}_4^2$ in *canonical form* [43] from the matrix multiplication $\mathbf{p}^T\mathbf{D}\mathbf{p} = \mathbf{p}^T(\mathbf{P}^T\mathbf{M}\mathbf{P})\mathbf{p}$:

$$p_{01}^2 + p_{02}^2 + p_{03}^2 - p_{23}^2 - p_{31}^2 - p_{12}^2 \;=\; 0. \tag{5.33}$$

Observing the signs on these six pure quadratic terms, one immediately sees that the Plücker quadric, $\mathcal{P}_4^2$, has the form of an hyperboloid in the five dimensional space. In this space, only the points on $\mathcal{P}_4^2$ represent lines.

5.8.4 Study's Soma

A general Euclidean displacement of reference frame E with respect to Σ, as given by Equation (5.25), depends on six independent parameters: three are required for the orientation of E and three for the position of O_E. Regarding this situation geometrically, distinct Euclidean displacements of E may be represented as distinct points in a six-dimensional space. Hence, a displacement is an element of a six-dimensional geometry. However, Study showed [37] that a coordinate space of dimension eight is necessary to ensure that all the relations among the entries of Equation (5.25) are fulfilled. Thus, an array of eight numbers can represent a displacement as a fundamental element in a seven dimensional homogeneous projective space. These eight numbers were termed *soma* by Study [45]. Similar to the Plücker array, Study's soma are

$$(c_0 : c_1 : c_2 : c_3 : g_0 : g_1 : g_2 : g_3).$$

The first four of Study's soma coordinates are the Euler-Rodrigues parameters, c_i, defined in Section 5.8.1. The remaining four, g_i $i \in \{0, \ldots, 3\}$, are linear combinations of the elements of $\mathbf{d}$, from Equation (5.25), and the c_i such that the following quadratic condition is satisfied:

$$c_0 g_0 + c_1 g_1 + c_2 g_2 + c_3 g_3 = 0. \tag{5.34}$$

Study defined these four parameters to be

$$\begin{aligned}
g_0 &= d_1 c_1 + d_2 c_2 + d_3 c_3, \\
g_1 &= -d_1 c_0 + d_3 c_2 - d_2 c_3, \\
g_2 &= -d_2 c_0 - d_3 c_1 + d_1 c_3, \\
g_3 &= -d_3 c_0 + d_2 c_1 - d_1 c_2.
\end{aligned} \tag{5.35}$$

Owing to the homogeneity of the Euler-Rodrigues parameters there is an additional quadratic constraint on the soma, stemming from the denominator of Equation (5.26), which is similar to the non-zero condition for the Plücker coordinates:

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 \neq 0. \tag{5.36}$$

Thus, of the eight soma coordinates only six are independent, but all eight are required to uniquely describe a displacement [37].

Equation (5.34) represents a six-dimensional quadric hyper-surface in a seven-dimensional space. It is called *Study's quadric*, S_6^2 [46]. Its form can be determined in a way analogous to that used for $\mathcal{P}_4^2$. After applying the same diagonalisation procedure to the quadratic form, the normal form of S_6^2 is revealed to be:

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 - g_0^2 - g_1^2 - g_2^2 - g_3^2 = 0.$$

We see immediately that S_6^2 has the form of an hyperboloid in the soma space. Of all the points in the soma space, only those on S_6^2 represent displacements.

5.8.5 Vectors in a Dual Projective Three-Space

Another way of looking at the eight soma coordinates is to consider them as two sets of four parameters, each of which can represent a vector in a four-dimensional coordinate space [47, 48]. A spatial Euclidean displacement can then be mapped into the set of two *Study vectors* in the four-dimensional space in an analogous way that a line in Euclidean space can be mapped to sets of two Plücker vectors. Employing this concept, Ravani [47] introduced the idea of representing a Euclidean displacement as a point in a dual projective three-space. This, however, leads directly to the representation of displacements in terms of dual quaternions, see Blaschke [49], Bottema and Roth [28], or McCarthy [50] for example.

Although this representation and that of Study are analytically identical, they represent completely different geometric interpretations. In the latter case, displacements are represented by points on Study's quadric in its seven-dimensional projective space, while the former represents displacements by two vectors in a dual projective three-space.

5.8.6 Transfer Principle

A representation identical to the one discussed in the last section can be obtained using the *transfer principle* (Bottema and Roth [28], Ravani and Roth [48]). Spherical displacements are readily represented using the four Euler-Rodrigues parameters. That is, if a spherical displacement is mapped into the points of a real three-dimensional projective space where the coordinates are four-tupples of Euler-Rodrigues parameters, then spatial displacements can be mapped into a similar, but dual, space. In other words, the representation of a spatial displacement is obtained simply by dualising the corresponding spherical displacement (Ravani and Roth [48]).

5.9 Kinematic Mappings of Displacements

So far in this chapter we have discussed various ways to represent displacements. In all of them, at least six independent numbers are required. This led Study, in 1903 [45], to the idea of mapping distinct displacements in Euclidean space to the points of a seven-dimensional projective image space. The homogeneous coordinates of the image space are the eight soma coordinates. As mentioned earlier, these eight coordinates are not independent. They are super-abundant by two. However, two quadratic constraints must be satisfied. The non-zero condition, Equation (5.36), and the displacement must be a point on S_6^2, Equation (5.34). It is natural to expect that a six-dimensional image space would suffice. However, as previously mentioned, Study [37] showed that an 8D coordinate space is required.

5.9.1 General Euclidean Displacements

NOTE: for the remainder of the chapter the material presented will use the North American convention for homogeneous coordinates.

Study's kinematic mapping of general Euclidean displacements is given by the following equations in terms of the eight Study soma $\{c_i : g_i\}$

$$(x_1 : x_2 : x_3 : x_4 : y_1 : y_2 : y_3 : y_4) = (c_1 : c_2 : c_3 : c_0 : \frac{g_1}{2} : \frac{g_2}{2} : \frac{g_3}{2} : \frac{g_0}{2}).$$

Equation (5.25) can always be represented as a linear transformation by making it *homogeneous*, see McCarthy [50] for example. Let the homogeneous coordinates of points in the fixed frame Σ be the ratios $[X : Y : Z : W]$, and those of points in the moving frame E be the ratios $[x : y : z : w]$. Then Equation (5.25) can be rewritten as

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \mathbf{Q} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \tag{5.37}$$

where

$$\mathbf{Q} = \Delta^{-1} \begin{bmatrix} c_0^2 + c_1^2 - c_2^2 - c_3^2 & 2(c_1c_2 - c_0c_3) & 2(c_1c_3 + c_0c_2) & d_1 \\ 2(c_1c_2 + c_0c_3) & c_0^2 - c_1^2 + c_2^2 - c_3^2 & 2(c_2c_3 - c_0c_1) & d_2 \\ 2(c_1c_3 - c_0c_2) & 2(c_2c_3 + c_0c_1) & c_0^2 - c_1^2 - c_2^2 + c_3^2 & d_3 \\ 0 & 0 & 0 & \Delta \end{bmatrix},$$

with $\Delta = c_0^2 + c_1^2 + c_2^2 + c_3^2$, and the d_i are the components of the position vector of $O_{E/\Sigma}$.

Let the transformation matrix $\mathbf{T}$ be the *image* of the elements of $\mathbf{Q}$ under the kinematic mapping. Since $\Delta \neq 0$ by one of the quadratic constraints, it's value is arbitrary and represents a scaling factor whose value is meaningless in a projective space. Recall, the homogeneous coordinates of $[\lambda x : \lambda y : \lambda z]$ and of $[\gamma x : \gamma y : \gamma z]$ represent the same point in the projective plane for any non-zero scalar constants λ and γ. Then we may write

$$\mathbf{T} = \begin{bmatrix} x_1^2 - x_2^2 - x_3^2 + x_4^2 & 2(x_1x_2 - x_3x_4) & 2(x_1x_3 + x_2x_4) & 2(y_4x_1 - y_3x_2 + y_2x_3 - y_1x_4) \\ 2(x_1x_2 + x_3x_4) & -x_1^2 + x_2^2 - x_3^2 + x_4^2 & 2(x_2x_3 - x_1x_4) & 2(y_3x_1 + y_4x_2 - y_1x_3 - y_2x_4) \\ 2(x_1x_3 - x_2x_4) & 2(x_2x_3 + x_1x_4) & -x_1^2 - x_2^2 + x_3^2 + x_4^2 & 2(-y_2x_1 + y_1x_2 + y_4x_3 - y_3x_4) \\ 0 & 0 & 0 & x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{bmatrix}.$$

This transforms the coordinates of points in frame E to coordinates of the same points in frame Σ (assuming that the two frames are initially coincident) after a given displacement in terms of the coordinates of a point on S_6^2.

5.9.2 Planar Displacements

The transformation matrix $\mathbf{T}$ simplifies considerably when we consider displacements that are restricted to the plane. Three DOF are lost and hence four Study parameters vanish. The displacements may be restricted to any plane. Without loss in generality, we may select one of the principal planes in Σ. Thus, we arbitrarily select the plane $Z = 0$. Since E and Σ are assumed to be initially coincident, this means

$$\begin{bmatrix} X \\ Y \\ 0 \\ W \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ 0 \\ w \end{bmatrix}. \tag{5.38}$$

This requires that $d_3 = 0$: since $Z = z = 0$, E can translate in neither the Z nor z directions. It also requires that $s_x = s_y = 0$, and $s_z = 1$ because the equivalent rotation axis is parallel to the Z and z axes. All of this, in turn, means

$$\begin{aligned}
c_1 &= 0,\\
c_2 &= 0,\\
c_3 &= \sin\varphi/2,\\
c_0 &= \cos\varphi/2,\\
g_1 &= -d_1c_0 - d_2c_3,\\
g_2 &= -d_2c_0 + d_1c_3,\\
g_3 &= 0,\\
g_0 &= 0,
\end{aligned}$$

which leaves us with only four soma coordinates to map:

$$(x_3 : x_4 : y_1 : y_2) \quad = \quad (c_3 : c_0 : \frac{g_1}{2} : \frac{g_2}{2}). \tag{5.39}$$

The corresponding homogeneous linear transformation matrix reduces to

$$\mathbf{T} \quad = \quad \begin{bmatrix} x_4^2 - x_3^2 & -2x_3x_4 & 0 & 2(y_2x_3 - y_1x_4) \\ 2x_3x_4 & x_4^2 - x_3^2 & 0 & -2(y_1x_3 + y_2x_4) \\ 0 & 0 & x_3^2 + x_4^2 & 0 \\ 0 & 0 & 0 & x_3^2 + x_4^2 \end{bmatrix}. \tag{5.40}$$

We may eliminate the third row and column because they only provide multiples of the trivial equation

$$Z \quad = z \;=\; 0. \tag{5.41}$$

Thus, $\mathbf{T}$ reduces to a 3×3 matrix,

$$\mathbf{T} \quad = \quad \begin{bmatrix} x_4^2 - x_3^2 & -2x_3x_4 & 2(y_2x_3 - y_1x_4) \\ 2x_3x_4 & x_4^2 - x_3^2 & -2(y_1x_3 + y_2x_4) \\ 0 & 0 & x_3^2 + x_4^2 \end{bmatrix}. \tag{5.42}$$

Planar displacements still map to points on S_6^2, but we need only consider a special sub-set of these points. In fact, we may change our geometric interpretation altogether. We see that planar displacements can be represented by points in a three-dimensional projective image space. The coordinates of the points are the four Study parameters $(x_3 : x_4 : y_1 : y_2)$. In this sub-space, the points are not restricted to a special quadric. They can take on any value with the exception that x_3 and x_4 are not simultaneously zero. Points on the real line defined by $x_3 = x_4 = 0$ are not the images of real planar displacements because this sub-space is still contained in the soma space, where the non-zero quadratic condition requires $x_1^2 + x_2^2 + x_3^2 + x_4^2 \neq 0$. It is easy to see that if $x_1 = x_2 = 0$ the quadratic non-zero condition can only be violated if $x_3 = x_4 = 0$. This condition is of little interest since we are only interested in *real* displacements.

5.10 Blaschke-Grunẅald Mapping of Plane Kinematics

Another mapping of planar displacements, which is seen to be isomorphic to the Study mapping, can be derived in a somewhat more intuitive way. Very detailed accounts may be found in Bottema and Roth [28], De Sa [20] and Ravani [47]. It was introduced in 1911 simultaneously, and independently, by Grünwald [51] and Blaschke [52].

The idea is to map the three independent quantities that describe a displacement to the points of a 3D projective image space called Σ'. A general displacement in the plane requires three independent parameters to fully characterise it. The position of a point in E relative to Σ can be given by the homogeneous linear transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & a \\ \sin\varphi & \cos\varphi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{5.43}$$

where the ratios $(x : y : z)$ represent the homogeneous coordinates of a point in E, $(X : Y : Z)$ are those of the same point in Σ. The Cartesian coordinates of the origin of E measured in Σ are (a, b), while φ is the rotation angle measured from the X-axis to the x-axis, the positive sense being counter-clockwise. Clearly, in Equation (5.43) the three characteristic displacement parameters are (a, b, φ). Image points, points in the 3D homogeneous projective image space, are defined in terms of the displacement parameters (a, b, φ) as

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} a\sin(\varphi/2) - b\cos(\varphi/2) \\ a\cos(\varphi/2) + b\sin(\varphi/2) \\ 2\sin(\varphi/2) \\ 2\cos(\varphi/2) \end{bmatrix}. \tag{5.44}$$

By virtue of the relationships expressed in Equation (5.44), the transformation matrix from Equation (5.43) may be expressed in terms of the homogeneous coordinates of the image space, Σ'. This yields a linear transformation to express a displacement of E with respect to Σ in terms of the image point:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (X_4^2 - X_3^2) & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & (X_4^2 - X_3^2) & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & (X_4^2 + X_3^2) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{5.45}$$

Comparing the elements of the 3×3 transformation matrix in Equation (5.45) with the one in Equation (5.42) it is a simple matter to show that the homogeneous coordinates of the image space Σ' and those of the soma space are related in the following way:

$$(X_1 : X_2 : X_3 : X_4) = (y_2 : -y_1 : x_3 : x_4). \tag{5.46}$$

Comparing Equation (5.44) with Equation (5.39) it is evident that the two transformations are isomorphic.

Since each distinct displacement described by (a, b, φ) has a corresponding unique image point, the inverse mapping can be obtained from Equation (5.44): for a given point of the image space, the displacement parameters are

$$\begin{aligned} \varphi &= 2\arctan(X_3/X_4), \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2), \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2). \end{aligned} \tag{5.47}$$

When computing the inverse tangent function to obtain a numerical value for φ, the two argument inverse tangent function [27], atan2(y/x), should be used since it accounts for the sines of the values of X_3 and X_4 and placed the angle in the correct quadrant in Σ. Equations (5.47) give correct results when either X_3 or X_4 is zero. Caution is in order, however, because the mapping is injective, not bijective: *there is at most one pre-image for each image point* [53]. Thus, not every point in the image space represents a displacement. It is easy to see that any image point on the real line $X_3 = X_4 = 0$ has no pre-image and therefore does not correspond to a real displacement of E. From Equation (5.47), this condition renders φ indeterminate and places a and b on the line at infinity.

5.10.1 Dervation of the Mapping

Recall from Chapter 2, the pole of a planar displacement is the real invariant point associated with the displacement transformation matrix for the given a, b, φ corresponding to it's sole eigenvalue. It's easy to show that:

$$\begin{aligned} X_p = x_p &= \frac{1}{2}a\sin(\varphi/2) - \frac{1}{2}b\cos(\varphi/2), \\ Y_p = y_p &= \frac{1}{2}a\cos(\varphi/2) + \frac{1}{2}b\sin(\varphi/2), \\ Z_p = z_p &= \sin(\varphi/2), \end{aligned} \tag{5.48}$$

where $Z_p = \sin(\varphi/2)$ is an artifact of the derivation.

The image of the pole of the displacement under the kinematic mapping is called the image point. The image point defined by the Blaschke-Grunẅald mapping is defined using the *pole coordinates*. The image space is a 3D projective space described by the homogeneous coordinates $(X_1 : X_2 : X_3 : X_4)$. The mapping is:

$$(X_1 : X_2 : X_3 : X_4) = (X_p : Y_p : Z_p : \tau Z_p),$$

where

$$(X_1 : X_2 : X_3 : X_4) \neq ((0 : 0 : 0 : 0),$$

$$\tau \equiv \cot(\varphi/2),$$

and (X_p, Y_p, Z_p) depend on (a, b, φ).

Using the mapping Equations (5.44) we can express the linear transformation matrix in Equation (5.43) in terms of the homogeneous coordinates of the image space. Let

$$\mathbf{A} = \begin{bmatrix} \cos\varphi & -\sin\varphi & a \\ \sin\varphi & \cos\varphi & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

We have $a_{11} = a_{22}$, which may be re-expressed in terms of X_3 and X_4. Recall $X_3 = 2\sin(\varphi/2),\ X_4 = 2\cos(\varphi/2)$. Then:

$$\begin{aligned} X_4^2 - X_3^2 &= (2\cos(\varphi/2))^2 - (2\sin(\varphi/2))^2, \\ &= 4\left(\frac{1+\cos\varphi}{2}\right) - 4\left(\frac{1-\cos\varphi}{2}\right), \\ &= 4\cos\varphi. \end{aligned}$$

In the above derivation we have used the identities:

$$\cos^2(\varphi/2) = \frac{1+\cos\varphi}{2}, \qquad \sin^2(\varphi/2) = \frac{1-\cos\varphi}{2}.$$

Next $a_{12} = -a_{21}$. We obtain a_{21} from

$$2X_3X_4 = 2(2\sin(\varphi/2))(2\cos(\varphi/2)).$$

We use the identity $2\sin(\varphi/2) = \dfrac{\sin\varphi}{\cos(\varphi/2)}$ giving:

$$2\left[\left(\frac{\sin\varphi}{\cos(\varphi/2)}\right)2\cos(\varphi/2)\right] = 4\sin\varphi.$$

a_{13} is obtained as:

$$\begin{aligned} 2(X_1X_3+X_2X_4) &= 2[(a\sin(\varphi/2)-b\cos(\varphi/2)2\sin(\varphi/2)+(a\cos(\varphi/2)+b\sin(\varphi/2))2\cos(\varphi/2)], \\ &= 4[a\sin^2(\varphi/2)-b\cos(\varphi/2)\sin(\varphi/2)+a\cos^2(\varphi/2)+b\cos(\varphi/2)\sin(\varphi/2)], \\ &= 4a[\cos^2(\varphi/2)+\sin^2(\varphi/2)], \\ &= 4a. \end{aligned}$$

a_{23} is:

$$\begin{aligned} 2(X_2X_3-X_1X_4) &= 2[(a\cos(\varphi/2)+b\sin(\varphi/2)2\sin(\varphi/2)-(a\sin(\varphi/2)-b\cos(\varphi/2))2\cos(\varphi/2)], \\ &= 4[a\cos(\varphi/2)\sin(\varphi/2)+b\sin^2(\varphi/2)-a\cos(\varphi/2)\sin(\varphi/2)+b\cos^2(\varphi/2)], \\ &= 4b[\cos^2(\varphi/2)+\sin^2(\varphi/2)], \\ &= 4b. \end{aligned}$$

a_{33} is

$$\begin{aligned} X_3^2 + X_4^2 &= (2\sin(\varphi/2))^2 + (2\cos(\varphi/2))^2, \\ &= 4. \end{aligned}$$

Notice that 4 is a common factor to all non-zero terms of $\mathbf{A}$ when transformed using the relations above. This just implies that

$$4\mathbf{X} = 4\mathbf{A}\mathbf{x}$$

and the constant 4 can safely be factored out of the equation. Substituting the above relations into $\mathbf{A}$ gives:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_4^2 + X_3^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (5.49)$$

This transforms the coordinates of points in E to those of Σ after a displacement specified by (a, b, φ). But, the transformation is in terms of the coordinates of the image space. The result is a projective parametric equation. i.e.:

$$\begin{aligned} X &= (X_4^2 - X_3^2)x - (2X_3X_4)y + 2(X_1X_3 + X_2X_4)z, \\ Y &= (2X_3X_4)x + (X_4^2 - X_3^2)y + 2(X_2X_3 - X_1X_4)z, \\ Z &= (X_4^2 + X_3^2)z. \end{aligned}$$

The right-hand side of the equations are composed of terms that are homogeneously linear in the homogenous coordinates $(x : y : z)$, and homogeneously quadratic in the image space coordinates.

5.11 Geometry of the Image Space

A group of collinieations that leaves the absolute quadric invariant gives rise to various Cayley-Klein geometries. The geometry is hyperbolic when the absolute quadric is real and elliptic when complex, and the quadric is second-order in terms of all coordinates.

The geometry of the planar kinematic mapping image space is determined by the invariants of the group of linear transformations described by Equation (5.49). They are:

1. Two complex conjugate planes:

$$\begin{aligned} V_1 : \ X_3 + iX_4 &= 0, \\ V_2 : \ X_3 - iX_4 &= 0. \end{aligned}$$

2. The real line of intersection of V_1 and V_2 described by the equations $X_3 = X_4 = 0$:

$$l = V_1 \cap V_2 = (X_3 = 0) \cap (X_4 = 0).$$

3. Two complex conjugate points on l:

$$\begin{aligned} J_1 &= (1 : i : 0 : 0), \\ J_2 &= (1 : -i : 0 : 0). \end{aligned}$$

The complex conjugate points J_1 and J_2 are analogous to the imaginary circular points in the plane. Every circle in planes parallel to $X_3 = 0$ contains them, as does every circle in planes parallel to $X_4 = 0$.

The planes V_1 and V_2 comprise a degenerate imaginary quadric containing two factors:

$$(X_3 + iX_4)(X_3 - iX_4) = X_3^2 + X_4^2 = 0.$$

Blaschke observed that this is a special limiting case of the elliptic absolute quadric

$$\rho(X_1^2 + X_2^2) + X_3^2 + X_4^2 = 0.$$

As $\rho \to 0$ the degenerate invariant quadric of the image space is obtained. Since this is a limiting case, the geometry of the image space is termed *quasi-elliptic* [20, 52]. The term quasi-elliptic owes its existence to Blaschke [31].

Distinct image space points not on the line $X_3 = X_4 = 0$ are distinct displacements. Of interest are two special cases:

1. The (180°) half-turns in E_2:

$$X_3 = \pm 1,\ X_4 = 0,$$

$$\Rightarrow\ \varphi = \pi.$$

2. (a) The pure rectilinear and curvilinear translations in E_2:

$$X_3 = 0\ \ \Rightarrow\ \ X_4 = 1,$$

$$\Rightarrow\ \varphi = 0.$$

   (b) Both X_1 and X_2 vary but $X_3 =$ non-zero constant and $X_4 =$ non-zero constant means that $\varphi =$ constant such that $0 < \varphi < 180°$. These are rectilinear and curvilinear translations in the Euclidean plane where the moving frame E maintains a constant non-zero angle with respect to the fixed frame Σ.

5.12 Kinematic Constraints

For planar displacements there are only two lower pairs: R- and P-pairs. This means there are only three practical planar dyads in a 4-bar linkage:

$$RR,\ PR,\ \text{and}\ RP$$

These 3-link chains are designated according to the type of joints connecting the links, and listed in series starting with the joint connected to the ground, each illustrated in Figure 5.16. When a pair of dyads are joined, a 4-bar linkage is obtained. However, the designation of the *output* dyad changes. For example, consider a planar 4-bar linkage composed of an RR-dyad on the *left-hand side* of the mechanism, and a PR-dyad on the *right-hand side*, where the *input* link is the *grounded* link in the RR-dyad.

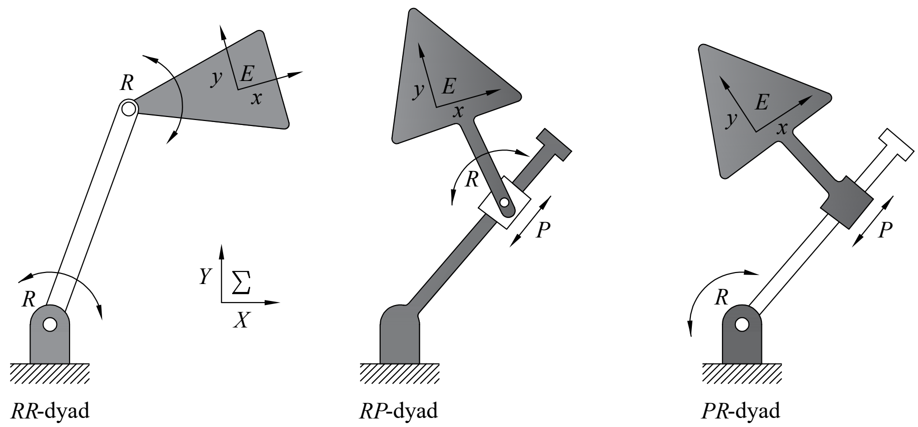


Figure 5.16: Types of dyads.

If R_1 is actuated by some form of torque supplied by an electric rotary motor transferred by a transmission in turn driving the input link, l_1. The linkage is designated by listing the joints in sequence from the ground fixed actuated joint, starting with the input link listing the joints in order. Thus, the mechanism composed of a driving RR-dyad, and an output PR-dyad is called an $RRRP$ linkage, see Figure 5.17, where the order of PR is switched to RP. If the output were an RP-dyad, the mechanism would be an $RRPR$ linkage. If the input were an RP-dyad while the output was an RR-dyad, the resulting mechanism would be an $RPRR$ linkage, with no noticeable alteration in the name.

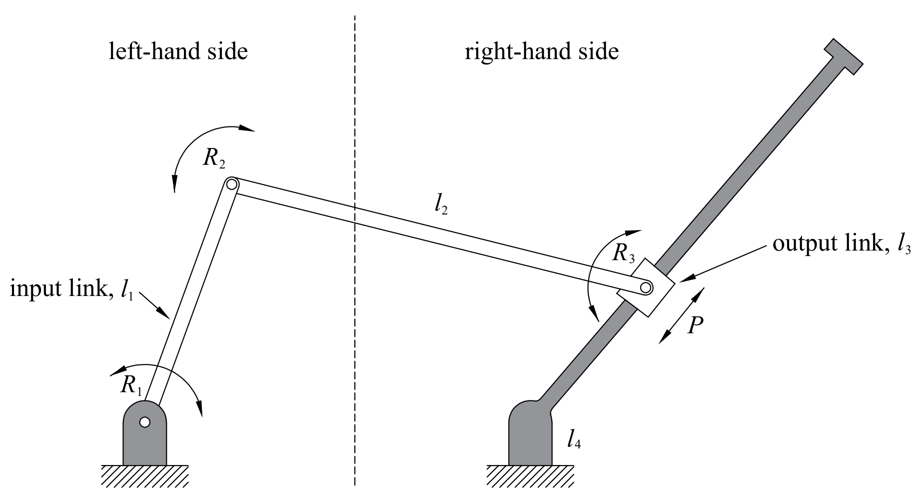


Figure 5.17: A four bar linkage with RR-dyad on the left-hand side and PR-dyad on the right-hand side.

Both synthesis and analysis require a geometric model of the mechanical system. To use kinematic mapping, we need a model that describes the displacement of moving frame E with respect to frame Σ.

A very nice feature of kinematic mapping is that the resulting transformation matrix form is independent of choice of E and Σ. Rather, the transformation converts coordinates of points given in E to those of the same point described in Σ.

The motion of the distal link in a dyad is two parameter motion. For instance, l_2 can rotate about the revolute center R_2, while R_2 is free to rotate about R_1 (assuming, for now, R_1 is conceptually passive).

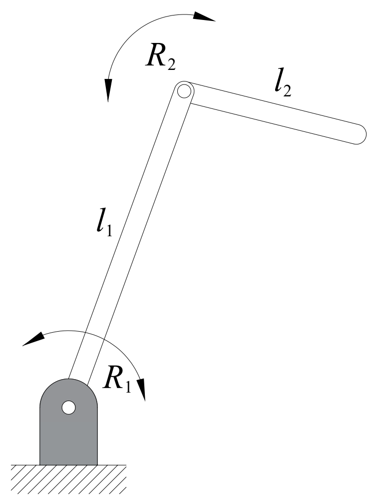


Figure 5.18: The motion of the distal link in a dyad is two parameter motion.

If we are going to use kinematic mapping, a good question to ask is "What does the motion, a continuum of constrained displacements, look like in the image space?" First let's assign coordinate systems E and Σ. This assignment is arbitrary, but we want to have the fewest terms in our resulting equations. So we place them as shown in Figure 5.19 for convenience. Notice that the origin of E, O_E, which has constant coordinates in E, is constrained to move on a circle centered at O_Σ with radius $r = l_1$. The transformation characterizing the displacement of E WRT Σ is a function of a, b, φ. To transform a point (x, y, z) in E to (X, Y, Z) in Σ, we write:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2X_4^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

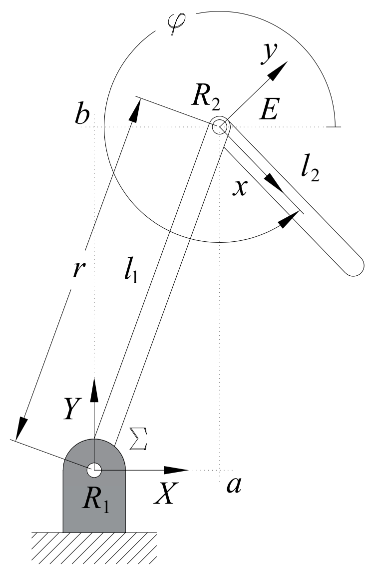


Figure 5.19: Assignment of coordinate systems E and Σ.

Expanding, we get:

$$\begin{aligned} X &= (X_4^2 - X_3^2)x - 2X_3X_4y + 2(X_1X_3 + X_2X_4)z, \\ Y &= 2X_3X_4x + (X_4^2 - X_3^2)y + 2(X_2X_3 - X_1X_4)z, \\ Z &= (X_3^2 + X_4^2)z. \end{aligned} \tag{5.50}$$

We can characterize the constraints imposed by the RR-dyad by observing that one point on l_2 is constrained to move on a circle, centered at O_Σ, with radius $r = l_1$.

The general equation of a circle in E_2 is:

$$(X - X_c)^2 + (Y - Y_c)^2 - r^2 = 0.$$

Set $X = \dfrac{X}{Z}$ and $Y = \dfrac{Y}{Z}$ and substitute:

$$\left(\frac{X}{Z} - X_c\right)^2 + \left(\frac{Y}{Z} - Y_c\right)^2 - r^2 = 0$$

Expand to get:

$$\left(\frac{X}{Z}\right)^2 - 2\left(\frac{X}{Z}\right)X_c + X_c^2 + \left(\frac{Y}{Z}\right)^2 - 2\left(\frac{Y}{Z}\right)Y_c + Y_c^2 - r^2 = 0.$$

Multiply by Z^2 to get:

$$X^2 + Y^2 - 2X_cXZ - 2Y_cYZ + X_c^2 + Y_c^2 - r^2Z^2 = 0. \tag{5.51}$$

Let

$$\begin{aligned}
k_0 &= \text{arbitrary non-zero real number,} \\
k_1 &= -X_c, \\
k_2 &= -Y_c, \\
k_3 &= X_c^2 + Y_c^2 - r^2 \;=\; k_1^2 + k_2^2 - r^2,
\end{aligned}$$

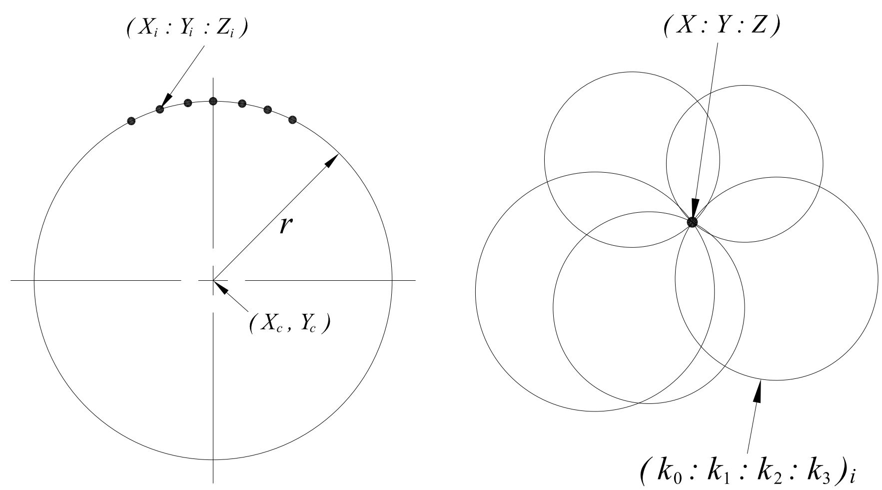
and substitute these into Equation (5.51):

$$k_0(X^2 + Y^2) + 2k_1XZ + 2k_2YZ + k_3Z^2 \quad = \quad 0. \tag{5.52}$$

When $k_0 = 0$, Equation (5.52) represents a line. When $k_0 = 1$, Equation (5.52) represents a circle. Moreover, when $k_0 = 1$, Equation (5.52) represents exactly the constrained motion of the point on l_2 that is forced to move on a circle of radius $r = l_1$, centered at (X_c, Y_c), which is in this particular case O_Σ.

$(X : Y : Z)$ are the homogenous coordinates of points on the circumference of a circle $k_0 = 1$ or a line $k_0 = 0$. These coordinates are expressed in Σ.

But, the constraint expressed by Equation (5.52) describes exactly that of an RR-dyad: a point with fixed point coordinates in frame E is constrained to move on a circle, with fixed circle coordinates in frame Σ. The homogenous point coordinates are $(X : Y : Z)$, while $(k_0 : k_1 : k_2 : k_3)$ are the homogenous circle coordinates in this dual expression.



(a) Family of all *points* on a fixed *circle*. (b) Family of all *circles* on a fixed *point*.

Figure 5.20: Duality in projective geometry.

Equation (5.52) is a projective dualistic expression that means the following.

1. The family of all points on the circumference of a fixed circle (fixed center and constant radius), see Figure 5.20a.

2. The family of all circles on a fixed point, see Figure 5.20b.

But, the RR-displacement constraint involved a fixed point in E that moves on a fixed circle in Σ. So, we transform the coordinates of the point from E to Σ.

To exploit the quasi-elliptic properties of the kinematic mapping image space, we can transform the coordinates using Equation (5.51). The kinematic constraint in terms of the displacement coordinates is obtained by substituting:

$$\begin{aligned} X &= (X_4^2 - X_3^2)x - 2X_3X_4y + 2(X_1X_3 + X_2X_4)z, \\ Y &= 2X_3X_4x + (X_4^2 - X_3^2)y + 2(X_2X_3 - X_1X_4)z, \\ Z &= (X_3^2 + X_4^2)z. \end{aligned}$$

into the *circle constraint* Equation (5.52). But before we look at the constraint equation in the image space, let's establish the constraints associated with PR- and RP-dyads .

The circular constraints are only created by planar RR-dyads and are concisely summarised in the following definition.

RR**-Dyad:** A point in E with fixed point coordinates, $(x : y : z)$ is constrained to move on a circle with fixed circle coordinates $(k_0 : k_1 : k_2 : k_3)$ in Σ.

Whereas PR- and RP-dyads generate linear constraints described as follows.

PR**-Dyad:** A point with fixed *point* coordinates $(x : y : z)$ in E is constrained to move on a *line* with fixed *line* coordinates $(k_0 : k_1 : k_2 : k_3)$ in Σ

RP**-Dyad:** A *line* with fixed *line* coordinates $(k_0 : k_1 : k_2 : k_3)$ in E is constrained to move on a *point* with fixed *point* coordinates $(x : y : z)$ in Σ.

For circular constraints $k_0 = 1$, and for linear constraints $k_0 = 0$. However, leaving k_0 unspecified will allow us to derive a single relation expressing all three constraints in terms of coordinates in the image space.

Note that the RP- and PR-dyad constraints are not only duals, but the roles of E and Σ are, in a sense, inverted. That is, if we consider E to be fixed and Σ to be moving for the PR-dyad, then PR-dyad and RP-dyad constraints are identical. This suggests using the inverse transform to obtain the coefficients of the constraint equation in the image space. If

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{T}^{-1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

Because **T** represents the group of Euclidean planar displacements, it is always invertible. Expanding both, we get:

$$\begin{aligned} X &= (X_4^2 - X_3^2)x - 2X_3X_4y + 2(X_1X_3 + X_2X_4)z, \\ Y &= 2X_3X_4x + (X_4^2 - X_3^2)y + 2(X_2X_3 - X_1X_4)z, \\ Z &= (X_3^2 + X_4^2)z, \end{aligned}$$

and

$$\begin{aligned} x &= (X_4^2 - X_3^2)X - 2X_3X_4Y + 2(X_1X_3 + X_2X_4)Z, \\ y &= 2X_3X_4X + (X_4^2 - X_3^2)Y + 2(X_2X_3 - X_1X_4)Z, \\ z &= (X_3^2 + X_4^2)Z. \end{aligned}$$

Substituting both sets of relations into the general circular constraint Equation (5.52) we get an equation that splits into two factors:

$$\begin{aligned} &\frac{1}{4}\left[X_3^2 + X_4^2\right]\big[k_0z^2(X_1^2 + X_2^2) + (-k_0x + k_1z)zX_1X_3 + (-k_0y + k_2z)zX_2X_3 \\ &\mp (k_0y + k_2z)zX_1X_4 \pm (k_0x + k_1z)zX_2X_4 \mp (k_1y - k_2x)zX_3X_4 \\ &+ \frac{1}{4}(k_0(x^2 + y^2) - 2z(k_1x + k_2y) + k_3z^2)X_3^2 \\ &+ \frac{1}{4}(k_0(x^2 + y^2) + 2z(k_1x + k_2y) + k_3z^2)X_4^2\big] = 0 \end{aligned}$$

The first factor is exactly $\frac{1}{4}$ times the non-zero condition, which must be satisfied for a point to be the image of a real displacement. This factor must be non-zero and may be safely *factored out*. What remains is a general second-order equation in the X_i.

The general quadric is the geometric image of the kinematic constraint. If the kinematic constraint is a fixed point in reference frame E bound to a circle ($k_0 = 1$) or a line ($k_0 = 0$) with fixed coordinates in Σ, then $(x : y : z)$ are the coordinates of the fixed point in E, and the upper signs apply. If the kinematic constraint is a fixed point in Σ bound to move on a circle ($k_0 = 1$) or line ($k_0 = 0$) with fixed coordinates in E, then (X:Y:Z) are the coordinates of the fixed reference point in Σ, which are substituted for $(x : y : z)$. Moreover, the lower signs apply.

The second factor can be simplified under the following assumptions:

1. No mechanism of practical significance will have a point at infinity, so we can set $z = 1$.

2. Half-turns (rotations of $\varphi = \pi$) have images in the plane $X_4 = 0$. Because the X_i are implicitly defined in terms of the pole of the displacement, when $\varphi = \pm\pi$ we get

$$(X_1 : X_2 : X_3 : X_4) = (a : b : \pm 2 : 0).$$

   When we remove the two parameter family of image points for orientations of $\pm\pi$, we can, for convenience, normalize the image space coordinates by setting $X_4 = 1$

However, let's be careful. We start with four homogenous coordinates in the 3D projective image space defined by:

$$\begin{aligned} X_1 &= a\sin(\varphi/2) - b\cos(\varphi/2), \\ X_2 &= a\cos(\varphi/2) + b\sin(\varphi/2), \\ X_3 &= 2\sin(\varphi/2), \\ X_4 &= 2\cos(\varphi/2). \end{aligned}$$

Setting $X_4 = 1$ means dividing all four coordinates by $2\cos(\varphi/2)$ (which is now possible because $\varphi = \pi$ has been removed). We now get:

$$\begin{aligned} X_1 &= \frac{1}{2}(\arctan(\varphi/2) - b), \\ X_2 &= \frac{1}{2}(a + b\tan(\varphi/2)), \\ X_3 &= \tan(\varphi/2), \\ X_4 &= 1. \end{aligned}$$

Applying all these assumptions gives the following second-order surface:

$$\begin{aligned} & k_0(X_1^2 + X_2^2) + (k_1 - k_0 x)X_1X_3 + (k_2 - k_0 y)X_2X_3 \\ & \mp(k_0 y + k_2)X_1 \pm (k_0 x + k_1)X_2 \mp (k_1 y - k_2 x)X_3 \\ & +\frac{1}{4}[k_0(x^2 + y^2) - 2(k_1 x + k_2 y) + k_3]X_3^2 \\ & +\frac{1}{4}[k_0(x^2 + y^2) + 2(k_1 x + k_2 y) + k_3] = 0. \end{aligned} \tag{5.53}$$

Equation (5.53) is a general expression that represents the kinematic constraints in the kinematic mapping image space (projected onto the hyperplane $X_4 = 1$) for RR, PR, and RP dyads. It can be shown that when $k_0 = 0$ the resulting equation is an hyperbolic paraboloid, otherwise the surface is an hyperboloid of one sheet. When $k_0 = 1$, the constraint surface corresponds to an RR-dyad.

5.13 RR-Dyad Constraint Surface

When we set $k_0 = 1$ and complete the squares in X_1 and X_2 in Equation (5.53) we get, after some algebra:

$$\begin{aligned} & \left[X_1 - \frac{1}{2}((x - k_1)X_3 + y + k_2)\right]^2 + \left[X_2 - \frac{1}{2}((y - k_2)X_3 - x - k_1)\right]^2 \\ = \; & \frac{1}{4}[(k_1^2 + k_2^2 - k_3)(X_3^2 + 1)]. \end{aligned} \tag{5.54}$$

In planes where $X_3 = \textit{constant}$, Equation (5.54) represents a circle of the form:

$$(X_1 - X_c)^2 + (X_2 - Y_c)^2 = R^2,$$

where:

$$
\begin{aligned}
X_c &= \frac{1}{2}((x-k_1)X_3 + y + k_2), \\
Y_c &= \frac{1}{2}((y-k_2)X_3 - x - k_1), \\
R &= \sqrt{\frac{1}{4}(k_1^2 + k_2^2 - k_3)(X_3^2 + 1)}.
\end{aligned}
$$

As X_3 is varied from $-\infty$ to ∞, the locus of circle centers is a line. Setting $X_3 = t$, the parametric equation of the locus of circle centers is:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} y + k_2 \\ -x - k_1 \\ 0 \end{bmatrix} + \frac{t}{2}\begin{bmatrix} x - k_1 \\ y - k_2 \\ 2 \end{bmatrix}.
$$

The quadric surface is a family of generally non-concentric circles whose center points are all collinear. By Equation (5.54) we see that the smallest circle in this family occurs in the hyper-plane $X_3 = 0$, and has radius:

$$
R_{min} = \sqrt{\frac{1}{4}[k_1^2 + k_2^2 - k_3]}.
$$

As X_3 increases in value the circles become larger, regardless of the sign of X_3. Thus the quadric surface extends to infinity in two directions.

There are only nine distinct types of quadric surfaces [3]:

1. spheres;
2. ellipsoids;
3. parabolic cylinders;
4. hyperbolic cylinders;
5. elliptic cylinders;
6. cones;
7. hyperboloid of one-sheet;
8. hyperboloid of two sheets;
9. hyperbolic paraboloids.

Spheres and ellipsoids contain circles, but are finite. Parabolic cylinders extend to infinity, but in only one direction, and no real plane intersections contain circles. Hyperbolic cylinders extend to infinity in two directions, but no real plane intersections contain circles. Elliptic cylinders contain real circles, but all have the same diameter. Cones also contain circles, but all contain one with vanishing diameter. Plane intersections with hyperbolic paraboloids are

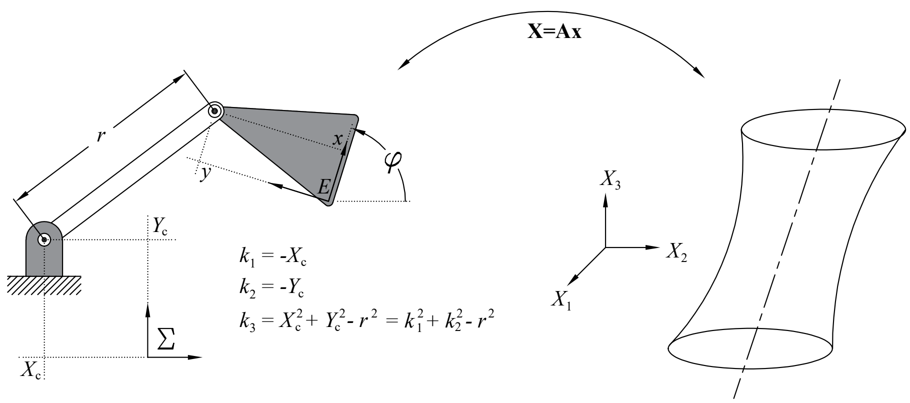


Figure 5.21: Graphical representation of how dyad constraints map to a constraint surface in the image space.

parabolas, hyperbolas, or two lines, but never circles. Hyperboloids of two sheets are not continuous. By process of elimination, the *RR*-dyad constraint surface must be an hyperboloid of one sheet.

The locus of circle centers is not generally perpendicular to the plane containing a circle. Thus, it is generally not a surface of revolution. However, the hyperboloid always intersects the planes parallel to $X_3 = constant$ in circles. Hence, it is simple to write a parametric equation for the constraint hyperboloid projected into the hyperplane $X_4 = 1$. All we have to do is write the parametric equation for the hyperboloid circle in any plane $X_3 = t$. Any point P on the

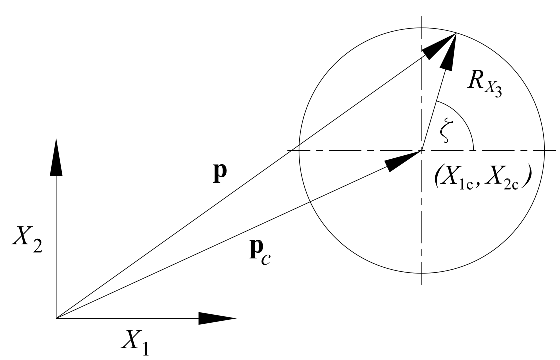


Figure 5.22: An arbitrary hyperboloid circle.

circle with radius R_{x_3} centered at P_c is simply:

$$\mathbf{p} = \mathbf{p}_c + \mathbf{R}_{x_3}(\xi).$$

Recall $R_{x_3} = \sqrt{\frac{1}{4}(k_1^2 + k_2^2 - k_3)(X_3^2 + 1)}$. Substitute $k_3 = k_1^2 + k_2^2 - r^2$ and

$X_3 = t$ to get:

$$R_{x_3} = \frac{1}{2} r \sqrt{t^2+1},$$

where r is the radius of the circle that the point with fixed coordinates in E is constrained to move on. R_{x_3} is the hyperboloid circle radius in the plane $X_3 = t$. Therefore the parametric equation of the image space constraint hyperboloid for an RR-dyad is:

$$\begin{aligned} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} &= (1/2) \begin{bmatrix} ((x-k_1)t + y + k_2) + (r\sqrt{t^2+1})\cos\xi \\ ((y-k_2)t - x - k_1) + (r\sqrt{t^2+1})\sin\xi \\ 2t \end{bmatrix}, \\ t &\ \epsilon\ \{-\infty, \cdots, \infty\}, \\ \xi &\ \epsilon\ \{0, \cdots, 2\pi\}. \end{aligned}$$

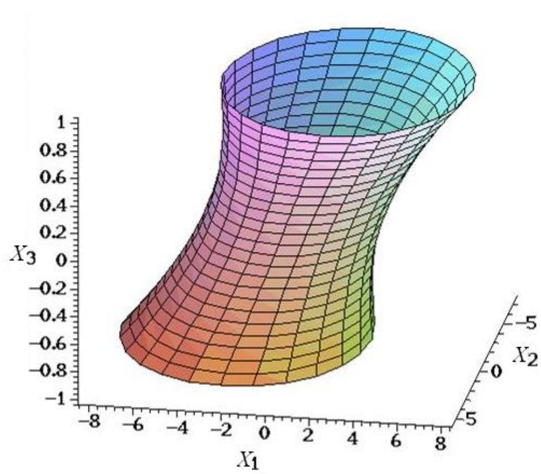


Figure 5.23: A projection of a constraint hyperboloid of one sheet in the hyperplane $X_4 = 1$.

5.14 PR-and RP-Dyad Constraint Surface

A fundamentally different constraint surface corresponds to the dual PR- and RP-dyad motion constraints. Recall the verbal description.

PR-Dyad: A point with fixed point coordinates in E moves on a line with fixed line coordinates in Σ.

RP-Dyad: A line with fixed line coordinates in E moves on a point with fixed point coordinates in Σ.

These conditions require the k_i to be proportional to planar line coordinates. Thus, we set $k_0 = 0$ in the general homogenous circle equation

$$k_0(X^2 + Y^2) + 2k_1XZ + 2k_2YZ + k_3Z^2 = 0.$$

This leaves the following quadratic:

$$Z(2k_1X + 2k_2Y + k_3Z) = 0. \tag{5.55}$$

The general quadratic equation splits into the two linear factors given in Equation (5.55): the line at infinity $Z = 0$, and the line involved in the PR- and RP-dyad constraints.

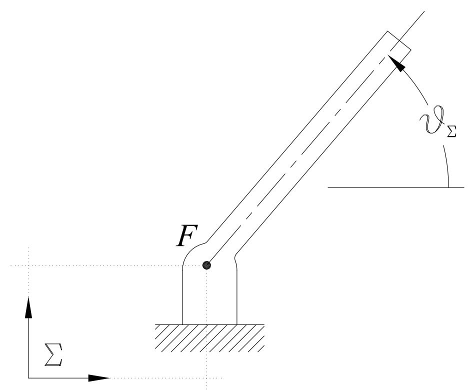


Figure 5.24: Any line in the plane can be characterized by its Grassmann line coordinates.

We can safely factor out $Z = 0$, because only finite lines need be considered for practical designs. The k_i are related to the Grassmann line coordinates by:

$$[k_1 : k_2 : k_3] = [\tfrac{1}{2}L_1 : \tfrac{1}{2}L_2 : L_3].$$

The k_i are determined by expansion of the determinant of two known points on the line. Two points on the line are $(F_{X/Z} : F_{Y/Z} : F_{Z/Z})$ and the point at infinity given by the direction of the line in Σ $\cos(\vartheta_E) : \sin(\vartheta_E) : 0)$ Because $F_{Z/\Sigma}$ is an arbitrary scaling factor, we can set it equal to 1. The Grassmann line coordinates are:

$$\begin{aligned}\begin{bmatrix} X & Y & Z \\ F_{X/Z} & F_{Y/Z} & 1 \\ \cos(\vartheta_\Sigma) & \sin(\vartheta_\Sigma) & 0 \end{bmatrix} &= L_1X + L_2Y + L_3Z, \\ &= \sin(\vartheta_E)X + \cos(\vartheta_E)Y + \\ &\quad \big(F_{(X/Z)}\sin(\vartheta_E) - F_{(Y/Z)}\cos(\vartheta_E)\big)\,Z.\end{aligned}$$

To use these line coordinates in the line equation derived from the circle equation we must divide L_1 and L_2 by 2, giving:

$$\begin{aligned}[k_1 : k_2 : k_3] &= [\tfrac{1}{2}L_1 : \tfrac{1}{2}L_2 : L_3], \\ &= \left[-\frac{1}{2}\sin(\vartheta_E) : \frac{1}{2}\cos(\vartheta_E) : \left(F_{(X/Z)}\sin(\vartheta_E) - F_{(Y/Z)}\cos(\vartheta_E)\right)\right].\end{aligned}$$

Substituting $k_0 = 0$ in Equation (5.53), the general constraint surface after setting $X_4 = Z = 1$ gives:

$$\begin{aligned}&(k_1X_3 \pm k_2)X_1 + (k_2X_3 \pm k_1)X_2 + \frac{1}{4}(k_3 - 2(k_1x + k_2y))X_3^2 \\ &\pm(k_2x - k_1y)X_3 + \frac{1}{4}(k_3 + 2(k_1x + k_2y)) = 0. \qquad (5.56)\end{aligned}$$

For PR-dyads use the upper signs. For RP-dyads the derivation is identical, except use use lower signs and substitute (X, Y) for (x, y).

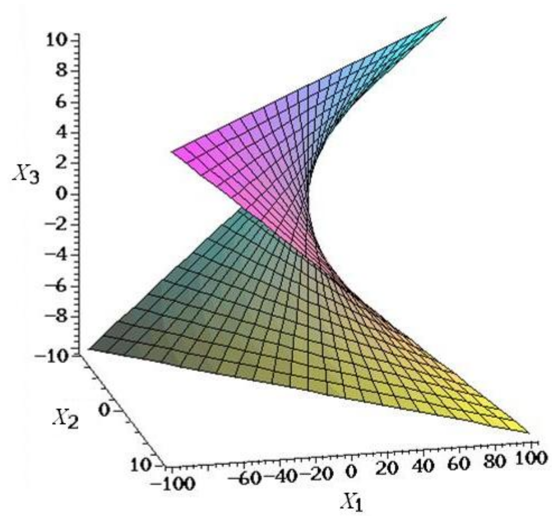


Figure 5.25: A projection of a constraint hyperbolic paraboloid in the hyperplane $X_4 = 1$.

Equation (5.56) is quadratic in the X_i, but very different in form from Equation (5.54). To compare them we investigate what happens when Equation (5.56) intersects planes parallel to $X_3 = constant$. We get:

$$a(X_3)X_1 + b(X_3)X_2 + c(X_3) = 0.$$

As X_3 is varied from $-\infty$ to ∞ we get a family of mutually skew lines, all parallel to a plane ($X_3 = constant$), but not to each other. The PR- and RP-dyad constraint surface is therefore an hyperbolic paraboloid. It is a little more involved to parameterize this surface compared to the RR-dyad hyperboloid of one sheet, but it is still easily done. Like the hyperboloid of one sheet, the hyperbolic paraboloid is a *doubly-ruled* surface. That means two distinct families of lines lie evenly on the surface.

Each family of lines is called a regulus, R, see Figure 5.26. All the lines in one regulus are mutually skew (no line in the regulus intersects any other line in the same regulus), and are all parallel to a plane. The lines in one regulus are parallel to $X_3 = 0$. The lines in the *opposite* regulus are all parallel to some other plane. But each line in one regulus intersects every line in the opposite regulus. Thus, each line in one regulus is a *directrix* for the opposite regulus. Let's define $\mathcal{L}_0$ to be the line of the constraint hyperbolic paraboloid in the

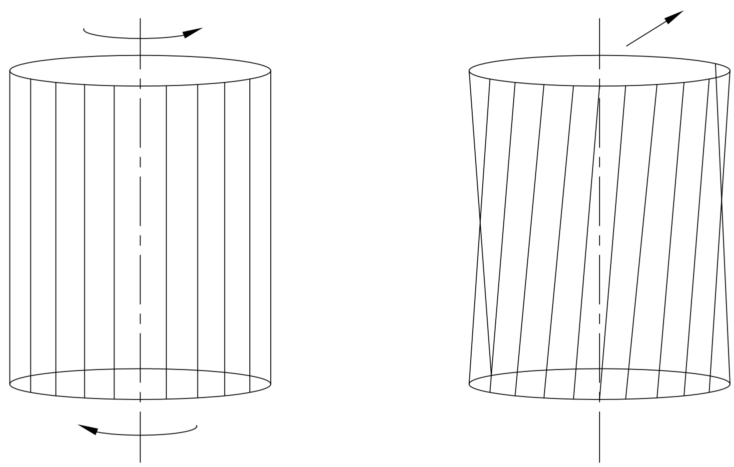


Figure 5.26: Regulus of a ruled surface.

plane $X_3 = 0$, see Figure 5.27. The regulus that $\mathcal{L}_0$ is a member of is R_0. Now, consider the plane π that also contains $\mathcal{L}_0$, but is perpendicular to $X_3 = 0$

The $X_3 - axis$ is parallel to π. There is one, and only one line $\mathcal{L}$ contained in the other regulus that intersects plane π. Because $\mathcal{L}_0$ and $\mathcal{L}$ are in opposite regulii they intersect. Since $\mathcal{L}$ intersects every line in the regulus R_0, each distinct point on $\mathcal{L}$ represents a distinct intersection of a distinct line $\mathcal{L}_i$ in R_0. We will use $\mathcal{L}$ as the directrix to generate the lines in R_0.

We start by observing that the locus of point on $\mathcal{L}$ is a function of $X_3 = t$. A general line in space can be represented parametrically by a fixed point on the line and a direction. For each i^{th} value of t, there is a unique point on directrix $\mathcal{L}$, which is the point of intersection with $\mathcal{L}_i \epsilon R_0$.

The direction of $\mathcal{L}_i$ is also a function of $X_3 = t$, since every line in R_0 must be parallel to $X_3 = t$. Stepping in the direction of the i^{th} line in R_0, $\mathcal{L}_i$, by varying a second linear parameter s yields the locus of points on $\mathcal{L}_i$:

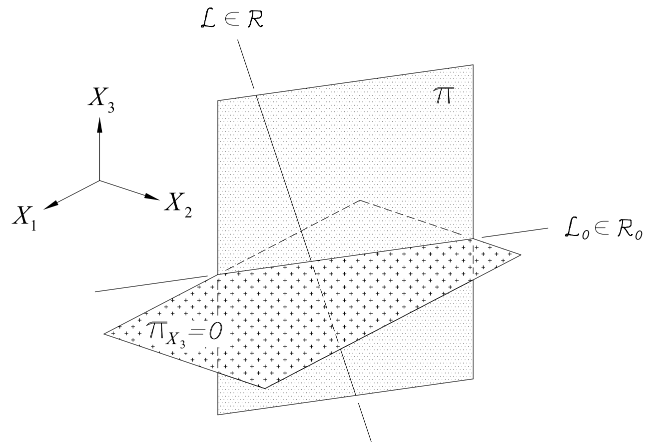


Figure 5.27: Constructing an hyperbolic paraboloid.

$$\mathcal{L}_i = \begin{bmatrix} f(t_i) \\ g(t_i) \\ t_i \end{bmatrix} + s \begin{bmatrix} a(t_i) \\ b(t_i) \\ 0 \end{bmatrix}.$$

This collection of lines is the regulus R_0. It is clearly quadratic by virtue of the second order bi-linear terms $sa(t_i)$ and $sb(t_i)$. Deriving the functions $f(t)$, $g(t)$, $a(t)$, $b(t)$ will yield a parametrization of the general constraint (PR-dyad) hyperbolic paraboloid.

An important feature of the parametrization is that it is free from representational singularities. Some components of the parametric equation usually involve ratios. The denominators must be free from dependence on the parameters. Using the proposed directrix gives a parametrization that can always be drawn since there can be no divide-by-zero conditions generated by the need to divide by $t = 0$ or $s = 0$.

5.15 Parametrization Steps

Step 1

Determine plane π perpendicular to $X_3 = 0$. We obtain the equation of $\mathcal{L}_\infty$ by setting $X_3 = 0$ in Equation (5.56):

$$\mathcal{L}_0 : -k_2X_1 + k_1X_2 + \frac{1}{4}(k_3 + 2[k_1x + k_2y]) = 0. \tag{5.57}$$

The line $\mathcal{L}_0$ is the axis of a pencil of planes. It is the line of intersection of the plane $X_3 = 0$ and the desired perpendicular plane π. Thus, we can solve

Equation (5.57) for either X_1 or X_2, and allowing X_3 and X_1 or X_2 to vary as well. Solving we get:

$$\pi = \begin{cases} X_1 = \dfrac{1}{k_2}\left(k_1 X_2 + \dfrac{1}{4}\left[k_3 + 2(k_1 x + k_2 y)\right]\right) \\ X_2 = X_2 \\ X_3 = X_3, \end{cases} \tag{5.58}$$

and

$$\pi = \begin{cases} X_1 = X_1 \\ X_2 = \dfrac{1}{k_1}\left(-k_2 X_1 + \dfrac{1}{4}[k_3 + 2(k_1 x + k_2 y)]\right) \\ X_3 = X_3 \end{cases} \tag{5.59}$$

Note, k_1 and k_2 cannot simultaneously vanish because they are proportional to real line coordinates. i.e. if $k_1 = 0 \Rightarrow\ k_2 \neq 0$. Similarly if $k_2 = 0 \Rightarrow\ k_1 \neq 0$.

Either representation for π may be used leading to identical results. Without loss in generality we may assume k_2 is sufficiently large for symbolic derivation.

Equations (5.58) mean that any point $[X_1 : X_2 : X_3 : 1] \;\epsilon\; \pi$ is given by choosing values for X_2 and X_3. X_1 is then a function of X_2. Since X_3 varies linearly, π is perpendicular to the plane $X_3 = 0$, and it contains $\mathcal{L}_0$ so it is the plane we wanted.

Step 2

Find an expression for $\mathcal{L} \;\epsilon\; R$. i.e. determine the line equation for the directrix, a single line in π. It is the unique line in R and π that intersect every line in R_0. The observation that $\mathcal{L}$ is the line of intersection of π and R makes this easy. Substitute Equation (5.58) into the implicit equation of the constraint hyperboloid, Equation (5.56). We get:

$$\frac{X_3}{4k_2}\big(4[k_1^2 + k_2^2]X_2 + [k_2 k_3 - 2(k_2^2 y + k_1 k_2 x)]X_3 + 2[k_1^2 + 2k_2^2]x - 2k_1 k_2 y + k_1 k_3\big) = 0,$$

assuming k_2 large enough. The two factors appear to be the plane $X_3 = 0$, and the desired line:

$$\mathcal{L}:\ 4[k_1^2 + k_2^2]X_2 + [k_2 k_3 - 2(k_2^2 y + k_1 k_2 x)]X_3 + 2[k_1^2 + 2k_2^2]x - 2k_1 k_2 y + k_1 k_3 = 0.$$

This appears to contradict the theorem that a plane must intersect a quadric surface in a curve of order two. In our case, this conic section (a planar curve of order two) should degenerate into two lines, namely $\mathcal{L}_0$ and $\mathcal{L}$. Well... in fact, it does. The factor $X_3 = 0$ is an artifact of representation. Recall that the plane $X_3 = 0$ contains $\mathcal{L}_0$, and no other line in R_0. The second factor *must* be an expression of $\mathcal{L}$, since it is a line contained in the intersection of plane π that is *not* $\mathcal{L}_0$.

Solve the equation of $\mathcal{L}$ for X_2 and set $X_3 = t$, giving the X_2 coordinate of the locus of points on $\mathcal{L} \;\epsilon\; R$:

$$g(t) = \frac{[(2(k_1 k_2 x + k_2^2 y) - k_2 k_3)t - 2(k_1^2 + 2k_2^2)x + 2k_1 k_2 y - k_1 k_3]}{4(k_1^2 + k_2^2)}.$$

The X_1 coordinate is obtained by substituting the expression for $X_2 = g(t)$ into the first of equations (5.58) (the equation for plane π), which yields another function of only t:

$$f(t) = \frac{[(2(k_1k_2 + k_1^2x) - k_1k_3)t + 2(2k_1^2 + k_2^2)y + 2k_1k_2x + k_2k_3]}{4(k_1^2 + k_2^2)}.$$

This gives the parametric equation for $\mathcal{L} = [f(t), g(t), t]^T$.

Note that the sum of squares $k_1^2 + k_2^2$ never vanishes. Now we require direction vectors for the $\mathcal{L}_i \epsilon R_0$. We can re-write Equation (5.56), the hyperbolic paraboloid constraint surface as:

$$aX_1 + bX_2 + cX_3^2 + dX_3 + e \quad = \quad 0,$$

where a and b are both functions of $X_3 = t$:

$$\begin{aligned} a(t) &= k_1t - k_2, \\ b(t) &= k_1 + k_2t. \end{aligned}$$

In an arbitrary plane $X_3 = t$, the direction of the corresponding line in R_0 is given by the coefficient ratio $-b/a$, i.e., the slope of the line in the given plane. In other words, the line $\mathcal{L}_i$ is parallel to the direction given by

$$a(t)X_1 + b(t)X_2 = 0.$$

Non trivial solutions require:

$$a(t) = X_2,\ b(t) = -X_1,$$

or

$$a(t) = -X_2,\ b(t) = X_1.$$

These are equivalent conditions because the linear sum vanishes. Thus, the locus of points on a line in the direction of $\mathcal{L}_i$ is:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = s \begin{bmatrix} -b(t) \\ a(t) \\ 0 \end{bmatrix} \quad = \quad \begin{bmatrix} -k_1 - k_2t \\ k_1t - k_2 \\ 0 \end{bmatrix}.$$

Combining the points on the directrix $\mathcal{L}$ with the loci of points in the directions of $\mathcal{L}_i$ in the opposite regulus R_0 gives the desired representational singularity-free parametric equation:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad = \quad s \begin{bmatrix} f(t) \\ g(t) \\ t \end{bmatrix} + \begin{bmatrix} -k_1 - k_2t \\ k_1t - k_2 \\ 0 \end{bmatrix}.$$

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