Chapter 6

Analytic Projective Geometry

6.1 Line and Point Coordinates: Duality in P_2

Consider a line in E_2 defined by the equation

$$X_0 + X_1 x + X_2 y = 0. \tag{6.1}$$

The locus that the variable points (x, y) trace out are characterised by the constants X_i. The equation contains two bilinear terms and one linear term. If we use homogeneous coordinates, we get

$$X_0 + X_1 \left(\frac{x_1}{x_0}\right) + X_2 \left(\frac{x_2}{x_0}\right) = 0,$$

or

$$X_0 x_0 + X_1 x_1 + X_2 x_2 = \sum_{i=0}^{2} X_i x_i = 0. \tag{6.2}$$

Equation (6.2) is now *homogeneous*, that is, all three terms are now bilinear. The equation is symmetric in both the X's and the x's and has a much more pleasing form than the original in Equation (6.1). Because of the duality in the projective plane P_2 we may consider this the equation of a line, or of a point. The numbers x_0, x_1, x_2 are called the coordinates of the point $(x_0 : x_1 : x_2)$. The numbers X_0, X_1, X_2 are the coordinates of the line $[X_0 : X_1 : X_2]$. There are two cases to consider.

1. **Line equation, line coordinates.** When $(x_0 : x_1 : x_2)$ is a variable point on a fixed line with coordinates $[X_0 : X_1 : X_2]$, then $\sum_{i=0}^{2} X_i x_i = 0$ is a line equation generating a range of points.

2. Point equation, point coordinates. When $[X_0 : X_1 : X_2]$ is a variable line on a fixed point with coordinates $(x_0 : x_1 : x_2)$, then $\sum_{i=0}^{2} X_i x_i = 0$ is a point equation generating a pencil of lines.

6.1.1 Collinear Points and Concurrent Lines

1. The line determined by the *distinct points* $(x_0 : x_1 : x_2)$ and $(y_0 : y_1 : y_2)$ yields two equations

$$\left.\begin{array}{rcl} X_0x_0 + X_1x_1 + X_2x_2 & = & 0, \\ X_0y_0 + X_1y_1 + X_2y_2 & = & 0, \end{array}\right\} \tag{6.3}$$

and three unknowns: the line coordinates $[X_0 : X_1 : X_2]$. Since $[0 : 0 : 0]$ is not in the set, we can always consider the ratios (X_1/X_0) and (X_2/X_0). This suggests we divide each equation by X_0 and then use Cramer's rule to solve for the two ratios.

We recall Cramer's rule here for convenience. If $\mathbf{Ax} = \mathbf{b}$ is a system of n linear equations in n unknowns such that the determinant $|\mathbf{A}| \neq 0$, then the system has a unique solution. This solution is

$$x_0 = \frac{|\mathbf{A_0}|}{|\mathbf{A}|}, \quad x_1 = \frac{|\mathbf{A_1}|}{|\mathbf{A}|}, \quad x_2 = \frac{|\mathbf{A_2}|}{|\mathbf{A}|}, \quad \cdots \quad x_n = \frac{|\mathbf{A_n}|}{|\mathbf{A}|}, \tag{6.4}$$

where $\mathbf{A}_i$ is the matrix obtained by replacing the elements in the i^{th} column of $\mathbf{A}$ by the elements in the vector

$$\mathbf{b} \quad = \quad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Hence, we rewrite Equations (6.3) as

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} X_1/X_0 \\ X_2/X_0 \end{bmatrix} \quad = \quad \begin{bmatrix} -x_0 \\ -y_0 \end{bmatrix}.$$

Now, using Cramer's rule we find

$$\frac{X_1}{X_0} = \frac{\begin{vmatrix} -x_0 & x_2 \\ -y_0 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}}, \quad \frac{X_2}{X_0} = \frac{\begin{vmatrix} x_1 & -x_0 \\ y_1 & -y_0 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}}.$$

Since switching two columns in a determinant will change its sign the solutions could be written as

$$\frac{X_1}{X_0} = \frac{\begin{vmatrix} x_2 & x_0 \\ y_2 & y_0 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}}, \quad \frac{X_2}{X_0} = \frac{\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}}.$$

Thus, the homogeneous coordinates of the line are

$$[X_0 : X_1 : X_2] = \left[\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} : \begin{vmatrix} x_2 & x_0 \\ y_2 & y_0 \end{vmatrix} : \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} \right], \tag{6.5}$$

assuming not all these determinants are zero. Since $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$ if and only if it's rows are proportional, we can deduce that if all the determinants were zero then the point coordinates $(x_0 : x_1 : x_2)$ and $(y_0 : y_1 : y_2)$ are proportional and hence, represent the same point. This contradicts the original statement that the points are distinct.

2. Consider the point of intersection determined by the two distinct lines $[X_0 : X_1 : X_2]$ and $[Y_0 : Y_1 : Y_2]$. By the principle of duality in P_2, we obtain the homogeneous coordinates of the point by simply replacing the x_i and y_i with X_i and Y_i in Equation (6.5) to obtain the expression for the homogeneous coordinates of the point of intersection, giving

$$(x_0 : x_1 : x_2) = \left(\begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} : \begin{vmatrix} X_2 & X_0 \\ Y_2 & Y_0 \end{vmatrix} : \begin{vmatrix} X_0 & X_1 \\ Y_0 & Y_1 \end{vmatrix} \right). \tag{6.6}$$

Examples

(a) The equation of a line with coordinates $[1 : 2 : 3]$ is $x_0+2x_1+3x_2 = 0$.

(b) The coordinates of the line $2x_0 - 4x_1 + 5x_2 = 0$ are $[2 : -4 : 5]$.

(c) The equation of the point $(2 : -1 : 0)$ is $2X_0 - X_1 = 0$.

(d) The coordinates of the point $X_0 - X_2 = 0$ are $(1 : 0 : -1)$.

(e) The point of intersection of the two lines

$$\begin{aligned} 3x_0 - 2x_1 + 4x_2 &= 0, \quad \text{and} \\ 4y_0 + 2y_1 - 3y_2 &= 0, \end{aligned}$$

is

$$\left(\begin{vmatrix} -2 & 4 \\ 2 & -3 \end{vmatrix} : \begin{vmatrix} 4 & 3 \\ -3 & 4 \end{vmatrix} : \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix} \right) =$$

$$(-2 : 25 : 14) = (x_0 : x_1 : x_2) = (y_0 : y_1 : y_2).$$

6.2 Plane and Point Coordinates: Duality in P_3

Duality in projective space P_3 concerns the dual elements point and plane, not point and line as in P_2. Regardless, as in the projective plane P_2, there are also two cases to consider in P_3.

1. **Plane equation, plane coordinates.** When $(x_0 : x_1 : x_2 : x_3)$ is a variable point on a fixed plane with coordinates $[X_0 : X_1 : X_2 : X_3]$, then

$$X_0x_0 + X_1x_1 + X_2x_2 + X_3x_3 \;=\; \sum_{i=0}^{3} X_i x_i \;=\; 0$$

   is a plane equation. This equation generates a two dimensional range of points.

2. **Point equation, point coordinates.** When $[X_0 : X_1 : X_2 : X_3]$ is a variable plane on a fixed point with coordinates $(x_0 : x_1 : x_2 : x_3)$, then

$$X_0x_0 + X_1x_1 + X_2x_2 + X_3x_3 \;=\; \sum_{i=0}^{3} X_i x_i \;=\; 0$$

   is a point equation. This equation generates a bundle of planes.

6.2.1 Concurrent Planes, Coplanar Points

The point coordinates of the point of intersection of three concurrent planes, and the plane coordinates of the plane defined by three non-collinear points can be determined using Cramer's rule in the same way as the point coordinates of the point of intersection of two lines, or the line determined by two points in the plane. The point of intersection of 3 non-collinear planes $[X_0 : X_1 : X_2 : X_3]$, $[Y_0 : Y_1 : Y_2 : Y_3]$, and $[Z_0 : Z_1 : Z_2 : Z_3]$ has point coordinates

$$(x_0 : x_1 : x_2 : x_3) =$$

$$\left(\begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix} : - \begin{vmatrix} X_0 & X_2 & X_3 \\ Y_0 & Y_2 & Y_3 \\ Z_0 & Z_2 & Z_3 \end{vmatrix} : \begin{vmatrix} X_0 & X_1 & X_3 \\ Y_0 & Y_1 & Y_3 \\ Z_0 & Z_1 & Z_3 \end{vmatrix} : - \begin{vmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \\ Z_0 & Z_1 & Z_2 \end{vmatrix} \right) =$$

$$\left(\begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix} : \begin{vmatrix} X_0 & X_3 & X_2 \\ Y_0 & Y_3 & Y_2 \\ Z_0 & Z_3 & Z_2 \end{vmatrix} : \begin{vmatrix} X_0 & X_1 & X_3 \\ Y_0 & Y_1 & Y_3 \\ Z_0 & Z_1 & Z_3 \end{vmatrix} : \begin{vmatrix} X_0 & X_2 & X_1 \\ Y_0 & Y_2 & Y_1 \\ Z_0 & Z_2 & Z_1 \end{vmatrix} \right), \quad (6.7)$$

where the negative signs have been eliminated by switching the last two columns in the determinants that correspond to x_1 and x_3.

Because of the duality in projective space, the plane coordinates of the plane determined by 3 non-collinear points $(x_0 : x_1 : x_2 : x_3)$, $(y_0 : y_1 : y_2 : y_3)$, and $(z_0 : z_1 : z_2 : z_3)$ are

$$[X_0 : X_1 : X_2 : X_3] =$$

$$\left[\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} : \begin{vmatrix} x_0 & x_3 & x_2 \\ y_0 & y_3 & y_2 \\ z_0 & z_3 & z_2 \end{vmatrix} : \right.$$

$$\left. \begin{vmatrix} x_0 & x_1 & x_3 \\ y_0 & y_1 & y_3 \\ z_0 & z_1 & z_3 \end{vmatrix} : \begin{vmatrix} x_0 & x_2 & x_1 \\ y_0 & y_2 & y_1 \\ z_0 & z_2 & z_1 \end{vmatrix} \right]. \quad (6.8)$$

6.2.2 Incidence and Intersection Conditions: Lines in P_2

The *rank* of a matrix plays an important role in what follows. The rank of an $m \times n$ matrix $\mathbf{A}$, indicated by the non-zero integer $r(\mathbf{A})$, is the dimension of the column space of $\mathbf{A}$. In other words, the rank is the maximum number of linearly independent columns of the matrix. A remarkable theorem in linear algebra is that the maximum number of linearly independent columns in an $m \times n$ matrix is the same as the maximum number of linearly independent rows [1]. If m is the number of rows and n is the number of columns, the rank of an $m \times n$ matrix cannot be larger than the smaller of m or n. Another way of looking at this is that the maximum rank of a rectangular matrix cannot exceed the dimension of the largest square submatrix that it contains. For example, the rank of a 4×2 matrix cannot be greater than 2, because the dimension of largest submatrix is 2×2. Similarly, a the rank of 3×4 cannot exceed 3. Clearly, the rank of a non-trivial matrix which contains at least one non-zero element must be a non-zero integer because the matrix has at least one non-zero element.

Given two lines $X_0x_0 + X_1x_1 + X_2x_2 = 0$ and $Y_0x_0 + Y_1x_1 + Y_2x_2 = 0$ in P_2 with line coordinates $[X_0 : X_1 : X_2]$ and $[Y_0 : Y_1 : Y_2]$, then the maximum rank of the system of lines is $r = 2$, and is denoted

$$rank \begin{pmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \end{pmatrix} = r.$$

For any 3×2 matrix, the non-zero integer value for the rank means the following.

1. If $r = 1$, both lines are identical.
2. If $r = 2$ the lines possess one real intersection determined by Equation (6.6).

6.2.3 Conditions for a Plane and a Line in P_3

In P_3 a line is defined as the intersection of two planes π_i and π_j, $l = \pi_i \cap \pi_j$. Consider these together with an arbitrary plane π

$$l \;\cdots\; \left\{ \begin{array}{lrcl} \pi_1 : & X_0x_0 + X_1x_1 + X_2x_2 + X_3x_3 & = & 0, \\ \pi_2 : & Y_0x_0 + Y_1x_1 + Y_2x_2 + Y_3x_3 & = & 0, \end{array} \right.$$

$$\pi : \quad Z_0x_0 + Z_1x_1 + Z_2x_2 + Z_3x_3 \;=\; 0.$$

Then the following significance is associated with the rank r of the coefficient matrix

$$r = rank \begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \\ Z_0 & Z_1 & Z_2 & Z_3 \end{pmatrix}.$$

1. If $r = 1$ the three planes are identical.
2. If $r = 2$, the line l lies in plane π.
3. If $r = 3$ the line l and plane π have exactly one intersection given by the intersection of the three planes π_1, π_1, and π, which can be determined with Equation (6.7).

6.2.4 Conditions for Planes in P_3

Given three planes in P_3:

$$\sum_{i=0}^{3} X_i x_i = 0, \; \sum_{i=0}^{3} Y_i x_i = 0, \; \sum_{i=0}^{3} Z_i x_i = 0.$$

The rank of the plane coordinate matrix is denoted

$$r = rank \begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \\ Z_0 & Z_1 & Z_2 & Z_3 \end{pmatrix}.$$

The integer value of r means the following.

1. If $r = 1$ the three planes are identical.
2. If $r = 2$ the planes belong to an axial pencil, that is they have one line in common and are but one trio in the infinite family of all planes containing the one line.
3. If $r = 3$ the planes possess exactly one real point of intersection given by Equation (6.7).

6.2.5 Conditions for Two Lines in P_3

Consider the two lines l_1 and l_2

$$l_1 \;\cdots\; \begin{cases} \pi_1: & X_{10}x_0 + X_{11}x_1 + X_{12}x_2 + X_{13}x_3 = 0, \\ \pi_2: & X_{20}x_0 + X_{21}x_1 + X_{22}x_2 + X_{23}x_3 = 0, \end{cases}$$

$$l_2 \;\cdots\; \begin{cases} \pi_3: & X_{30}x_0 + X_{31}x_1 + X_{32}x_2 + X_{33}x_3 = 0, \\ \pi_4: & X_{40}x_0 + X_{41}x_1 + X_{42}x_2 + X_{43}x_3 = 0. \end{cases}$$

The following is true for l_1 and l_2 in P_3 according to the rank of the 4×4 plane coordinate matrix

$$rank \begin{pmatrix} X_{10} & \cdots & X_{13} \\ \vdots & \ddots & \vdots \\ X_{40} & \cdots & X_{43} \end{pmatrix} = r.$$

1. If $r = 1$ the four planes are identical.
2. If $r = 2$ both lines are identical.
3. If $r = 3$ the lines possess one intersection which is determined by Equation (6.7).
4. If $r = 4$ both lines are skew and possess no point of intersection.

6.2.6 General Parametric Equation for a Line in 3D Space

A line l can be described in a Cartesian coordinate system in vectorial parametric form as

$$\mathbf{x} = \mathbf{a} + t\mathbf{b}, \tag{6.9}$$

where $\mathbf{x}$ is the position vector of any point on the line, $\mathbf{a}$ is the position vector of a particular point on the line, $\mathbf{b}$ is a vector parallel to the line, and $t \in \mathbb{R}$. This is called a parametric equation because the line l is traced out by $\mathbf{x}$ as t varies between $-\infty$ and ∞, which is illustrated in Figure 6.1. But, as indicated in Section 6.2.3, in P_3 a line can be represented as the intersection of two planes

$$l \;\cdots\; \begin{cases} \pi_1: & X_0x_0 + X_1x_1 + X_2x_2 + X_3x_3 = 0, \\ \pi_2: & Y_0x_0 + Y_1x_1 + Y_2x_2 + Y_3x_3 = 0. \end{cases}$$

We can always redefine the point coordinates as Cartesian coordinates by dividing the homogeneous coordinates by x_0 which changes the two equations to be

$$l \;\cdots\; \begin{cases} \pi_1: & X_0 + X_1x + X_2y + X_3z = 0, \\ \pi_2: & Y_0 + Y_1x + Y_2y + Y_3z = 0. \end{cases}$$

Now, without loss in generality, we can set one of the Cartesian coordinates to be the parameter, let's say $z = t$. This gives two equations in the remaining

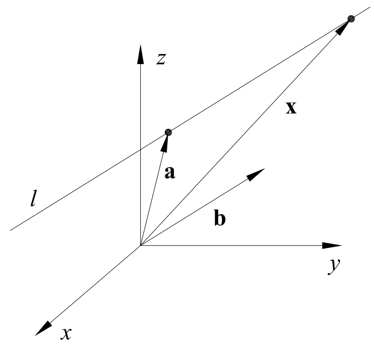

Figure 6.1: Parametric representation of a line.

two unknowns x and y, which we can express in vector-matrix form, giving

$$\begin{aligned} \pi_1: \quad X_0 + X_1 x + X_2 y + X_3 t &= 0, \\ \pi_2: \quad Y_0 + Y_1 x + Y_2 y + Y_3 t &= 0, \\ \Rightarrow \begin{bmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= -\begin{bmatrix} X_0 + X_3 t \\ Y_0 + Y_3 t \end{bmatrix}. \end{aligned}$$

We can solve simultaneously for x and y using Cramer's rule. To do so, it must be that

$$\begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} = \Delta \neq 0. \tag{6.10}$$

If $\Delta = 0$ but

$$r\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{pmatrix} = 2,$$

then the line is parallel to either the x or y basis vector direction. In this case, choose another pair of variables to solve for, either x and z, or y and z, since the two planes must intersect because $r = 2$. Assuming $\Delta \neq 0$, one finds

$$\begin{aligned} x &= \frac{1}{\Delta}\begin{vmatrix} -(X_0 + X_3 t) & X_2 \\ -(Y_0 + Y_3 t) & Y_2 \end{vmatrix} \\ &= \frac{1}{\Delta}\left(-Y_2(X_0 + X_3 t) + X_2(Y_0 + Y_3 t)\right), \\ &= \frac{1}{\Delta}\left(X_2 Y_0 - X_0 Y_2 + t(X_2 Y_3 - X_3 Y_2)\right), \end{aligned}$$

and

$$\begin{aligned} y &= \frac{1}{\Delta}\begin{vmatrix} X_1 & -(X_0 + X_3 t) \\ Y_1 & -(Y_0 + Y_3 t) \end{vmatrix} \\ &= \frac{1}{\Delta}\left(-X_1(Y_0 + Y_3 t) + Y_1(X_0 + X_3 t)\right), \\ &= \frac{1}{\Delta}\left(X_0 Y_1 - X_1 Y_0 + t(X_3 Y_1 - X_1 Y_3)\right). \end{aligned}$$

Thus the general parametric equation for the line in a Cartesian coordinate space has the form

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} X_2Y_0 - X_0Y_2 \\ X_0Y_1 - X_{11}Y_0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{\Delta}(X_2Y_3 - X_3Y_2) \\ \frac{1}{\Delta}(X_3Y_1 - X_1Y_3) \\ 1 \end{bmatrix}. \tag{6.11}$$

6.2.7 Arithmetic Examples

1. Investigate the mutual location(s) shared by the three planes

$$\begin{aligned} \pi_1 : &\quad -11x_0 + x_1 + x_2 + 2x_3 &= 0, \\ \pi_2 : &\quad -45x_0 + 3x_1 + 7x_2 + 6x_3 &= 0, \\ \pi_3 : &\quad 16x_0 + x_1 - 8x_2 + 2x_3 &= 0. \end{aligned}$$

<u>**Solution**</u>

The mutual locations shared by the planes is investigated by evaluating the rank of the plane coordinate coefficient matrix, in this case by performing elementary row reduction operations on the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 2 & -11 \\ 3 & 7 & 6 & -45 \\ 1 & -8 & 2 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & -11 \\ 0 & 4 & 0 & -12 \\ 0 & -9 & 0 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & -11 \\ 0 & 1 & 0 & -3 \\ 0 & -1 & 0 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 2 & -11 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r = 2.$$

Hence $r = 2$ and the three planes form an axial pencil of planes.

2. Find a parametric equation for the axis of the pencil from the previous example.

<u>**Solution**</u>

The axis can be described as the line of intersection of any two of the specified planes, for instance, $\pi_1 \cap \pi_3$. Next, check the numerical value of Δ.

$$\Delta = \begin{vmatrix} X_{11} & X_{12} \\ X_{31} & X_{32} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -8 \end{vmatrix} = -9 \neq 0. \tag{6.12}$$

Since $\Delta \neq 0$ we can use Equation 6.11 directly to obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

3. In projective space P_3 two lines are given as $l_1 = \pi_1 \cap \pi_2$ and $l_2 = \pi_3 \cap \pi_4$. Determine if the lines intersect, and if they do what are the coordinates of the point of intersection?

$$l_1 \cdots \begin{cases} \pi_1: & x_0 - 2x_1 - 2x_2 & = & 0, \\ \pi_2: & -x_0 + 3x_1 + 2x_2 + 4x_3 & = & 0, \end{cases}$$

$$l_2 \cdots \begin{cases} \pi_3: & x_0 - 2x_1 + x_2 - 2x_3 & = & 0, \\ \pi_4: & x_1 + 2x_2 + 3x_3 & = & 0. \end{cases}$$

Solution

First, determine the rank of the coefficient matrix.

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ -1 & 3 & 2 & 4 \\ 1 & -2 & 1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r = 3.$$

Since $r = 3$ both lines l_1 and l_2 intersect, which agrees with Section 6.2.5. To determine the homogeneous coordinates of the point of intersection we can select any three of the four given planes and use Equation (6.7). For planes π_1, π_2, and π_3 we obtain

$$x_0 = \begin{vmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{vmatrix} = \begin{vmatrix} -2 & -1 & 0 \\ 3 & 2 & 4 \\ -2 & 1 & -2 \end{vmatrix} = 18,$$

$$x_1 = \begin{vmatrix} X_{10} & X_{13} & X_{12} \\ X_{20} & X_{23} & X_{22} \\ X_{30} & X_{33} & X_{32} \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ -1 & 4 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 10,$$

$$x_2 = \begin{vmatrix} X_{10} & X_{11} & X_{13} \\ X_{20} & X_{21} & X_{23} \\ X_{30} & X_{31} & X_{33} \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 \\ -1 & 3 & 4 \\ 1 & -2 & -2 \end{vmatrix} = -2,$$

$$x_3 = \begin{vmatrix} X_{10} & X_{12} & X_{11} \\ X_{20} & X_{22} & X_{21} \\ X_{30} & X_{32} & X_{31} \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \\ 1 & 1 & -2 \end{vmatrix} = -2,$$

Assembling the individual determinants gives the coordinate ratios

$$(x_0 : x_1 : x_2 : x_3) = (18 : 10 : -2 : -2) = (9 : 5 : -1 : -1).$$

6.2.8 Skew Lines

In three-dimensional geometry, skew lines are two lines that do not intersect and are not parallel, see Figure 6.2. An example of a pair of skew lines are two distinct lines in the same regulus of a hyperbolic paraboloid, or a hyperboloid of one sheet. Two lines that both lie in the same plane must either cross each other or be parallel, so skew lines can exist only in three or more dimensions. Two lines are skew if and only if they are not coplanar.

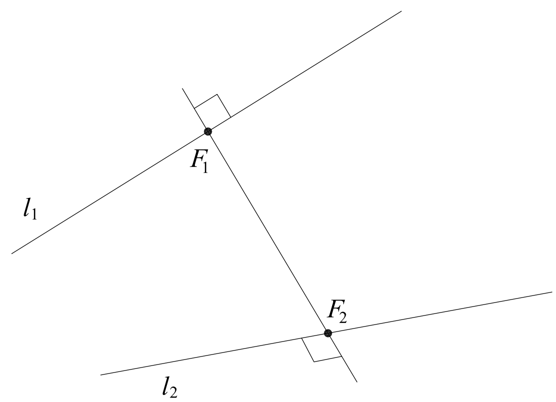

Figure 6.2: Two skew lines in E_3.

A pair of skew lines is always defined by a set of four non-coplanar points that form the vertices of a tetrahedron which possesses non-zero volume. Let the position vectors of the four non-coplanar 3D points be **a**, **b**, **c**, and **d**. The volume of the corresponding tetrahedron is the positive number given by

$$V = \left\| \frac{1}{6} \left| \begin{array}{ccc} \mathbf{a}-\mathbf{d} & \mathbf{b}-\mathbf{d} & \mathbf{c}-\mathbf{d} \end{array} \right| \right\|, \tag{6.13}$$

or any other combination of pairs of vertices that form a simply connected graph. For example consider the following four points

$$\mathbf{a} = \begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 11 \\ -3 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}.$$

Applying Equation (6.13) to the four position vectors yields

$$V = \left\| \frac{1}{6} \begin{vmatrix} -7 & -4 & -5 \\ -2 & 9 & -2 \\ 8 & -4 & 11 \end{vmatrix} \right\| = \frac{341}{6},$$

which is a positive non-zero number, leading to the conclusion that the four points are indeed contained on two skew lines. But, the question naturally arises: what pairing of the four points are on the two skew lines? Clearly, the vertices of a tetrahedron of non-zero volume represent six lines where each vertex is the intersection of three of the lines. However, inspection of the tetrahedron in Figure 6.3 reveals that every pair of opposite edges forms a pair of skew lines.

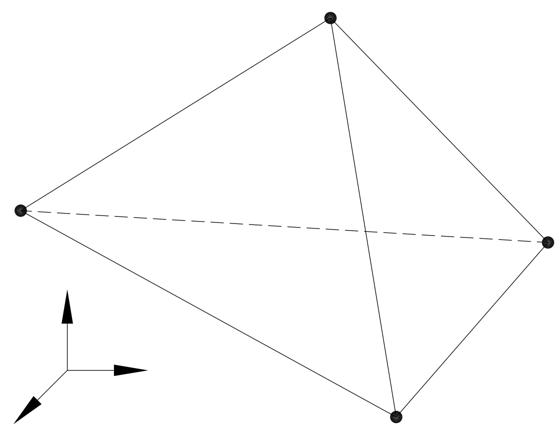

Figure 6.3: Tetrahedron in E_3.

If l_1 and l_2 are two skew lines in E_3 with parametric equations

$$\begin{aligned} l_1: \quad \mathbf{x}_1 &= \mathbf{a}_1 + t_1\mathbf{b}_1, \\ l_2: \quad \mathbf{x}_2 &= \mathbf{a}_2 + t_2\mathbf{b}_2, \end{aligned}$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the lines, then a line F_1F_2 exists which is mutually orthogonal to l_1 and l_2, where l_1 contains point F_1 and l_2 contains point F_2. The line segment F_1F_2 is called the *common normal* of l_1 and l_2 and represents the shortest distance between the skew lines.

The direction of the common normal is determined by the cross product of the direction vectors of the lines

$$\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2.$$

The unit vector in this direction is

$$\mathbf{u} = \frac{\mathbf{b}_1 \times \mathbf{b}_2}{|\mathbf{b}_1 \times \mathbf{b}_2|}.$$

The length of the common normal, d, is obtained with

$$d = \|\mathbf{u} \cdot (\mathbf{a}_2 - \mathbf{a}_1)\|. \tag{6.14}$$

Clearly, a condition for intersection of l_1 and l_2 is that d from Equation (6.14) equates to zero. Then, another test for skewness of two lines specified by two parametric equations is that

$$d = \|\mathbf{u} \cdot (\mathbf{a}_2 - \mathbf{a}_1)\| \neq 0. \tag{6.15}$$

The length of the common normal is the distance between points F_1 and F_2, the pair of points that are nearest to each other on each line. To determine the position vectors of F_1 and F_2 we first need to define two new vectors, namely

$$\begin{aligned} \mathbf{n}_2 &= \mathbf{b}_2 \times \mathbf{n}, \\ \mathbf{n}_1 &= \mathbf{b}_1 \times \mathbf{n}. \end{aligned}$$

The position vector of F_1 is $\mathbf{f}_1$ and is obtained with

$$\mathbf{f}_1 = \mathbf{a}_1 + \frac{(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{n}_2}{\mathbf{b}_1 \cdot \mathbf{n}_2} \mathbf{b}_1. \tag{6.16}$$

Similarly, the position vector of F_2 is $\mathbf{f}_2$ and is obtained with

$$\mathbf{f}_2 = \mathbf{a}_2 + \frac{(\mathbf{a}_1 - \mathbf{a}_2) \cdot \mathbf{n}_1}{\mathbf{b}_2 \cdot \mathbf{n}_1} \mathbf{b}_2. \tag{6.17}$$

Formal proofs for these relations may be found in [2].

Example

Find the respective locations of the end points F_1 and F_2, and the length of the shortest connecting line segment, if they exist, on each of the two lines l_1 and l_2 determined by the two parametric linear equations

$$\begin{aligned} l_1: \quad \mathbf{x}_1 &= \begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \\ l_2: \quad \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 11 \\ -3 \end{bmatrix} + t_2 \begin{bmatrix} 4 \\ -9 \\ 3 \end{bmatrix}. \end{aligned}$$

Solution

First, we check if the lines are indeed skew by computing length d using Equation (6.14) and find

$$\begin{aligned} d &= \left\| \frac{1}{33} \begin{bmatrix} 27 \\ 6 \\ -18 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 11 \\ -12 \end{bmatrix} \right\|, \\ &= 11, \end{aligned}$$

which is a real non-zero number, and hence the two lines are indeed skew.

The locations of the points F_1 and F_2 are determined using Equations (6.16) and (6.17), yielding

$$\begin{aligned} \mathbf{f}_1 &= \begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix} + \frac{\left(\begin{bmatrix} 1 \\ 11 \\ -3 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix} \right) \cdot \begin{bmatrix} 144 \\ 153 \\ 267 \end{bmatrix}}{\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 144 \\ 153 \\ 267 \end{bmatrix}} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \\ &= \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \mathbf{f}_2 &= \begin{bmatrix} 1 \\ 11 \\ -3 \end{bmatrix} + \frac{\left(\begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 11 \\ -3 \end{bmatrix} \right) \cdot \begin{bmatrix} -18 \\ 117 \\ 12 \end{bmatrix}}{\begin{bmatrix} 4 \\ -9 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -18 \\ 117 \\ 12 \end{bmatrix}} \begin{bmatrix} 4 \\ -9 \\ 3 \end{bmatrix}, \\ &= \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}. \end{aligned}$$

It is easy to check that $\mathbf{f}_1$ and $\mathbf{f}_2$ require that $t_1 = -1$ and that $t_2 = 1$. Moreover, it is simple to confirm the length of the common normal being $d = 11$ by computing the length of the distance between $\mathbf{f}_1$ and $\mathbf{f}_2$:

$$\|\mathbf{f}_2 - \mathbf{f}_1\| = 11.$$

6.3 Plücker Coordinates

We will now examine some arithmetic applications using Plücker coordinates [3, 4], a special case of Grassmann coordinates [5] which were introduced in Chapter 4. Recall from that chapter these line coordinates can be considered in two ways.

1. The line between two points giving *Plücker line coordinates*, or *ray coordinates* as they are sometimes called.
2. The line of intersection between two planes giving *Plücker axial coordinates.*

The Plücker line coordinates are the six numbers that are generated from the homogeneous coordinates of two points in 3D space

$$p_{ik} = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix} \qquad i, k \in \{0, \ldots, 3\},\ i \neq k.$$

Of the twelve possible sub-determinants, only six are independent, since $p_{ik} = -p_{ki}$. Historically, the following six are used for line coordinates

$$(p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}). \tag{6.18}$$

Recall also that the Plücker coordinates are super-abundant by two because only four generalised coordinates are required to determine a unique line in three dimensions. Hence, there must be two constraints on the six parameters. First, because the coordinates are homogeneous, there are only five independent ratios. It necessarily follows that

$$(p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}) \quad \neq \quad (0 : 0 : 0 : 0 : 0 : 0). \tag{6.19}$$

Second, the six numbers must obey the following quadratic condition

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} \quad = \quad 0. \tag{6.20}$$

The first three elements of the Plücker coordinates can be thought of as a position vector pointing in the direction of the line, known as a *spear*, while the last three represent the moment that the line makes with respect to the coordinate system origin in which the points that are used to generate the line coordinates are described. The two sets of three numbers may be described as two dual vectors **d** and **m**, and together comprise the Plücker array of six numbers:

$$(\overbrace{p_{01} : p_{02} : p_{03}}^{\mathbf{d}} : \overbrace{p_{23} : p_{31} : p_{12}}^{\mathbf{m}}). \tag{6.21}$$

6.3.1 Normalised Plücker Coordinates

Plücker line coordinates are normalised in the following way

$$\mathbf{p} \quad = \quad \frac{(p_{01} : p_{02} : p_{03})}{\sqrt{p_{01}^2 + p_{02}^2 + p_{03}^2}}, \tag{6.22}$$

$$\overline{\mathbf{p}} \quad = \quad \frac{(p_{23} : p_{31} : p_{12})}{\sqrt{p_{01}^2 + p_{02}^2 + p_{03}^2}}. \tag{6.23}$$

The Euclidean interpretation of normalised Plücker line coordinates is that, now as a unit vector, $\mathbf{p}$ represents the unit direction vector of the line and $\overline{\mathbf{p}}$, which is not necessarily a unit vector, represents the moment of the line about the origin

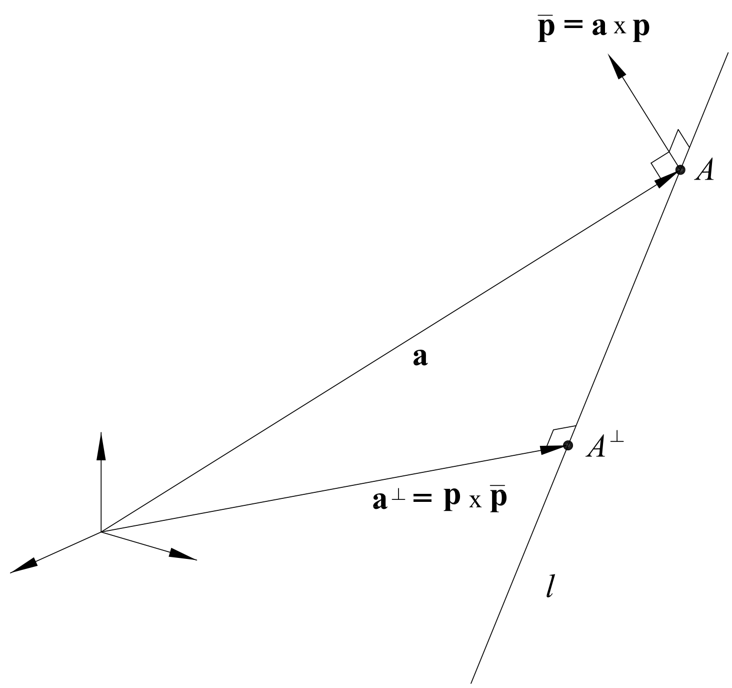

Figure 6.4: Normalised Plücker line coordinates.

in the coordinate system in which the points defining the line are described, see Figure 6.4 for example. It is clear that

$$\mathbf{p} \cdot \mathbf{p} = 1.$$

While in the notation of Figure 6.4

$$\overline{\mathbf{p}} = \mathbf{a} \times \mathbf{p},$$

and because $\mathbf{p}$ has unit length

$$\mathbf{a}^{\perp} = \mathbf{p} \times \overline{\mathbf{p}},$$

where $\mathbf{a}^{\perp}$ is perpendicular to the line l, and the magnitude of $\mathbf{a}^{\perp}$ is equal to the magnitude of $\overline{\mathbf{p}}$

$$\begin{aligned} \mathbf{a}^{\perp} &\perp l, \\ \|\mathbf{a}^{\perp}\| &= \|\overline{\mathbf{p}}\|. \end{aligned}$$

The free vector of the moment, $\overline{\mathbf{p}} = \mathbf{a} \times \mathbf{p}$, is perpendicular to both $\mathbf{a}$ and $\mathbf{p}$. Hence

$$\mathbf{p} \cdot \overline{\mathbf{p}} = \mathbf{p} \cdot (\mathbf{a} \times \mathbf{p}) = 0.$$

Example

Determining Plücker line coordinates given two points $\mathbf{x}$ and $\mathbf{y}$ in E_3, where $\mathbf{x} = (3,0,2)^T$, $\mathbf{y} = (4,1,0)^T$, and demonstrate that they satisfy the Plücker identity given by Equation (6.20) thereby meaning the six coordinates indeed represent a line. Additionally, show that the relation $\overline{\mathbf{p}} = \mathbf{a} \times \mathbf{p}$ holds.

Solution

Define $\mathbf{x}$ and $\mathbf{y}$ as homogeneous coordinates where

$$x = \frac{x_1}{x_0},\ y = \frac{x_2}{x_0},\ z = \frac{x_3}{x_0},\ \text{and set } x_0 = 1,$$

$$\Rightarrow\ \mathbf{x} = (1:3:0:2),\ \mathbf{y} = (1:4:1:0).$$

Consider the sub-determinants $p_{ik} = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}$, $i \neq k$ for

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 1 & 4 & 1 & 0 \end{bmatrix},$$

$$\begin{aligned}
p_{01} &= \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1,\\
p_{02} &= \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1,\\
p_{03} &= \begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2,\\
p_{23} &= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2,\\
p_{31} &= \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} = 8,\\
p_{12} &= \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3.
\end{aligned}$$

Thus, the Plücker line coordinates are

$$(p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}) = (1:1:-2:-2:8:3).$$

1. The question asking if the coordinates satisfy the Plücker condition is really asking are they a line?

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = -2 + 8 - 6 = 0.$$

We can conclude that the six numbers sattisfy the Plücker condition, Equation (6.20), hence they represent a line.

2. Is it true that $\overline{\mathbf{p}} = \mathbf{a} \times \mathbf{p}$? This relation holds even without normalising the coordinates because of orthoganality.

$$\mathbf{x} \times \mathbf{d}_p = \begin{vmatrix} i & j & k \\ 3 & 0 & 2 \\ 1 & 1 & -2 \end{vmatrix} = \begin{bmatrix} -2 \\ 8 \\ 3 \end{bmatrix},$$
$$\mathbf{y} \times \mathbf{d}_p = \begin{vmatrix} i & j & k \\ 4 & 1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = \begin{bmatrix} -2 \\ 8 \\ 3 \end{bmatrix}.$$

6.3.2 Axial Coordinates

The axial coordinates are differentiated from Plücker coordinates by denoting them as $\hat{p}_{ik}$. They are derived by considering the line of intersection between two planes having plane coordinates X and Y

$$\begin{bmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{bmatrix}.$$

The coordinates are obtained by expanding the array of plane coordinates of two planes with the 2×2 sub-determinants

$$\hat{p}_{ik} = \begin{vmatrix} X_i & X_k \\ Y_i & Y_k \end{vmatrix},$$

leading to

$$(\hat{p}_{01} : \hat{p}_{02} : \hat{p}_{03} : \hat{p}_{23} : \hat{p}_{31} : \hat{p}_{12}). \tag{6.24}$$

It can be shown, but only after a significant amount of subscript manipulation, that axial coordinates and Plücker coordinates have the same components but in different order [2]. It turns out that axial coordinates of a set of two plane coordinates are related to the Plücker coordinates of two distinct points on the line of intersection of the two planes in the following way:

$$(\hat{p}_{01} : \hat{p}_{02} : \hat{p}_{03} : \hat{p}_{23} : \hat{p}_{31} : \hat{p}_{12}) = (p_{23} : p_{31} : p_{12} : p_{01} : p_{02} : p_{03}). \tag{6.25}$$

6.4 Operations With Plücker and Axial Coordinates

Plücker line and axial coordinates lead to very convenient operations in line space that may be implemented in much less cumbersome ways than in Euclidean point space. In what follows several will be discussed and demonstrated.

6.4.1 Angle Between Lines

A well known theorem in linear algebra is that the magnitude of the cross product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is proportional to the angle ϑ between them:

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin\vartheta \tag{6.26}$$

Consider two Plücker arrays p and q representing two distinct lines where $\mathbf{d}_p$ and $\mathbf{d}_q$ are the vectors comprising the first three elements of p and q, respectively. If the lines intersect, the angle between the two lines can be determined using descriptive geometry in a view where the plane comprising the two intersecting lines appears in true shape. The angle between two nonintersecting lines is the same as the angle between two intersecting lines that are, respectively, parallel to the nonintersecting skew lines. The angle can be computed using $\mathbf{d}_p$ and $\mathbf{d}_q$ as

$$\vartheta = \sin^{-1}\left(\frac{\|\mathbf{d}_p \times \mathbf{d}_q\|}{\|\mathbf{d}_p\|\|\mathbf{d}_q\|}\right) \tag{6.27}$$

If the Plücker arrays p and q are normalised, then the angle between the two lines is simply

$$\vartheta = \sin^{-1}\|\mathbf{p} \times \mathbf{q}\| \tag{6.28}$$

6.4.2 The Shortest Distance Between Two Lines

Consider two lines given by Plücker arrays p and q. Do the lines intersect, or is there a shortest, perpendicular distance between them? We can answer these questions by first defining

$$\Omega(p,q) = p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} + p_{23}q_{01} + p_{31}q_{02} + p_{12}q_{03}. \tag{6.29}$$

The set of six bilinear terms are obtained as the Laplacian expansion along the top two rows of the 4×4 matrix of homogeneous coordinates x_i and y_i on p and s_i and t_i on q with $i \in \{0,1,2,3\}$, by some computation analogous to Equation (4.32) in Chapter 4. The array is organised as

$$\Omega(p,q) = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ s_0 & s_1 & s_2 & s_3 \\ t_0 & t_1 & t_2 & t_3 \end{vmatrix}. \tag{6.30}$$

If p and q intersect, then

$$\Omega(p,q) = 0, \tag{6.31}$$

because p and q are coplanar. Otherwise $\Omega(p,q) \neq 0$ because p and q are skew lines. It can be shown that the magnitude of the distance is given by [6]

$$d(p,q) = \frac{\|\Omega(p,q)\|}{\|\mathbf{d}_p \times \mathbf{d}_q\|}, \tag{6.32}$$

where

$$\begin{aligned} \mathbf{d}_p &= (p_{01}, p_{02}, p_{03})^T, \\ \mathbf{d}_q &= (q_{01}, q_{02}, q_{03})^T. \end{aligned}$$

Note that here, there is no need for $\mathbf{d}_p$ and $\mathbf{d}_q$ to be normalised.

Example

Let line p be the y-axis and line q be parallel to the x-axis through the point $(0, 0, 1)$, as illustrated in Figure 6.5. Compute the shortest distance between the lines, as well as the angle ϑ between the lines.

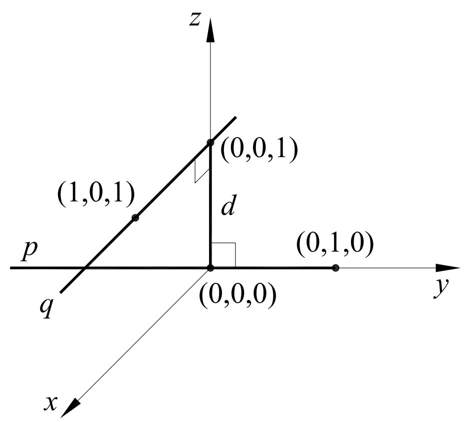

Figure 6.5: Two perpendicular lines p and q.

Solution

To use $\Omega(p, q)$, the Plücker coordinates for each line must first be determined. The Cartesian coordinates of two convenient points on p are selected to be $\mathbf{x} = (0, 0, 0)$ and $\mathbf{y} = (0, 1, 0)$. The Cartesian point coordinates are homogenised with $x_0 = 1$, giving

$$p: \quad \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The Plücker coordinates are computed as

$$\begin{aligned} p_{01} &= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0, \\ p_{02} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \\ p_{03} &= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0, \\ p_{23} &= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \\ p_{31} &= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0, \\ p_{12} &= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0. \end{aligned}$$

Similarly for line q, two convenient points are selected to be $\mathbf{s} = (0, 0, 1)$ and $\mathbf{t} = (1, 0, 1)$. Homogenising the points with $x_0 = 1$ gives

$$q : \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{aligned} q_{01} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \\ q_{02} &= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0, \\ q_{03} &= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \\ q_{23} &= \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0, \\ q_{31} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \\ q_{12} &= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0. \end{aligned}$$

Thus the Plücker coordinates of the two lines are

$$\begin{aligned} p &: \quad (0 : 1 : 0 : 0 : 0 : 0), \\ q &: \quad (1 : 0 : 0 : 0 : 1 : 0). \end{aligned}$$

The shortest, perpendicular distance, or length of the common normal is given by the absolute value of the ratio of

$$\begin{aligned} \Omega(p, q) &= 0 + 1 + 0 + 0 + 0 + 0 = 1, \\ \mathbf{d}_p \times \mathbf{d}_q &= \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \\ \|\mathbf{d}_p \times \mathbf{d}_q\| &= 1, \end{aligned}$$

where i, j, and k are unit basis vectors in the directions of coordinate axes x, y, and z respectively. Hence, the length of the common normal of lines p and q is

$$d(p, q) = \left\| \frac{1}{1} \right\| = 1.$$

The angle between the two lines is

$$\vartheta = \sin^{-1}\left(\frac{\|\mathbf{d}_p \times \mathbf{d}_q\|}{\|\mathbf{d}_p\| \|\mathbf{d}_q\|} \right) = \sin^{-1}\left(\frac{1}{(1)(1)} \right) = 90^\circ.$$

It will be helpful to examine one more example with a predictable, though different outcome.

Example

Let p be the x-axis, and q be on the points $(0,2,1)$ and $(1,2,1)$, as illustrated in Figure 6.6. Compute the shortest distance between the lines as in the previous example.

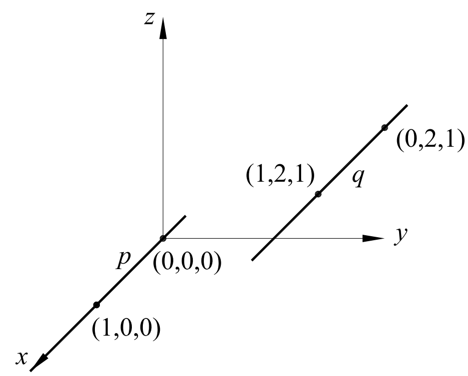

Figure 6.6: Two parallel lines p and q.

Solution

The Cartesian coordinates of two convenient points on p are selected to be $\mathbf{x} = (0,0,0)$ and $\mathbf{y} = (1,0,0)$. They are homogenised with $x_0 = 1$ giving

$$p: \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} p_{01} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \\ p_{02} &= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0, \\ p_{03} &= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0, \\ p_{23} &= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0, \\ p_{31} &= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0, \\ p_{12} &= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0. \end{aligned}$$

For line q, the two specified points are $\mathbf{s} = (0,2,1)$ and $\mathbf{t} = (1,2,1)$. Homogenising these Cartesian point coordinates with $x_0 = 1$ gives

$$q: \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

$$
\begin{aligned}
q_{01} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \\
q_{02} &= \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0, \\
q_{03} &= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \\
q_{23} &= \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0, \\
q_{31} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \\
q_{12} &= \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = -2.
\end{aligned}
$$

However, when we check the intersection condition in Equation (6.31) condition we find

$$
\begin{aligned}
\Omega(p,q) &= p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} + p_{23}q_{01} + p_{31}q_{02} + p_{12}q_{03}, \\
&= 0+0+0+0+0+0 = 0.
\end{aligned}
$$

This result should not be surprising since lines p and q are parallel, and thus intersect in a point at infinity, where the distance between the lines is, of course, zero. Moreover, the angle between the two parallel lines is seen to be $\vartheta = 0°$ since

$$
\vartheta = \sin^{-1}\left(\frac{\|\mathbf{d}_p \times \mathbf{d}_q\|}{\|\mathbf{d}_p\|\|\mathbf{d}_q\|}\right) = \sin^{-1}\left(\frac{0}{(1)(1)}\right) = 0°.
$$

Example

In this example we will confirm the length of the common normal from the example in Section 6.2.8 by using Equation (6.32), thereby demonstrating a pair of concomitant methods for computing the shortest distance between two skew lines. Additionally, compute the angle between the two lines. The two skew lines are specified as the two parametric equations in E_3

$$
\begin{aligned}
l_1: \quad \mathbf{x}_1 &= \begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \\
l_2: \quad \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 11 \\ -3 \end{bmatrix} + t_2 \begin{bmatrix} 4 \\ -9 \\ 3 \end{bmatrix}.
\end{aligned}
$$

Solution

To use Equation (6.32) the two parametric equations must be re-expressed in the form of Plücker line coordinates. To do that, two points on each line are needed. We will assign the name p to line l_1 and q to line l_2. Clearly, from the parametric equations we have the coordinates of a point on each line, we establish another by assigning values to the parameters $t_1 = t_2 = 1$. This yields required point coordinates which are homogenised with $x_0 = y_0 = 1$

$$\begin{aligned} p: \quad \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} &= \begin{bmatrix} 1 & -2 & 0 & 9 \\ 1 & 0 & 0 & 12 \end{bmatrix}, \\ q: \quad \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 11 & -3 \\ 1 & 5 & 2 & 0 \end{bmatrix}. \end{aligned}$$

The Plücker coordinates of the two lines are determined to be

$$\begin{aligned} p: &\quad (2:0:3:0:24:0), \\ q: &\quad (4:-9:3:6:-15:-53). \end{aligned}$$

The length of the common normal is given by the absolute value of the ratio of

$$\Omega(p,q) = 12 + 0 - 159 + 0 - 216 + 0 = -363,$$

and

$$\begin{aligned} \mathbf{d}_p \times \mathbf{d}_q &= \begin{vmatrix} i & j & k \\ 2 & 0 & 3 \\ 4 & -9 & 3 \end{vmatrix} = \begin{bmatrix} 27 \\ 6 \\ -18 \end{bmatrix}, \\ \|\mathbf{d}_p \times \mathbf{d}_q\| &= 33. \end{aligned}$$

Hence, the length of the common normal of lines p and q is

$$d(p,q) = \left\| \frac{-363}{33} \right\| = 11,$$

which agrees with the results in the example from Section 6.2.8.

The angle ϑ between the two lines is

$$\vartheta = \sin^{-1}\left(\frac{\|\mathbf{d}_p \times \mathbf{d}_q\|}{\|\mathbf{d}_p\|\|\mathbf{d}_q\|}\right) = \sin^{-1}\left(\frac{33}{\sqrt{13}\sqrt{106}}\right) = 62.7447°.$$

6.4.3 Cylinder Collision Detection

An algorithm for determining if two cylindrical Gough-Stewart platform legs collide can be structured as a two stage test. First the infinite cylinders to which the legs belong are examined. Then, if the legs fail the infinite cylinder test the finite cylinder sections must be considered.

Infinite Cylinder Test

It is generally not possible for any pair of legs in a Gough-Stewart platform to be parallel. For distinct nonparallel lines p and q in space, the perpendicular (shortest) distance between them is given by Equation (6.32), $d(p, q)$. If the lines p and q are regarded as the centre lines of the cylindrical prismatic legs with radii r_1 and r_2, then clearly if

$$d(p, q) \quad > \quad r_1 + r_2 \tag{6.33}$$

no collision between the two legs occurs.

However, if

$$d(p, q) \quad \leq \quad r_1 + r_2 \tag{6.34}$$

then somewhere along the infinite length of the two cylinders a collision occurs. In this case, we must determine if the collision occurs on the finite portions of the cylinders comprising the two Gough-Stewart platform legs.

Finite Cylinder Test

Consider two finite length cylinders whose axes are represented by lines p and q, as illustrated in Figure 6.7. Each cylinder can be described by a starting point where the axis intersects the beginning of the cylinder, identified by position vectors $\mathbf{c}$ and $\mathbf{d}$ respectively. Vectors $\mathbf{r}$ and $\mathbf{s}$ have magnitudes equal to the lengths of the finite cylinders. The endpoints of the cylinder segments are located with position vectors $\mathbf{f}$ and $\mathbf{g}$.

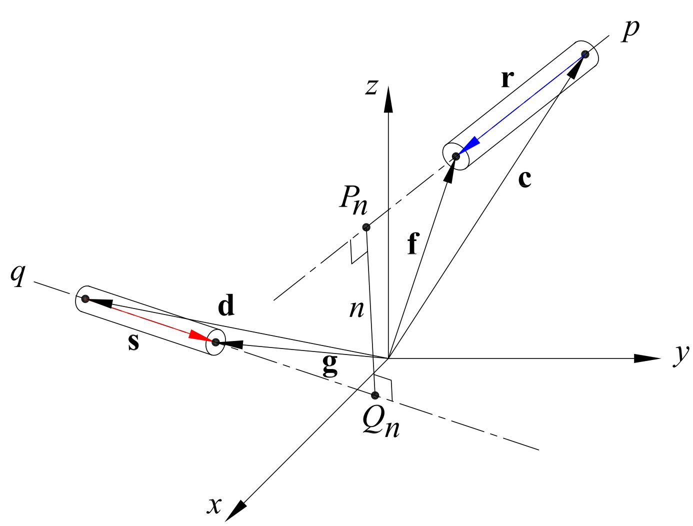

Figure 6.7: Two finite cylinders.

The common normal between lines p and q is labelled line segment n. Line n intersects lines p and q in points P_n and Q_n, respectively. The position vectors of these two points are described by the following two parametric equations

$$\mathbf{p}_n = \mathbf{c} + t_1\mathbf{r}, \tag{6.35}$$

$$\mathbf{q}_n = \mathbf{d} + t_2\mathbf{s}, \tag{6.36}$$

where

$$t_1 = \frac{((\mathbf{d}-\mathbf{c})\times\mathbf{s})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}, \tag{6.37}$$

$$t_2 = \frac{((\mathbf{d}-\mathbf{c})\times\mathbf{r})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}, \tag{6.38}$$

$$\mathbf{n} = \mathbf{r}\times\mathbf{s}. \tag{6.39}$$

The values of either parameter t_1 or t_2 can be used to determine if the common normal intersection points are within the finite cylinder sections, thereby indicating the cylindrical legs will collide. For the cylinder on line p three possibilities exist:

1) $t_1 \leq 0 \quad \Leftrightarrow$ P_n occurs before the start point $\mathbf{c}$ and no collision occurs;

2) $0 < t_1 < 1 \quad \Leftrightarrow$ P_n occurs between the start and endpoints $\mathbf{c}$ and $\mathbf{f}$ and therefore the legs collide;

3) $t_1 \geq 1 \quad \Leftrightarrow$ P_n occurs beyond the endpoint $\mathbf{f}$ and no collision occurs.

Similar conditions apply to the location of point Q_n on line q for the computed value of t_2. That is, if Q_n lies between $\mathbf{d}$ and $\mathbf{g}$ then a collision between the cylindrical legs will occur if

$$d(p,q) \leq r_1 + r_2.$$

6.4.4 Location of Point of Intersection of Two Lines

Let p and q be distinct lines in P_3 containing the points

$$X(x_0 : x_1 : x_2 : x_3),\ Y(y_0 : y_1 : y_2 : y_3) \in p$$

and

$$S(s_0 : s_1 : s_2 : s_3),\ T(t_0 : t_1 : t_2 : t_3) \in q.$$

The lines p and q intersect if, and only if the points X, Y, S, and T are coplanar. The four points may be represented as

$$\mathbf{AX} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ s_0 & s_1 & s_2 & s_3 \\ t_0 & t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0,$$

where $[X_0 : X_1 : X_2 : X_3]$ are the plane coordinates of the plane in which the two lines lie. Let $\Delta(\mathbf{A})$ be the determinant of coefficient matrix $\mathbf{A}$. This system of homogeneous linear equations will have a non-trivial solution if, and only if $\Delta(\mathbf{A}) = 0$, meaning there is a linear dependency among the lines because they both share one location and therefore generate plane $\mathbf{X}$. Hence a condition for the lines to intersect is

$$\Delta(\mathbf{A}) = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ s_0 & s_1 & s_2 & s_3 \\ t_0 & t_1 & t_2 & t_3 \end{vmatrix} = 0. \tag{6.40}$$

This determinant can be calculated using the Laplacian expansion theorem, but it can also be formed in terms of Plücker line coordinates

$$p_{ik} = \begin{vmatrix} x_1 & x_k \\ y_i & y_k \end{vmatrix}, \quad q_{ik} = \begin{vmatrix} s_i & s_k \\ t_i & t_k \end{vmatrix},$$

where it can be shown that the condition for intersection can also be expressed as

$$\Delta(\mathbf{A}) = \Omega(p,q) = p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} + p_{23}q_{01} + p_{31}q_{02} + p_{12}q_{03} = 0. \tag{6.41}$$

If, given four planes and the corresponding coefficient matrix possesses rank $r = 3$, thereby satisfying the condition in Equation (6.41), then Equation (6.7) can be used to efficiently determine the location of the point of intersection.

Another way to find the point of intersection is to construct parametric equations for each line constructed as in Equation (6.9). Since the lines have a point in common, we equate the parametric equations, solve for either parameter, then determine the point coordinates. Assuming we have the Plücker line coordinates for lines p and q, the corresponding parametric equations will have the following form

$$\mathbf{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \upsilon \begin{bmatrix} p_{01} \\ p_{02} \\ p_{03} \end{bmatrix}, \tag{6.42}$$

$$\mathbf{q} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + \omega \begin{bmatrix} q_{01} \\ q_{02} \\ q_{03} \end{bmatrix}, \tag{6.43}$$

where $(p_{01} : p_{02} : p_{03})$ and $(q_{01} : q_{02} : q_{03})$ can be considered as vectors parallel to the respective lines, and X and S are position vectors of points on the respective lines that were used to compute the Plücker coordinates. It is not required for $\mathbf{p}$ and $\mathbf{q}$ to be normalised because it is sufficient that they are both vectors parallel to the direction of their respective lines.

The point of intersection occurs where $\mathbf{p} = \mathbf{q}$. At the point of intersection Equations (6.42) and (6.43) represent three equations in the two unknown parameters υ and ω. To determine the two unknowns we can use any two of the three, so without loss in generality we select the first two equations and rearrange them, similar to the derivation of Equation (6.11), as

$$\begin{bmatrix} p_{01} & -q_{01} \\ p_{02} & -q_{02} \end{bmatrix} \begin{bmatrix} \upsilon \\ \omega \end{bmatrix} = \begin{bmatrix} s_1 - x_1 \\ s_2 - x_2 \end{bmatrix}.$$

Now Cramer's rule can be used to solve for υ and/or ω

$$\begin{aligned} \upsilon &= \frac{\begin{vmatrix} s_1 - x_1 & -q_{01} \\ s_2 - x_2 & -q_{02} \end{vmatrix}}{\begin{vmatrix} p_{01} & -q_{01} \\ p_{02} & -q_{02} \end{vmatrix}}, \\ \omega &= \frac{\begin{vmatrix} p_{01} & s_1 - x_1 \\ p_{02} & s_2 - x_2 \end{vmatrix}}{\begin{vmatrix} p_{01} & -q_{01} \\ p_{02} & -q_{02} \end{vmatrix}}, \end{aligned}$$

which in turn reveals

$$\upsilon = \frac{q_{02}(x_1 - s_1) + q_{01}(s_2 - x_2)}{p_{02}q_{01} - p_{01}q_{02}}, \tag{6.44}$$

$$\omega = \frac{p_{01}(x_1 - s_1) + p_{02}(s_2 - x_2)}{p_{02}q_{01} - p_{01}q_{02}}. \tag{6.45}$$

Now either υ or ω can be used to compute $\mathbf{p}$ or $\mathbf{q}$ in either of Equations (6.42) or (6.43), which should result in identical values.

6.4.5 Intersection S of a line p with a Plane π

The line p is specified as Plücker coordinates (p_{ik}), the plane π is specified as plane coordinates $\pi[X_0 : X_1 : X_2 : X_3]$. Let $X(x_i)$, and $Y(y_i)$, $i \in \{0, 1, 2, 3\}$ be two distinct points on p and the number pair (X_0, Y_0) describe the points of intersection $S = X_0X + Y_0Y$ of line p with plane π. From the plane equation

$$\sum_{k=0}^{3} X_k x_k = 0$$

one can calculate

$$\begin{aligned}
\sum_{k=0}^{3} X_k(X_0 x_k + Y_0 y_k) &= X_0 \sum_{k=0}^{3} X_k x_k + Y_0 \sum_{k=0}^{3} X_k y_k = 0,\\
\Rightarrow \frac{X_0}{Y_0} &= \frac{\sum_{k=0}^{3} X_k y_k}{-\sum_{k=0}^{3} X_k x_k},\\
\Rightarrow s_i &= \left(\sum_{k=0}^{3} X_k y_k\right) x_i - \left(\sum_{k=0}^{3} X_k y_k\right) y_i,\\
&= \sum_{k=0}^{3} X_k (y_k x_i - y_i x_k) = \sum_{k=0}^{3} X_k p_{ik}, \quad i \in \{0,1,2,3\}.
\end{aligned}$$

Considering the homogeneous components of $S(s_0 : s_1 : s_2 : s_3)$ we have derived the very convenient formula

$$S(s_i) = \pi \cap p \Rightarrow \sum_{k=0}^{3} X_k p_{ik}, \quad i \in \{0,1,2,3\}, \quad k \neq i. \tag{6.46}$$

Note: for $k = i$, $p_{ii} = 0$ since $(y_i x_i - x_i y_i) = 0 \; \forall \, i$.

Example

Line p is specified as the Plücker coordinates $(1 : 1 : 2 : -3 : 8 : 3)$ and plane π is specified as the plane coordinates $[1 : 4 : 2 : 3]$. Determine the location of the point of intersection $S(s_i)$ of line p and plane π.

Solution

The location of the point $S(s_i)$ is determined with Equation (6.46),

$$S(s_i) = \pi \cap p \Rightarrow \sum_{k=0}^{3} X_k p_{ik}, \quad i \in \{0,1,2,3\}, \quad k \neq i.$$

The equation is used four times incrementing i for each of the four s_i, $i \in \{0,1,2,3\}$.

$$\begin{aligned}
i=0: \quad s_0 &= X_0p_{00} + X_1p_{01} + X_2p_{02} + X_0p_{03} \\
&= 0 + 4(1) + 2(1) + 3(2) \\
&= 12, \\
i=1: \quad s_1 &= X_0p_{10} + X_1p_{11} + X_2p_{12} + X_0p_{13} \\
&= 1(1) + 0 + 2(3) - 3(8) \\
&= -17, \\
i=2: \quad s_2 &= X_0p_{20} + X_1p_{21} + X_2p_{22} + X_0p_{23} \\
&= -1(1) - 4(3) + 0 - 3(3) \\
&= -22, \\
i=3: \quad s_3 &= X_0p_{30} + X_1p_{31} + X_2p_{32} + X_0p_{33} \\
&= -1(2) + 4(8) + 2(3) + 0 \\
&= 36.
\end{aligned}$$

Assembling these numbers as a set of point coordinates, the point of intersection of $\pi \cap p$ is

$$S(s_i) \quad = \quad (12 : -17 : -22 : 36).$$

6.4.6 The Plane π Determined by Point X and Line p

Consider a point in P_3 given by $X(x_0 : x_1 : x_2 : x_3)$ where X is not on line p, which is given by its axial line coordinates $\hat{p}_{ik}$. Then there exists a unique plane $\pi[X_0 : X_1 : X_2 : X_3]$. Because of the duality of Equation (6.46), the plane coordinates are given by it's dual

$$X_i = \sum_{k=0}^{3} x_k \hat{p}_{ik}, \quad i \in \{0,1,2,3\}. \tag{6.47}$$

6.4.7 Condition for Incidence of Line p and Plane π

If line $p(p_{01}, \cdots, p_{12})$ lies entirely in a plane $\pi[X_0, \cdots, X_3]$, then no unique intersection S exists. That is, the point coordinates of S are indeterminate. Hence, the incidence condition is

$$\sum_{k=0}^{3} x_k \hat{p}_{ik} = 0, \quad i \in \{0,1,2,3\}. \tag{6.48}$$

Bibliography

[1] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, England, 1985.

[2] M.L. Husty, A. Karger, H. Sachs, and W. Steinhilper. *Kinematik und Robotik*. Springer-Verlag, Berlin, Germany, 1997.

[3] J. Plücker. "Über ein neues Coordinatensystem". *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, vol. 5: pages 1–36, 1830.

[4] J. Plücker. "On a New Geometry of Space". *Philosophical Transactions of the Royal Society of London*, vol. 155: pages 725–791, 1865.

[5] H. Grassmann. *A New Branch of Mathematics,* English translation 1995 by Kannenberg, L. C. Open Court, Chicago, Ill., U.S.A., 1844.

[6] H. Pottmann and J. Wallner. *Computational Line Geometry.* Springer-Verlag Berlin Heidelberg, 2001.