

# Direct Kinematic Mapping for General Planar Parallel Manipulators

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**Abstract:** Planar kinematic mapping yields a highly compact general symbolic univariate polynomial solution to direct kinematics problems for all three legged, three degree of freedom planar parallel manipulators. Direct kinematics solutions of planar parallel robots with arbitrary mixed leg architecture are exposed for the first time. Circle and line constraints in *both* the moving frame as well as the fixed one are used as required.

**Résumé:** Tracer cinématique planaire rapporte une solution polynôme univariable symbolique générale fortement compacte aux problèmes directs de cinématique pour chacun des trois à jambes, trois degrés de manipulateurs parallèles planaires de liberté. Des solutions directes de cinématique des robots parallèles planaires avec l'architecture mélangée arbitraire de jambe sont exposées pour la première fois. Des contraintes de cercle et de ligne dans tous les *deux* la trame mobile aussi bien que la fixe sont utilisées comme exigées.

**Introduction:** Planar kinematic mapping was introduced independently by Blaschke and Grünwald in 1911 (Blaschke, 1911; Grünwald, 1911). But, their writings are difficult. In North America Roth, De Sa, Ravani (De Sa and Roth, 1981; Ravani and Roth, 1983), as well as others, have made contributions. However, we choose to build upon interpretations by Husty (1995, 1994),

who used the accessible language of Bottema and Roth (1990). Since Husty showed how to solve direct kinematics of planar three legged platform manipulators with kinematic mapping why do we reexamine what usually entails putting each vertex of a given triangle on one of three circles in the plane? There are several reasons.

Firstly, Husty's approach using arbitrary real number platform design parameters and joint variables, leads to numerical precision requirements and computations ill suited to real-time applications. Compact, manageable symbolic coefficient representation in the sixth order planar univariate polynomial is therefore required. Moreover, his approach is limited to specific symmetric architectures (the legs must be kinematically identical).

Secondly, Merlet (1996) addressed the direct kinematics problem of all possible three legged lower pair jointed planar platforms. However, because plane trigonometry is used to formulate the constraint equations, distinct architectures require distinct sets of equations, which are further dependent of platform geometry.

Thirdly, Hayes, Husty and Zsombor-Murray (1999), presented a unified approach to the direct kinematics of three legged platforms, but the resulting univariate could only be applied to those possessing topological symmetry (three kinematically identical legs).

Fourthly, although Hayes (1999) at-

tempted to formulate a single univariate to treat all symmetric and mixed leg planar architectures, he omitted some cases with free prismatic P-joints, one attached to end effector(EE), the other to base(FF). Furthermore he overlooked an important opportunity, in formulating the three constraint equations, which leads to sixth order univariate coefficients, substantially more compact than those of his reasonably-optimized computation.

In this paper, we present a single set of constraint equations that can be used to solve the direct kinematics problem of all possible three legged planar platforms possessing three degrees of freedom. We describe in great detail certain pre- and post-computational tasks, some tedious, and tabulate a great many symbolic coefficient definitions. This is done in order to make formulation and solution of constraint equations, necessary to do direct kinematics for all three-legged planar platform architectures, convenient and attractive and to show that the final symbolic univariate equation is compact and practical for purposes of computation. Finally, it is shown how to set up a typical one degree of freedom six-bar mechanism for input-output analysis. The three points on three circles paradigm is employed to demonstrate that kinematic mapping is an efficient analytic tool, especially in the case of elementary problems which are usually treated with iterative computational methods.

**Planar Mapping:** As regards the plane, recall the general homogeneous transformation of a point  $\{x : y : z\}$  in the moving, end effector EE frame measured as  $\{X : Y : Z\}$  in the fixed reference frame FF expressed in terms of the kinematic mapping image space coordinates (Bottema and Roth, 1990).

$$\lambda \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (1)$$

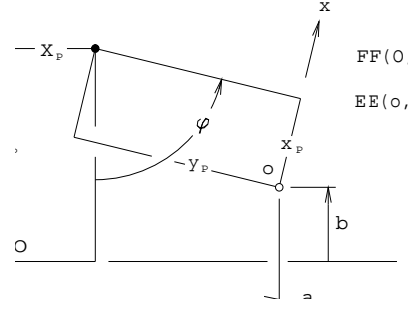


Figure 1. POLE POSITION.

where

$$\mathbf{T} = \begin{bmatrix} X_4^2 - X_3^2 & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix}$$

The inverse transformation can be obtained with the inverse of the  $3 \times 3$  matrix in Eq. (1) as follows.

$$\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{T}^{-1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (2)$$

with

$$\mathbf{T}^{-1} = \begin{bmatrix} X_4^2 - X_3^2 & 2X_3X_4 & 2(X_1X_3 - X_2X_4) \\ -2X_3X_4 & X_4^2 - X_3^2 & 2(X_2X_3 + X_1X_4) \\ 0 & 0 & X_3^2 + X_4^2 \end{bmatrix}$$

where the product of these matrices is not a unit matrix but yields

$$\begin{bmatrix} (X_3^2 + X_4^2)^2 & 0 & 0 \\ 0 & (X_3^2 + X_4^2)^2 & 0 \\ 0 & 0 & (X_3^2 + X_4^2)^2 \end{bmatrix}$$

However this is a diagonal matrix with identical elements and homogeneous coordinates may be multiplied by any value  $X_3^2 + X_4^2 \neq 0$  while preserving the unique point they represent. This argument also provides the necessary non-zero condition. Furthermore  $\lambda$ , above and below, is some non-zero constant.

**Planar Image Coordinates and Pole Position:** The kinematic mapping image coordinates are defined, with respect to  $P$ ,

the pole of a general displacement in the plane shown in Fig. 1, as follows.

$$\{X_1 : X_2 : X_3 : X_4\} = \left\{ a \sin \frac{\phi}{2} - b \cos \frac{\phi}{2} : a \cos \frac{\phi}{2} + b \sin \frac{\phi}{2} \right. \\ \left. : 2 \sin \frac{\phi}{2} : 2 \cos \frac{\phi}{2} \right\}$$

The pole position is invariant with respect to EE and FF, *i.e.*,  $X_P = x_P$  and  $Y_P = y_P$ . Thus using homogeneous coordinates  $P\{X_P : Y_P : Z_P\} \equiv P\{x_P : y_P : z_P\}$ . This means

$$P\{-b \sin \phi - a(\cos \phi - 1) : -b(\cos \phi - 1) \\ + a \sin \phi : -2(\cos \phi - 1)\}$$

Recall

$$\cos \phi = \cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2}, \quad \sin \phi = 2 \cos \frac{\phi}{2} \sin \frac{\phi}{2} \\ \cos \phi - 1 = \cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2} - 1 = -2 \sin^2 \frac{\phi}{2}$$

Then

$$P \left\{ -2 \frac{X_2 X_3 - X_1 X_4}{X_3^2 + X_4^2} 2 \cos \frac{\phi}{2} \sin \frac{\phi}{2} - 2 \frac{X_1 X_3 + X_2 X_4}{X_3^2 + X_4^2} (-2 \sin^2 \frac{\phi}{2}) \right. \\ \left. : -2 \frac{X_2 X_3 - X_1 X_4}{X_3^2 + X_4^2} (-2 \sin^2 \frac{\phi}{2}) + 2 \frac{X_1 X_3 + X_2 X_4}{X_3^2 + X_4^2} 2 \cos \frac{\phi}{2} \sin \frac{\phi}{2} \right. \\ \left. : -2(-2 \sin^2 \frac{\phi}{2}) \right\}$$

Dividing by  $4 \cos \sin \frac{\phi}{2}$  produces

$$P \left\{ -\frac{X_2 X_3 - X_1 X_4}{X_3^2 + X_4^2} + \frac{X_1 X_3 + X_2 X_4}{X_3^2 + X_4^2} \tan \frac{\phi}{2} \right. \\ \left. : \frac{X_2 X_3 - X_1 X_4}{X_3^2 + X_4^2} \tan \frac{\phi}{2} + \frac{X_1 X_3 + X_2 X_4}{X_3^2 + X_4^2} : \tan \frac{\phi}{2} \right\}$$

Since

$$\tan \frac{\phi}{2} = \frac{X_3}{X_4} \quad (3)$$

$$P \left\{ \frac{X_1 X_3 + X_2 X_4 - (X_2 X_3 - X_1 X_4) \frac{X_4}{X_3}}{X_3^2 + X_4^2} \right. \\ \left. : \frac{X_2 X_3 - X_1 X_4 + (X_1 X_3 + X_2 X_4) \frac{X_4}{X_3}}{X_3^2 + X_4^2} : 1 \right\}$$

Which simplifies to

$$P \left\{ \frac{X_1}{X_3} : \frac{X_2}{X_3} : 1 \right\} \quad (4)$$

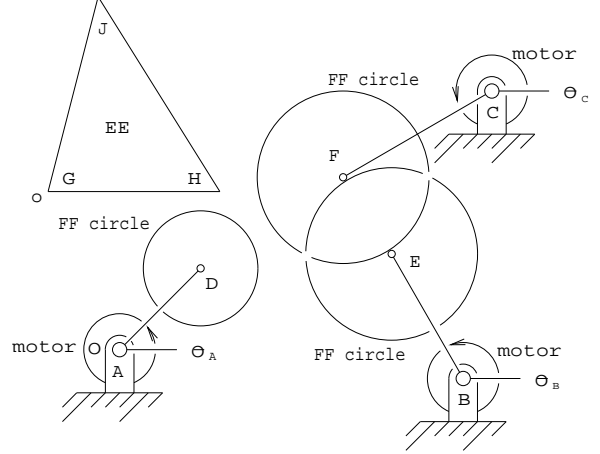


Figure 2. FIXED FRAME FF AND END EFFECTOR EE.

Or homogeneously

$$P\{X_1 : X_2 : X_3\}$$

Armed with Eqs. (3) and 4 any solution in terms of  $X_1, X_2, X_3, X_4$  can be conveniently converted to the required displacement of EE.

### Univariate Polynomials

**Legs, Circles and Points:** Consider a common platform architecture with three R-R-R jointed legs as shown in Fig. 2. Regardless of which joint is actuated, there is always an unactuated R-R joint serial dyad which renders a point on EE free to move on the circumference of a circle in FF.

**The Fixed Frame FF:** In Fig. 3 one sees the three possible actuator locations on any given leg.

1. In Fig. 3 -i-, since  $\theta_A$  is known, the fixed circle on FF is centred on  $D$  and has radius  $r_D$ , the link length spanning  $D \rightarrow G$  where  $G$  is the moving point on EE.
2. In Fig. 3 -ii-, since  $\theta_D$  is known, the fixed circle on FF is centred on  $A$  and has radius

$$r_{AG} = \sqrt{(r_A + r_D \cos \theta_D)^2 + (r_D \sin \theta_D)^2}$$

the length spanning  $A \rightarrow G$  where  $G$  is the moving point on EE.  $r_A$  is the link length  $A \rightarrow D$ .

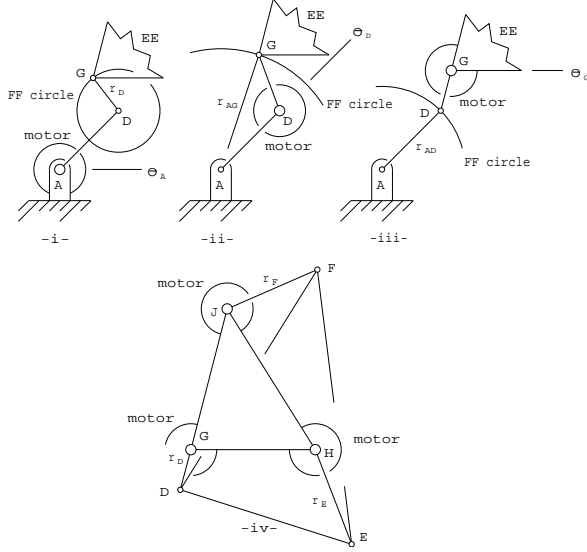


Figure 3. CIRCLES DEPEND ON ACTUATOR POSITION.

3. In Fig. 3 -iii-, since  $\theta_G$  is known, the fixed circle on FF is again centred on  $A$  and has radius  $r_{AD}$ .

In order to invoke the ideal frames referred to in Section the FF origin must be placed on  $D$  for base actuated legs like -i- above but remain on  $A$  with intermediate or platform joint actuation like -ii- and -iii-. For the example shown in Fig. 2, the centre  $E$  of the second circle in FF would have to be rotated onto the  $X$ -axis and the centre  $F$  of the third rotated by the same angle  $\psi$  so as to yield circle centre coordinates in FF. Then with  $A(0, 0)$ ,  $B(12, -1)$ ,  $C(13, 9)$ , as shown in Fig. 2, we get

$$D(0, 0), \quad E(X_E, 0), \quad F(X_F, Y_F)$$

where

$$\begin{aligned} X_E &= [(12 + r_{BE} \cos \theta_B - r_{AD} \cos \theta_A)^2 - \\ &\quad (-1 + r_{BE} \sin \theta_B - r_{AD} \sin \theta_A)^2]^{-\frac{1}{2}} \\ X_F &= (13 + r_{CF} \cos \theta_C - r_{AD} \cos \theta_A) \cos \psi \\ Y_F &= (9 + r_{CF} \sin \theta_C - r_{AD} \sin \theta_A) \sin \psi \\ \tan \psi &= \frac{9 + r_{CF} \sin \theta_C - r_{AD} \sin \theta_A}{13 + r_{CF} \cos \theta_C - r_{AD} \cos \theta_A} \end{aligned}$$

This procedure applies if all legs are base (FF -i-) actuated but requires only minor modification to accommodate intermediate (-ii-) or platform (EE -iii-) actuation.

**The Moving End Effector Frame EE:** The three EE point coordinates  $(x, y)$  for FF (-i) and intermediate joint (-ii) actuated legs are available by inspection of Fig. 2.

$$\begin{aligned} G(0, 0), \quad H(j, 0), \quad J(x_J, y_J), \\ x_J = \frac{h^2 - g^2 + j^2}{2j}, \quad y_J = \sqrt{h^2 - x_J^2} \end{aligned}$$

But all platform actuated joints move EE triangle vertices to their corresponding intermediate joint. In the example below we see three platform actuators and the EE triangle  $G H J$  is replaced by  $D E F$ .

$$\begin{aligned} x_D &= r_D \cos \theta_G, \quad y_D = r_D \sin \theta_G, \\ x_E &= j + r_E \cos \theta_H, \quad y_E = r_E \sin \theta_H, \\ x_F &= x_j + r_E \cos \theta_J, \quad y_F = y_j + r_E \sin \theta_J, \\ d &= \sqrt{(x_F - x_E)^2 + (y_F - y_E)^2}, \\ e &= \sqrt{(x_D - x_F)^2 + (y_D - y_F)^2}, \\ f &= \sqrt{(x_E - x_D)^2 + (y_E - y_D)^2} \end{aligned}$$

**Planar Constraint Equations:** Consider the case where the manipulator's three leg chains all contain unactuated R-R joint dyads. Therefore three points on EE move on three circles on FF. Substituting the three equations  $X_i = X_i(x, y, z)$  implied by Eq. (1) into the circle equation  $C_0(X^2 + Y^2) + 2C_1X + 2C_2Y + C_3 = 0$  produces a hyperboloid of one sheet in the image space, see Fig. 4.

$$\begin{aligned} C_0 z^2 (X_1^2 + X_2^2) + (-C_0 x + C_1 z) z X_1 X_3 \\ + (-C_0 y + C_2 z) z X_2 X_3 + (-C_0 y - C_2 z) z X_1 X_4 \\ + (C_0 x + C_1 z) z X_2 X_4 + (-C_1 y + C_2 x) z X_3 X_4 \\ + \frac{1}{4} [C_0 (x^2 + y^2) - 2C_1 x z - 2C_2 y z + C_3 z^2] X_3^2 \\ + \frac{1}{4} [C_0 (x^2 + y^2) + 2C_1 x z + 2C_2 y z + C_3 z^2] X_4^2 = 0 \end{aligned} \quad (5)$$

Note that setting  $C_0 = z = X_4 = 1$  while  $C_1 = -X_m$ ,  $C_2 = -Y_m$ ,  $C_3 = X_m^2 + Y_m^2 - r^2$ , the circle centre coordinates and its radius,

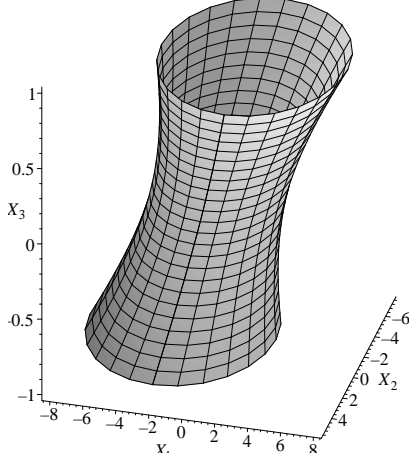


Figure 4. A HYPERBOLOID OF ONE SHEET.

produces

$$\begin{aligned}
& (X_1^2 + X_2^2) + (C_1 - x)X_1X_3 + (C_2 - y)X_2X_3 \\
& \mp (C_2 + y)X_1 \pm (C_1 + x)X_2 \pm (C_2x - C_1y)X_3 \\
& + \frac{1}{4}[(x^2 + y^2) - 2C_1x - 2C_2y + C_3]X_3^2 \\
& + \frac{1}{4}[(x^2 + y^2) + 2C_1x + 2C_2y + C_3] = 0 \quad (6)
\end{aligned}$$

where  $(x, y)$  are the coordinates of the moving point on EE with  $z = 1$  and the upper signs apply. If the constraint is intended to express the inverse, a point on FF bound to a circle in EE, then the lower signs apply and  $x, y$  or  $z$  is substituted wherever  $X, Y$  or  $Z$  appears. The inverse situation of a circle moving on a point is never required in problem formulation. However if a point is bound to a line, *i.e.*, in the case of a free prismatic leg joint and if one desires to treat all mixed leg configurations, the line may be either on FF *or* EE. Eq. (6) reduces to Eq. (7) if a point is bound to a line and  $C_0 = 0$ . This produces a hyperbolic paraboloid in the image space, see Fig. 5.

$$\begin{aligned}
& C_1X_1X_3 + C_2X_2X_3 \mp C_2X_1 \\
& \pm C_1X_2 \pm (C_2x - C_1y)X_3 \\
& - \frac{1}{4}[2C_1x + 2C_2y - C_3]X_3^2 \\
& + \frac{1}{4}[2C_1x + 2C_2y + C_3] = 0 \quad (7)
\end{aligned}$$

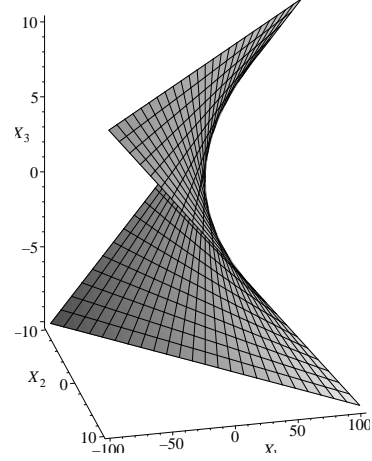


Figure 5. A HYPERBOLIC PARABOLOID.

It has been shown by Hayes and Husty (2001) that the hyperboloid of one sheet and the hyperbolic paraboloid are the only possible kinematic mapping image space constraint surfaces for all possible planar three-legged platforms.

#### Simplified Planar Image Space Constraints:

Ideal frames are used to represent fixed circles on FF and moving points on EE. The first circle is centred on the origin in FF and the second on the  $X$ -axis. The first moving point is on the origin in EE and the second on the  $x$ -axis. Therefore three hyperboloids in the kinematic map are represented by three equations of the form of Eq. (6), as follows.

$$\begin{aligned}
& X_1^2 + X_2^2 + a_6X_3^2 + a_7 = 0 \\
& X_1^2 + X_2^2 + b_1X_1X_3 + b_4X_2 + b_6X_3^2 + b_7 = 0 \\
& X_1^2 + X_2^2 + c_1X_1X_3 + c_2X_2X_3 + c_3X_1 \\
& + c_4X_2 + c_5X_3 + c_6X_3^2 + c_7 = 0
\end{aligned}$$

Eliminating  $X_1^2 + X_2^2$  from the last two and  $X_3^2$  from the third of this set reduces it to the following which is solved for  $X_3$ .

$$\begin{aligned}
& X_1^2 + X_2^2 + a_6X_3^2 + a_7 = 0 \\
& b_1X_1X_3 + b_4X_2 + e_6X_3^2 + e_7 = 0 \\
& f_1X_1X_3 + f_2X_2X_3 + f_3X_1 + f_5X_3 + f_7 \\
& = 0 \quad (8)
\end{aligned}$$

Notice that

$$\begin{aligned} e_6 &= b_6 - a_6, \quad e_7 = b_7 - a_7, \quad e_8 = c_6 - a_6, \\ e_9 &= c_7 - a_7, \quad f_1 = c_1 e_6 - b_1 e_8, \quad f_2 = c_2 e_6, \\ f_3 &= c_3 e_6, \quad f_4 = c_4 e_6 - b_4 e_8, \quad f_5 = c_5 e_6, \\ f_7 &= e_6 e_9 - e_7 e_8 \end{aligned}$$

**Planar Sixth Order Univariate:** Symbolic algebra software produces a Gröbner basis which contains a sixth order factor in  $X_3$  only. It can be abbreviated as

$$\sum A_i X_3^i, \quad i = 0, \dots, 6 \quad (9)$$

wherein the coefficients are given as

$$\begin{aligned} A_6 &= e_6^2(f_1^2 + f_2^2) + a_6 b_1^2 f_2^2 \\ A_5 &= 2[a_6 b_1 f_2(b_1 f_4 - b_4 f_1) - b_1 e_6 f_1 f_5 \\ &\quad + e_6^2(f_1 f_3 + f_2 f_4)] \\ A_4 &= a_6(b_1^2 f_4^2 + b_4^2 f_1^2) + b_1^2(a_7 f_2^2 + f_5^2) + e_6^2(f_3^2 + f_4^2) \\ &\quad - 2\{a_6 b_1 b_4(f_1 f_4 + f_2 f_3) \\ &\quad + e_6[b_1(f_1 f_7 + f_3 f_5) + b_4 f_2 f_5 - e_7(f_1^2 + f_2^2)]\} \\ A_3 &= 2\{b_4[a_6 f_3(b_4 f_1 - b_1 f_4) - e_6(f_2 f_7 + f_4 f_5)] \\ &\quad - b_1[f_1(e_7 f_5 + a_7 b_4 f_2) + e_6 f_3 f_7 \\ &\quad - b_1(f_5 f_7 + a_7 f_2 f_4)] + 2e_6 e_7(f_1 f_3 + f_2 f_4)\} \\ A_2 &= b_4^2(a_7 f_1^2 + f_5^2 + a_6 f_3^2) + b_1^2(f_7^2 + a_7 f_4^2) \\ &\quad + e_7^2(f_1^2 + f_2^2) + 2\{e_6 e_7(f_3^2 + f_4^2) \\ &\quad - b_4[a_7 b_1(f_1 f_4 + f_2 f_3) + (e_6 f_4 f_7 + e_7 f_2 f_5)] \\ &\quad - b_1 e_7(f_1 f_7 + f_3 f_5)\} \\ A_1 &= 2\{e_7^2(f_1 f_3 + f_2 f_4) - b_1 f_3(e_7 f_7 + a_7 b_4 f_4) \\ &\quad + b_4[b_4(f_5 f_7 + a_7 f_1 f_3) - e_7(f_2 f_7 + f_4 f_5)]\} \\ A_0 &= (e_7 f_4 - b_4 f_7)^2 + f_3^2(e_7^2 + a_7 b_4^2) \quad (10) \end{aligned}$$

**Backsubstitution:** Once the six values of  $X_3$  are available by numerically solving Eq. (9) one must ensure that each is matched with a unique pair of  $X_1$  and  $X_2$ .

**Planar Direct Kinematic Backsubstitution:** The system of three constraint equations in the image space appears in Eqs. (8). Having solved Eq. (9), using coefficients obtained with Eqs. (10),  $X_1$  and  $X_2$  are found for each value of  $X_3$  with the last two of Eqs. (8) which are now linear because  $X_3$  has been absorbed into the coefficients. These become

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13} &= 0, \\ a_{21}X_1 + a_{22}X_2 + a_{23} &= 0 \end{aligned}$$

where

$$\begin{aligned} a_{11} &= b_1 X_3, \quad a_{12} = b_4, \quad a_{13} e_6 X_3^2 + e_7 \\ a_{21} &= f_1 X_1 + f_3, \quad a_{22} f_2 X_3 + f_4, \\ a_{23} &= f_5 X_3 + f_7 \end{aligned}$$

**General Backsubstitution:** One is not always so lucky. There is no immediately obvious path to linear backsubstitution when dealing with a general system of three simultaneous quadrics with two remaining unknowns. For this reason it is worthwhile to examine a simple elimination procedure which leads to desired results. Here is the system in  $X_1$  and  $X_2$  with some value of  $X_3$  absorbed in the coefficients.

$$a_{ij} \begin{bmatrix} X_1^2 \\ X_2^2 \\ X_1 X_2 \\ X_1 \\ X_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$a_{11}X_1^2 + a_{12}X_2^2 + a_{13}X_1X_2 + a_{14}X_1 + a_{15}X_2 + a_{16} = 0 \quad (11)$$

$$a_{21}X_1^2 + a_{22}X_2^2 + a_{23}X_1X_2 + a_{24}X_1 + a_{25}X_2 + a_{26} = 0 \quad (12)$$

$$a_{31}X_1^2 + a_{32}X_2^2 + a_{33}X_1X_2 + a_{34}X_1 + a_{35}X_2 + a_{36} = 0 \quad (13)$$

First  $a_{21}$ ,  $a_{31}$ ,  $a_{32}$  are eliminated from Eqs. 12 and 13 to produce Eqs. 14 and 15.

$$b_{22}X_2^2 + b_{23}X_1X_2 + b_{24}X_1 + b_{25}X_2 + b_{26} = 0 \quad (14)$$

$$c_{33}X_1X_2 + c_{34}X_1 + c_{35}X_2 + c_{36} = 0 \quad (15)$$

where

$$\begin{aligned} b_{2j} &= a_{21}a_{1j} - a_{11}a_{2j}, \quad j = 2, \dots, 6 \\ c_{3j} &= b_{32}b_{2j} - b_{22}b_{3j}, \quad j = 3, \dots, 6 \\ b_{3j} &= a_{31}a_{1j} - a_{11}a_{3j}, \quad j = 2, \dots, 6 \end{aligned}$$

Substituting for  $X_1$  in Eqs. 11 and 14 with results from Eq. 15 yields a quartic, Eq. 16, and a cubic, Eq. 17, univariate in  $X_2$ .

$$d_4X_2^4 + d_3X_2^3 + d_2X_2^2 + d_1X_2 + d_0 = 0 \quad (16)$$

$$e_3X_2^3 + e_2X_2^2 + e_1X_2 + e_0 = 0 \quad (17)$$

where

$$\begin{aligned} d_4 &= a_{12}c_{33}^2, \\ d_3 &= (2a_{12}c_{34} - a_{13}c_{35} + a_{15}c_{33})c_{33} \\ d_2 &= (a_{16}c_{33} - a_c + 2a_{15}c_{34})c_{33} + \\ &\quad (a_{12}c_{34} - a_{13}c_{35})c_{34} + a_{11}c_{35}^2 \\ d_1 &= (a_{15}c_{34} - a_c + 2a_{16}c_{33})c_{34} + \\ &\quad (2a_{11}a_{35} - a_{14}c_{33})c_{36} \\ d_0 &= (a_{11}c_{36} - a_{14}c_{34})c_{36} + a_{16}c_{34}^2, \\ a_c &= a_{14}c_{35} + a_{13}c_{36} \\ e_3 &= b_{22}c_{33}, \\ e_2 &= b_{22}c_{34} - b_{23}c_{35} + b_{25}c_{33} \\ e_1 &= b_{25}c_{34} + b_{26}c_{33} - b_{23}c_{36} - b_{24}c_{35}, \\ e_0 &= b_{26}c_{34} - b_{24}c_{36} \end{aligned}$$

Raising Eq. 17 to a quartic and forming the difference with Eq. 16 gives another cubic.

$$f_3X_2^3 + f_2X_2^2 + f_1X_2 + f_0 = 0 \quad (18)$$

where

$$\begin{aligned} f_3 &= e_3d_3 - d_4e_2, \quad f_2 = e_3d_2 - d_4e_1, \\ f_1 &= e_3d_1 - d_4e_0, \quad f_0 = e_3d_0 \end{aligned}$$

The difference between Eqs. 17 and 18 eliminates  $X_2^3$  in Eq. 19.

$$g_2X_2^2 + g_1X_2 + g_0 = 0 \quad (19)$$

where

$$\begin{aligned} g_2 &= f_3e_2 - e_3f_2, \quad g_1 = f_3e_1 - e_3f_1, \\ g_0 &= f_3e_0 - e_3f_0 \end{aligned}$$

Then  $X_2^3$  is eliminated by raising Eq. 19 to a cubic and forming the appropriate difference with Eq. 18.

$$h_2X_2^2 + h_1X_2 + h_0 = 0 \quad (20)$$

where

$$h_2 = g_2f_2 - f_3g_1, \quad h_1 = g_2f_1 - f_3g_0, \quad h_0 = g_2f_0$$

Finally the required equation, Eq. 21, is obtained with the difference between Eqs. 19 and 20.

$$(h_2g_1 - g_2h_1)X_2 + (h_2g_0 - g_2h_0) = 0 \quad (21)$$

The last remaining step is to get  $X_1$  with Eq. 15. This guarantees that evaluating a unique pair  $X_1$  and  $X_2$  will cost no more than 141 FLOPS.

**Conclusion:** Rather than extol the virtues of kinematic mapping and the contributions claimed to have been made herein, an application exercise, which may be appreciated and understood by all, is described instead. Consider Fig. 6 which shows a four-bar planar mechanism driven through a five-bar loop with an input angle given as  $\theta$ . Furthermore all link length parameters are specified too. The problem is to find the pose of the coupler  $GHJ$ . One may immediately recognize this as planar three-leg manipulator where the platform points,  $G$ ,  $H$  and  $J$ , all in a line in this case, move on circles of radius 8, 12 and 16, respectively. For the first circle we have

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = -8^2, \quad x = 0, \quad y = 0$$

For the second,

$$\begin{aligned} C_1 &= -20, \quad C_2 = 0, \quad C_3 = 20^2 - 12^2, \\ x &= 14, \quad y = 0 \end{aligned}$$

Then the third circle parameters are

$$\begin{aligned} C_1 &= 6 - 5 \cos 58.33^\circ, \\ C_2 &= -14 - 5 \sin 58.33^\circ, \\ C_3 &= C_1^2 + C_2^2 - 16^2, \quad x = 6, \quad y = 0 \end{aligned}$$

With Eq. (6) we compute the  $a$ -,  $b$ - and  $c$ -coefficients. Then with the expressions below Eq. (8) the  $d$ - and  $e$ -coefficients are obtained. Finally, Eqs. (10) produce the seven coefficients of the sextic univariate of Eq (9) which is solved numerically for  $X_3$ . Section gives

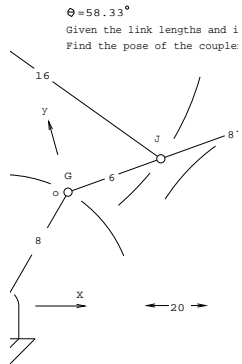


Figure 6. A ONE DEGREE OF FREEDOM MECHANISM.

the means to get  $X_1$  and  $X_2$ . Eqs. (3), 4 are all that is necessary to provide the means to move the coupler, from its “home” position, which is horizontal,  $G$  on  $O$  and  $H$  to the right, to its desired assembly mode(s). It is hoped that with this simple example, which many of us have tediously solved in class by plotting the trajectory of point  $J$ , under the motion of coupler  $GH$ , and intersecting it with the third circle, we have managed to expose the value of a kinematic mapping approach.

An example solving the direct kinematics of a two degree of freedom device is illustrated in Fig. 7. The mechanism’s assemblies can be solved with the three-legged platform direct kinematics using image space kinematic mapping. It has been set up in ideal  $EE(ox)$  and  $FF(OX)$  frames.

## References

[1] Blaschke, W., “Euklidische Kinematik und nichteuklidische Geometrie”, *Zeitschr. Math. Phys.*, Vol. 60, pp. 61-91 and 203-204, 1911.

[2] Bottema, O. & Roth, B., *Theoretical Kinematics*, Dover, ISBN 0-486-66346-9, ch.XI, pp.393-445.

[3] Gosselin, C., Sefrioui, J., “Polynomial Solutions For the Direct Kinematic Problem of Planar Three-Degree-of-Freedom Parallel Manipulators”, *Proc. 5th Int. Conf. on Adv. Rob. (ICAR)*, Pisa, Italy, pp. 1124-1129, 1991.

[4] Grünwald, J., “Ein Abbildungsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen

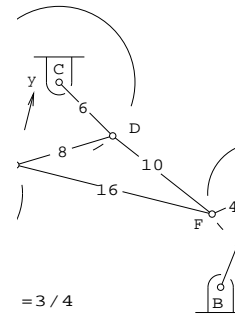


Figure 7. A TWO DEGREE OF FREEDOM MECHANISM.

Geometrie verknüpft”, *Sitzber. Ak. Wiss. Wien*, Vol. 120, pp. 677-741, 1911.

[5] Bulca, F. & Husty, M.L., “Kinematic Mapping of Spherical Three-Legged Platforms”, *Proc. of 15th Cdn. Conf. of Appl. Mech.*, Victoria, 95-05-28→06-01, v.2, ISBN 0920049-06, pp. 874-875.

[6] Daniali, H.R.M, *Contribution to the Kinematic Synthesis of Parallel Manipulators*, Ph.D. thesis, McGill University, 1995, pp.141-142.

[7] De Sa, S., Roth, B., “Kinematic Mappings. Part 1: Classification of Algebraic Motions in the Plane”, *ASME, J. of Mech. Design*, Vol. 103, pp. 585-591, 1981.

[8] Hayes, M.J.D., “Kinematics of General Planar Stewart-Gough Platforms”, Ph.D. Thesis, McGill University, 1999, pp.87-152.

[9] Hayes, M.J.D., Husty, M.L. & Zsombor-Murray, “Kinematic Mapping of Planar Stewart-Gough Platforms”, *Proc. of 17th Cdn. Conf. of Appl. Mech.*, Hamilton, 99-05-30→06-03, pp. 319-320.

[10] Hayes, M.J.D., Husty, M.L., “On the Kinematic Constraint Surfaces of General Three-Legged Planar Robot Platforms”, submitted to *Mechanism and Machine Theory*, 2001.

[11] Husty, M.L., “Kinematic Mapping of Planar Tree(sic.)-Legged Platforms”, *Proc. of 15th Cdn. Conf. of Appl. Mech.*, Victoria, 95-05-28→06-01, v.2, ISBN 0920049-06, pp. 876-877.

[12] Husty, M.L., “An Algorithm for Solving the Direct Kinematic of Stewart-Gough-Type Platforms”, McGill Centre for Intelligent Machines Technical Report, 1994, 25pp.

[13] Merlet, J-P., “Direct Kinematics of Planar Parallel Manipulators”, *IEEE Int. Conf. on Robotics and Automation*, Minneapolis, U.S.A., pp. 3744-3749, 1996.

[14] Ravani, B., Roth, B., “Motion Synthesis Using Kinematic Mappings”, *ASME, J. of Mechanisms, Transmissions, & Automation in Design*, Vol. 105, pp. 460-467, 1983.