Design Parameter Space of Spherical Four-bar Linkages

M.J.D. Hayes[1], **M. Rotzoll**[2], **C. Ingalls**[3], **and M. Pfurner**[4]

Abstract. Four link twist angles are the design parameters for spherical 4R linkages: changing the magnitudes of the twist angles changes the motion characteristics of the linkage. A new quartic algebraic input-output equation for spherical four-bar linkages, obtained in another paper, contains four terms which each factor into pairs of distinct cubics in the link twist parameters. These eight cubic factors possess a symmetry that suggest they combine to form a shape that, at least locally, bears a remarkable resemblance to a pair of dual tetrahedra in the design parameter space of the link twists. In this paper we show that the location of points relative to the eight distinct cubic surfaces implies a complete classification scheme for all possible spherical 4R linkages. Moreover, we show that the design parameter spaces of both the spherical and planar 4R linkages, with suitable scaling, intersect in 12 lines which form the 12 edges of a pair of dual tetrahedra.

Key words: Spherical four-bar linkages; design parameter space; uniform polyhedral compound.

1 Introduction

Over the millennia four-bar linkages have become ubiquitous, with applications ranging from aircraft landing gear deployment systems to beer bottle cap clamps. One might, however naïvely, be led to the conclusion that all is known. Nonetheless, commencing with the ground breaking work of Ferdinand Freudenstein in the 1950s [5], new discoveries and new insight continue to be obtained, often with surprising results. See [10] for a comprehensive collection of detailed examples and results offered by a vast array of investigators over the last 175 years.

The algebraic input-output (IO) equation for any planar four-bar linkage is a polynomial equation in the variable input link (driver) and output link (follower) angle parameters expressed in terms of the link lengths. Because the link lengths impose mobility constraints on the input and output links, they are considered de-

---

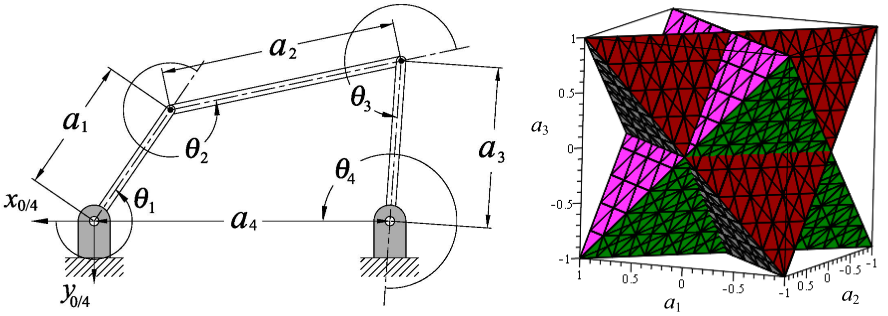
[1] Mech. and Aero. Engineering, Carleton University, Canada, e-mail: john.hayes@carleton.ca
[2] Mech. and Aero. Engineering, Carleton University, Canada, e-mail: mirja.rotzoll@carleton.ca
[3] School of Math. and Stats., Carleton University, Canada, e-mail: colin.ingalls@carleton.ca
[4] Unit Geometry and Surveying, University of Innsbruck, e-mail: martin.pfurner@uibk.ac.at

Table 1 Denavit-Hartenberg parameters for a planar 4R chain.

joint axis i	link length a_i	link angle θ_i	link offset d_i	link twist τ_i
1	a_1	θ_1	0	0
2	a_2	θ_2	0	0
3	a_3	θ_3	0	0
4	a_4	θ_4	0	0

sign parameters. Since the coupler motion is embedded in the polynomial, the IO equation is well suited to function generation synthesis. Moreover, it is an algebraic equation so the theory of algebraic geometry [2] can be applied to reveal characteristics of the IO relationship that may otherwise be occluded by trigonometry.

Individual link coordinate systems are assigned according to the original Denavit-Hartenberg (DH) convention [4]. Link parameters of length, a_i, joint angle, θ_i, link offset, d_i, and link twist angle, τ_i, are all defined relative to these coordinate systems. For a planar 4R linkage the design parameters are the four link lengths, a_1, a_2, a_3, and a_4, see Fig. 1(a), because the relative lengths determine the mobility capability of the linkage. The relative angles between the links θ_1, θ_2, θ_3, and θ_4, are variables in the IO equation. The link offsets and twist angles are all identically zero, see Table 1. Note that the base coordinate system illustrated in Fig. 1(a) is an artifact of the method used to derive the algebraic IO equation, see [13] for the details. Regardless, only the coincident origins and directions of the $z_{0/4}$-axes are fixed by the DH convention while the direction of the coincident $x_{0/4}$-axes are rotated by π radians compared to the usual representation, and the $y_{0/4}$-axes complete the two coincident right-handed coordinate systems.



(a) (b)

Fig. 1 Planar 4R chain and associated design parameter tetrahedra.

The algebraic IO equation for a planar 4R linkage is a planar quartic curve in the IO angle parameters $v_1 = \tan\theta_1/2$ and $v_4 = \tan\theta_4/2$ [6]. The design parameters are embedded in four quadratic terms that are each comprised of two factors that are linear sums and differences of link lengths. The algebraic IO equation, as derived in [13], is

$$Av_1^2v_4^2+Bv_1^2+Cv_4^2-8a_1a_3v_1v_4+D=0, \tag{1}$$

where

$$\begin{aligned} A &= (a_1-a_2+a_3-a_4)(a_1+a_2+a_3-a_4) &= A_1A_2,\\ B &= (a_1+a_2-a_3-a_4)(a_1-a_2-a_3-a_4) &= B_1B_2,\\ C &= (a_1-a_2-a_3+a_4)(a_1+a_2-a_3+a_4) &= C_1C_2,\\ D &= (a_1+a_2+a_3+a_4)(a_1-a_2+a_3+a_4) &= D_1D_2. \end{aligned}$$

The overall scale of the linkage is irrelevant since we are dealing with function generators. Without loss in generality, we can normalise the four link lengths by a_4, the distance between the centres of the two ground fixed R-pairs, thereby setting $a_4 = 1$. Projected into this hyperplane, the remaining three lengths can be used to establish three mutually orthogonal basis vectors. The eight symmetric linear factors, having the form $(a_1 \pm a_2 \pm a_3 \pm 1)$, can be considered as eight planes in the a_i for the eight permutations in sign. These eight planes intersect in the 12 edges of a pair of dual regular tetrahedra [7] while the plane segments bounded by the 12 edges are the tetrahedra faces, see Fig. 1(b).

These two tetrahedra belong to the only uniform polyhedral compound, called the stellated octahedron, which has order 48 octahedral symmetry [3]. This double tetrahedron has a regular octahedron at its core and shares its eight vertices with the cube [3]. Distinct points in this design parameter space represent distinct function generators and the locations of the points relative to the eight planes containing the faces of the double tetrahedron completely determine the mobility of the input and output links. There are 27 types of mobility conditions, determined using the techniques found in [7, 11], which depend on the signs of the sums of lengths in the three terms A_1, B_1, and C_1 from Eq. (1).

The focus of this paper is the design parameter space corresponding to spherical 4R linkages. Thus, the quartic algebraic IO equation for spherical 4R mechanisms, as derived in [13], is manipulated to examine the design parameter space implied by the magnitudes of the link twist angle parameters defined as $\alpha_i = \tan(\tau_i/2)$, where τ_i specifies the twist angles according to the original Denavit-Hartenberg convention [4]. For a spherical 4R the design parameters are therefore the four link twist angle parameters, α_i, while the relative link angles are the four variable θ_i. The link lengths and offsets are identically zero, see Table 2. In comparison with the design parameter space of planar 4R mechanisms [7] we see some startling similarities. But first, the spherical 4R algebraic IO equation requires some discussion.

Table 2 DH parameters a spherical 4R chain.

joint axis i	link length a_i	link angle θ_i	link offset d_i	link twist τ_i
1	0	θ_1	0	τ_1
2	0	θ_2	0	τ_2
3	0	θ_3	0	τ_3
4	0	θ_4	0	τ_4

2 The Spherical 4R Algebraic IO Equation

The R-pair axes of a spherical 4R mechanism all intersect at the centre of the sphere. Those of a planar 4R mechanism are all parallel; they can be thought of as intersecting in a common point at infinity of the projective extension of the Euclidean plane of the planar 4R. As shown in [9, 13], this means that the planar 4R mechanism is a special case of the spherical 4R. In the limit, as the radius of the sphere tends towards infinity, the algebraic IO equations of the spherical and planar 4R mechanisms are projectively equivalent. This suggests that there should be some similarities between the respective design parameter spaces.

A new and general method for deriving an algebraic form of the spherical 4R mechanism IO equation is presented in [13]. This method, using Study's kinematic mapping [1, 15], can also be used to derive the algebraic IO equation for planar 4R mechanisms, and we are working towards applying it to spatial linkages. Regardless, the algebraic IO equation for spherical 4R's has the form

$$Av_1^2v_4^2+Bv_1^2+Cv_4^2+8\alpha_1\alpha_3\left(\alpha_4^2+1\right)\left(\alpha_2^2+1\right)v_1v_4+D=0, \tag{2}$$

where

$$\begin{aligned}
A &= (\alpha_1\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4-\alpha_2\alpha_3\alpha_4+\alpha_1-\alpha_2+\alpha_3-\alpha_4)\\
&\quad (\alpha_1\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_4-\alpha_1\alpha_3\alpha_4-\alpha_2\alpha_3\alpha_4-\alpha_1-\alpha_2-\alpha_3+\alpha_4),\\
B &= (\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4-\alpha_1\alpha_3\alpha_4-\alpha_2\alpha_3\alpha_4+\alpha_1+\alpha_2-\alpha_3-\alpha_4)\\
&\quad (\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4-\alpha_2\alpha_3\alpha_4-\alpha_1+\alpha_2+\alpha_3+\alpha_4),\\
C &= (\alpha_1\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_4-\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4-\alpha_1+\alpha_2+\alpha_3-\alpha_4)\\
&\quad (\alpha_1\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_1+\alpha_2-\alpha_3+\alpha_4),\\
D &= (\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\\
&\quad (\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4-\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_1-\alpha_2+\alpha_3+\alpha_4).
\end{aligned}$$

In this equation the joint angle parameters are $v_i = \tan\theta_i/2$, where the IO angle parameter pair are v_1 and v_4, while the four link twist angle parameters are $\alpha_i = \tan\tau_i/2$. The link twist angles, τ_i, are defined using the original Denavit-Hartenberg assignment convention [4]. It can be shown that Eq. (2) is identical to the corresponding trigonometric IO equation for spherical four-bar linkages found in [11].

2.1 Interpreting the Spherical 4R Algebraic IO Equation

Analysing Eq. (2) using the theory of planar algebraic curves [12] one can see that it has characteristics which are independent of the constant design parameters α_i.

Clearly, Eq. (2) is of degree $n = 4$ in variables v_1 and v_4. It is also of interest to determine the planar curve's double, or singular, points: locations where the curve self-intersects. To identify the double points of Eq. (2) it must first be homogenised. We arbitrarily select w to be the homogenising coordinate, which gives

$$k_h : \; Av_1^2v_4^2 + Bv_1^2w^2 + Cv_4^2w^2 + 8\alpha_1\alpha_3(\alpha_4^2+1)(\alpha_2^2+1)v_1v_4w^2 + Dw^4 \;=\; 0. \tag{3}$$

The double points are revealed by the locations where the Jacobian ideal vanishes [12]. This ideal is generated by

$$\left\langle \frac{\partial k_h}{\partial v_1}, \frac{\partial k_h}{\partial v_4}, \frac{\partial k_h}{\partial w} \right\rangle. \tag{4}$$

Solving the system of four equations implied by Eq.s (3, 4) for v_1, v_4, and w reveals two double points located at infinity along the v_1- and v_4-axes, which exactly mirrors the results reported in [8] for planar 4R mechanisms:

$$(v_1 : v_4 : w) \;=\; (1:0:0);\;\; (0:1:0). \tag{5}$$

These two double points are common to all algebraic IO curves for every spherical 4R four-bar mechanism. Each of these double points can have real or complex tangents depending on the values of the four constant link twist parameters, α_i, which in turn determines the nature of the mobility of the input and output links.

The discriminant of Eq. (3), evaluated at a double point, reveals whether that double point has a pair of real or complex conjugate tangents [2] in turn yielding information about the topology of the mechanism [8]. The discriminant and the meaning of its value are [2]

$$\Delta = \left(\frac{\partial^2 k_h}{\partial v_i \partial w}\right)^2 - \frac{\partial^2 k_h}{\partial v_i^2}\frac{\partial^2 k_h}{\partial w^2} \begin{cases} > 0 \Rightarrow \text{two real distinct tangents (crunode)}, \\ = 0 \Rightarrow \text{two real coincident tangents (cusp)}, \\ < 0 \Rightarrow \text{two complex conjugate tangents (acnode)}. \end{cases}$$

For the homogeneous IO equation of an arbitrary spherical 4R linkage, Eq. (3), the discriminant of the point at infinity $(v_1 : v_4 : w) = (1:0:0)$ on the v_1-axis is obtained by setting $i = 4$ in the discriminant equation, i.e. ∂v_4, while the discriminant of the other point at infinity on the v_4-axis is obtained by setting $i = 1$ in the discriminant equation, i.e. ∂v_1, giving

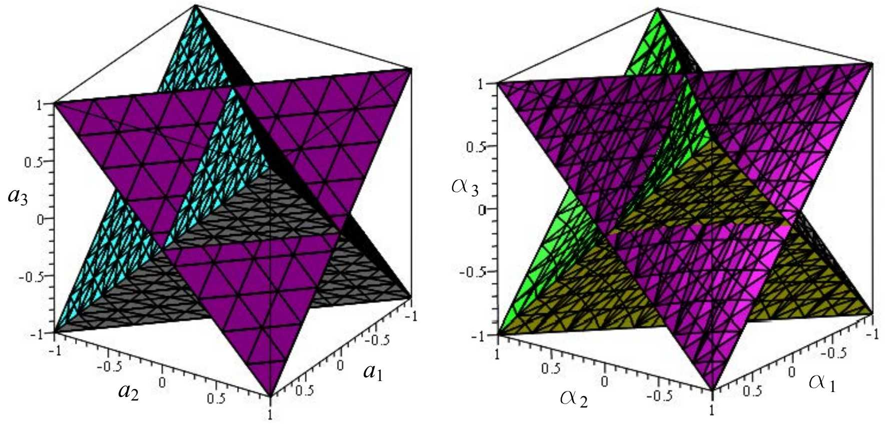
$$\Delta_{v_1} = -4AB, \quad \Delta_{v_4} = -4AC. \tag{6}$$

Since the signed numerical values of Eq. (6) depend on the products and sums of link twist angle parameters their values may be either greater than, less than, or identically equal to zero. Certainly, the classification of the mobility of the input and output links is determined by these values.

Finally, because an equation of degree $n = 4$ can have a maximum of three double points, the algebraic IO equation possesses genus 1 since it has only two. Because of this, it cannot be parameterised by rational functions, and is defined to be an *elliptic* curve [12]. Moreover, since the curve has genus 1 for every spherical 4R linkage, there are, at most, two assembly modes roughly corresponding to the “elbow-up” and “elbow-down” configurations [8].

3 Spherical 4R Design Parameter Space

The eight factors in the four coefficients A, B, C, and D in Eq. (2) are cubics in the α_i design constant twist angle parameters and have an intoxicating symmetric structure. When α_1, α_2, and α_3 are projected into the hyperplane $\alpha_4 = 1$ for a spherical 4R function generator, we can treat the three twist angle parameters α_1, α_2, and α_3 as mutually orthogonal basis vector directions. Figs. 2(a) and (b) illustrate the eight factors in each of the planar and spherical 4R algebraic IO equations where the surfaces are plotted in the ranges $a_i = \pm 1$ and $\alpha_i = \pm 1$ in the respective projections $a_4 = \alpha_4 = 1$. The planar 4R surface is a regular double tetrahedron with the special property of being the only uniform polyhedral compound [3]. The eight spherical 4R cubic surfaces have the appearance of being a double tetrahedron in the range of $\alpha_i = \pm 1$, but they are not planar and therefore are not tetrahedron faces.



(a) (b)

Fig. 2 Design parameter space surfaces: (a) planar 4R; (b) spherical 4R.

Cubic surfaces have fascinated mathematicians for several centuries. Clearly, the eight cubic factors in Eq. (2) possess some special properties. The first cubic factor in coefficient A from Eq. (2), which we will name A_1, after normalising with α_4, can be homogenised with coordinate w to reveal

$$A_{1,h} : \ \alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2 w + \alpha_1\alpha_3 w - \alpha_2\alpha_3 w + \alpha_1 w^2 - \alpha_2 w^2 + \alpha_3 w^2 - w^3. \tag{7}$$

The double points for this cubic are revealed by the locations of where the Jacobian ideal generated by

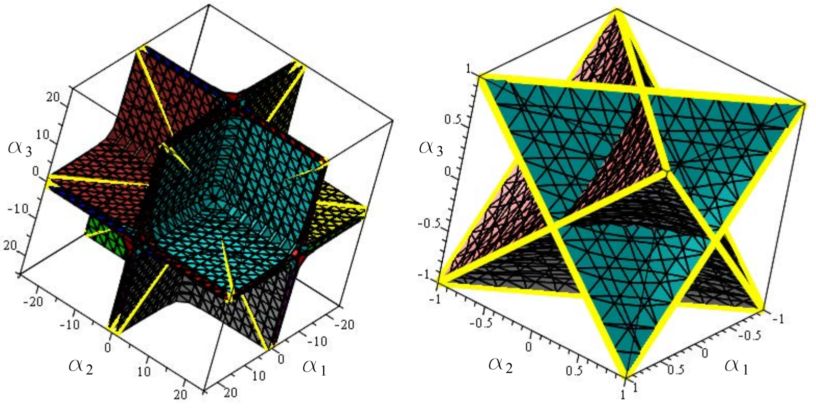
$$\left\langle \frac{\partial A_{1,h}}{\partial \alpha_1}, \frac{\partial A_{1,h}}{\partial \alpha_2}, \frac{\partial A_{1,h}}{\partial \alpha_3}, \frac{\partial A_{1,h}}{\partial w} \right\rangle \tag{8}$$

vanishes. It turns out that all eight cubics share the same three double points, namely

$$(\alpha_1 : \alpha_2 : \alpha_3 : w) = (1:0:0:0);\ (0:1:0:0);\ (0:0:1:0). \tag{9}$$

The discriminant evaluated at each of the three double points, common to all eight cubics, is $\Delta = 4$ for each double point. Since this discriminant is always greater than zero, the double points are all ordinary, or crunodes [2], because there are two distinct, real tangents at each double point. Alternately, we observe that each cubic surface meets the plane at infinity in the three lines $\alpha_1 = \alpha_2 = \alpha_3 = 0$. The double points are the vertices of this triangle. It can be shown that the two lines through each vertex are in the tangent singular cone at the vertex, and because the Hessian of $A_{1,h}$ is non-zero at each vertex then each one is an ordinary double point.

It is well known that cubic surfaces can contain as many as 27 lines [14]. It is also shown in [14] that a cubic surface possessing three ordinary double points can have, at most, 12 lines. The procedure for determining the lines is not particularly germane to this paper, nonetheless it can be shown that of these 12 lines six are complex and six are real. Of the six real lines three are at infinity. The remaining three lines on each surface intersect each other in an equilateral triangle. Moreover, different pairs of the cubics share a line, meaning that there are only 12 distinct finite lines among the eight cubics. The set of 12 distinct lines on each of the eight surfaces intersect to form the 12 edges of a double tetrahedron! This double tetrahedron can be regarded as the intersection of the planar and spherical 4R design parameter spaces. Treating the α_i as directed distances, each distinct point in this space determines a unique function generator, as well as the mobility of it's input, and output links.



(a) (b)

Fig. 3 12 distinct lines, three on each of eight cubics: (a) zoomed out; (b) zoomed in.

4 Conclusions

In this paper we have shown that there is a profound relationship between the design parameter spaces of planar and spherical 4R linkages. Indeed, if we ignore the difference between units of length for the a_i and measures of angle for the α_i and simply consider the magnitudes, we see that the design parameter spaces of planar and spherical 4R linkages intersect in the edges of the only uniform polyhedral compound. It is called the stellated octahedron, which has order 48 octahedral symmetry: a regular double tetrahedron that intersects itself in a regular octahedron. We believe that there is something of remarkable beauty in this new and elegant result: the design parameter spaces of these two classes of mechanism intersect along the edges of the only uniform polyhedral compound in the universe of polyhedra!

References

1. Bottema, O., Roth, B.: Theoretical Kinematics. Dover Publications, Inc., New York, N.Y. (1990)
2. Cipolla, R., Giblin, P.: Visual Motion of Curves and Surfaces. Cambridge University Press, Cambridge, U.K. (2000)
3. Coxeter, H.S.M.: Regular Polytopes, 3^{rd} Edition. Dover Publications, Inc., New York, N.Y., U.S.A. (1973)
4. Denavit, J., Hartenberg, R.S.: A Kinematic Notation for Lower-pair Mechanisms Based on Matrices. Trans ASME J. Appl. Mech. **23**, 215221 (1955)
5. Freudenstein, F.: "An Analytical Approach to the Design of Fourlink Mechanisms.". Trans. ASME **vol 77**, pages 483–492 (1954)
6. Hayes, M.J.D., Husty, M.L., Pfurner, M.: "Input-output Equation for Planar Four-bar Linkages". pp. 12–19. *16^{th} Advances in Robotic Kinematics*, eds. Lenarčič, J. and Parenti-Castelli, V., Springer, New York (2018)
7. Hayes, M.J.D., Rotzoll, M., Husty, M.L., Pfurner, M.: "Design Parameter Space of Planar Four-bar Linkages". Proceedings of the 15^{th} IFToMM World Congress (June 30-July 4, 2019)
8. Husty, M.L., Pfurner, M.: "An Algebraic Version of the Input-Output Equation of Planar Four-Bar Mechanisms". pp. 746–757. *International Conference on Geometry and Graphics*, Milan, Italy (2018)
9. McCarthy, J.M.: "Planar and Spatial Rigid Motion as Special Cases of Spherical and 3-Spherical Motion". Journal of Mechanisms, Transmissions, and Automation in Design **105**(3), 569–575 (1983)
10. McCarthy, J.M., Soh, G.S.: Geometric Design of Linkages, 2nd Edition *Interdiciplinaty Applied Mathematics*. Springer, New York, N.Y. (2011)
11. Murray, A.P., Larochelle, P.M.: "A Classification Scheme for Planar 4*R*, Spherical 4*R*, and Spatial *RCCR* Linkages to Facilitate Computer Animation". *Proceedings of 1998 ASME Design Engineering Technical Conferences (DETC'98)*, Atlanta, Georgia, U.S.A. (September 13-16, 1998)
12. Primrose, E.: Plane Algebraic Curves. MacMillan (1955)
13. Rotzoll, M., Hayes, M.J.D., Husty, M.L., Pfurner, M.: "A General Method for Determining Algebraic Input-output Equations for Planar and Spherical 4R Linkages". Accepted for publication in the *17^{th} International Symposium: Advances in Robotic Kinematics*, Ljubljana, Slovenia (June 28-July 2, 2020)
14. Segre, B.: The Non-singular Cubic Surfaces; a New Method of Investigation with Special Reference to Questions of Reality. Oxford, The Clarendon Press (1942)
15. Study, E.: Geometrie der Dynamen. Teubner Verlag, Leipzig, Germany (1903)