Elementary Kinematics from an Advanced Standpoint

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1. INTRODUCTION

At the beginning of the 20$^{\text{th}}$ century the celebrated German mathematical physicist Felix Klein (1849-1925) presented a series of lectures to high-school math and science teachers during the Christmas and Easter periods when there were no classes. The intent was to make the teachers aware of the necessity for them to be familiar with their subjects at the highest possible level of authority and rigor so as to be able to explain the material in the clearest, most effective way. His lecture notes were later published as books and translated to English [1, 2]. The two volumes are called "Elementary Mathematics from an Advanced Standpoint" and subtitled "Geometry" and "Arithmetic, Algebra and Analysis", respectively. These continue to provide inspiration. It is in this vein that the following simple example in planar kinematics is treated using the projective geometry of point, line and conic and a common substitution to convert trigonometric to algebraic functions.

2. INCLINED "SCOTCH YOKE"

Figure 1 shows a "Scotch yoke" mechanism wherein a slotted slider is driven by crank-pin P which rotates about centre O. The block which slides in the slot is attached to the crank-pin with a revolute joint. Distance OP is r. The crank angle is θ measured counterclockwise positive from a line o on O and in the direction of slider displacement. The motion of P relative to the slot is along line b inclined at angle β to o. This mechanism can be described as an RRPP closed, single loop chain; a four-bar mechanism variant. Consider the following aspects of the kinematics of this device.

- What is the displacement b_1 of point $B = b \cap o$, a convenient point attached to the reciprocating slider? This will be taken relative to the crank centre point O, as a function of the crank angle θ.
- What is the angular displacement θ as a function of b_1?
- What are the limits of b_1?

The pertinent unifying kinematic geometry focuses on the "fixed frame" FF defined by point O and line o and the "end effector" EE defined by B and b. The point P, although movable, becomes an "honourary member" of FF by virtue of its known position in terms of *joint variable* θ and *design parameter* r. The situation can be summed up by the following line and point constraint equations.

$$B = b \cap o, \;\; b = P \cap C, \;\; o = O \cap A$$

Note that $A\{0 : 1 : 0\}$ and $C\{0 : \cos\beta : \sin\beta\}$ are the respective points at infinity which close o and b while P is $\{1 : r\cos\theta : r\sin\theta\}$ and O is $\{1 : 0 : 0\}$. These are given by their homogeneous coordinates.

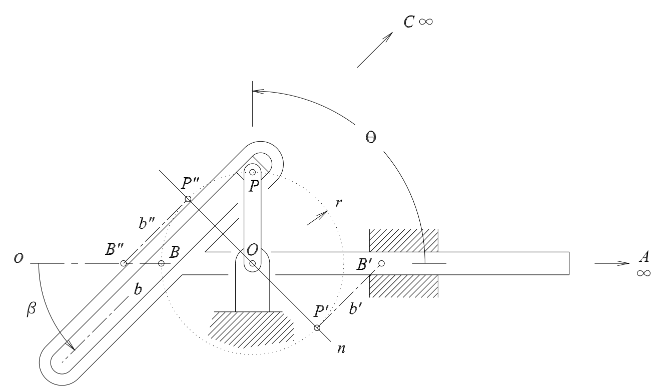

Figure 1. Inclined Scotch Yoke.

3. DIRECT KINEMATICS

Point B can be obtained after defining lines o and b.

$$o : \begin{vmatrix} w & x & y \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0 \Rightarrow \{0 : 0 : 1\}$$

$$b : \begin{vmatrix} w & x & y \\ 1 & r\cos\theta & r\sin\theta \\ 0 & \cos\beta & \sin\beta \end{vmatrix} = 0$$

$$\Rightarrow \{r(\sin\beta\cos\theta - \cos\beta\sin\theta) : -\sin\beta : \cos\beta\}$$

$$B : \begin{vmatrix} W & X & Y \\ 0 & 0 & 1 \\ r(\sin\beta\cos\theta - \cos\beta\sin\theta) & -\sin\beta & \cos\beta \end{vmatrix} = 0$$

$$\Rightarrow \{\sin\beta : r(\sin\beta\cos\theta - \cos\beta\sin\theta) : 0\}$$

Now the displacement equation is obtained with $B\{b_0 : b_1 : b_2\}$, dehomogenized by dividing through by b_0.

$$B\{1 : r(\cos\theta - \cot\beta\sin\theta) : 0\}$$

Not surprisingly B, representing slider displacement, is confined to move along o. Direct kinematic displacement, b_1, velocity, $\dot{b}_1$, and acceleration, $\ddot{b}_1$, are written below.

$$\begin{aligned}b_1 &= r(\cos\theta - \cot\beta\sin\theta)\\ \dot b_1 &= -r(\sin\theta + \cot\beta\cos\theta)\dot\theta\\ \ddot b_1 &= -r[(\cos\theta - \cot\beta\sin\theta)\dot\theta^2 + (\sin\theta + \cot\beta\cos\theta)\ddot\theta]\end{aligned}$$

4. INVERSE KINEMATICS

What if one wishes to obtain $\theta(b_1)$, *etc.*? Things are much easier if sines and cosines are replaced by $\tan(\theta/2)$. Then

$$\text{with } u = \tan\frac{\theta}{2},\ \cos\theta = \frac{1-u^2}{1+u^2} \text{ and } \sin\theta = \frac{2u}{1+u^2}$$

so

$$r(1-u^2-2\cot\beta\, u) - b_1(1+u^2) = 0$$

This can be solved for u to produce two values of $\tan(\theta/2)$, hence θ, in terms of the nondimensionalized displacement ratio $\rho = b_1/r$.

$$u = \frac{-\cot\beta \pm \sqrt{\cot^2\beta + 1 - \rho^2}}{1+\rho}$$

The inverse velocity is produced immediately by taking time derivatives of both sides of the inverse displacement equation above.

$$\left(\frac{\sec^2\frac{\theta}{2}}{2}\right)\dot\theta = \left(\mp\frac{\rho}{(1+\rho)\sqrt{\cot^2\beta+1-\rho^2}} + \frac{\cot\beta \mp \sqrt{\cot^2\beta+1-\rho^2}}{(1+\rho)^2}\right)\dot\rho$$

After defining two often repeated dimensionless groups

$$\rho_1 \equiv (1+\rho) \text{ and } \rho_2 \equiv \sqrt{\cot^2\beta + 1 - \rho^2}$$

inverse acceleration is written as follows.

$$\left(\frac{\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}}{2}\right)\dot\theta^2 + \left(\frac{\sec^2\frac{\theta}{2}}{2}\right)\ddot\theta =$$

$$\left(2\frac{-\cot\beta \pm \rho_2}{\rho_1^3} \pm 2\frac{\rho}{\rho_1^2\rho_2} \mp \frac{\rho^2}{\rho_1\rho_2^3} \mp \frac{1}{\rho_1\rho_2}\right)\dot\rho^2$$
$$\left(\mp\frac{\rho}{\rho_1\rho_2} + \frac{\cot\beta \mp \rho_2}{\rho_1^2}\right)\ddot\rho$$

So given r, β, b_1, $\dot b_1$ and $\ddot b_1$, then θ, $\dot\theta$ and $\ddot\theta$ can be computed in a straightforward manner while keeping the result pairings properly matched.

5. POLARITY AND DISPLACEMENT LIMITS

It is shown in Figure 1 that the limits of slider displacement b_1 are established where line b is tangent to the circle traced by the motion of crank-pin P. One could simply intersect a line n, on O and normal to b, with the circle and define where the two parallel lines b' and b'' intersect o. However if the locus of P were a general conic, instead of an origin centred circle of given radius, then the method outlined below would be quite useful to obtain n. This makes use of the so-called conic polarity relationship which defines the planar line polar to any point in the plane and with respect to a conic at hand. Conics may be defined with quadratic point forms on a homogeneous symmetric 3×3 matrix, *viz.*,

$$[p_0\ \ p_1\ \ p_2]\begin{bmatrix} a_{00} & a_{01} & a_{02}\\ a_{01} & a_{11} & a_{12}\\ a_{02} & a_{12} & a_{22}\end{bmatrix}\begin{bmatrix} p_0\\ p_1\\ p_2\end{bmatrix} = 0$$

or

$$a_{00}p_0^2 + 2a_{01}p_0p_1 + 2a_{02}p_0p_2 + a_{11}p_1^2 + 2a_{12}p_1p_2 + a_{22}p_2^2 = 0$$

If the matrix $[a_{ij}]$ is given along with a point, say $C\{c_0 : c_1 : c_2\}$ then a linear equation in coordinates p_j is obtained and its coefficients

$$n\{N_0 : N_1 : N_2\} \equiv \{c_0(a_{01} + a_{01} + a_{02}) :$$
$$c_0(a_{01} + a_{11} + a_{12}) : c_0(a_{02} + a_{12} + a_{22})\}$$

are the homogeneous coordinates of the line n subtended by the tangents from C and intersecting the conic on the points of tangency. With $C\{0 : \cos\beta : \sin\beta\}$ and the circle radius r one gets

$$[p_0\ \ p_1\ \ p_2]\begin{bmatrix} -r^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 0\\ \cos\theta\\ \sin\theta\end{bmatrix} = 0$$

$$0p_0 + \cos\theta\ p_1 + \sin\theta\ p_2 = 0$$

This linear equation is solved simultaneously with that of the conic to yield tangent points P' and P'', which, when paired with C, produce b' and b'', respectively. These intersect o on B' and B'', the required limiting excursions of B.

6. CONCLUSION

The three topics introduced here in the context of a simple kinematic analysis, *viz.*, projective geometry, trigonometric to algebraic conversion and the theory of conics, were not the easiest way to solve the problem posed. Nevertheless by introducing these methods along with an easy-to-follow example it is hoped that the reader sees how to use them to formulate robust constraint equations which lead to clear solutions amenable to efficient computation and unambiguous results in other, more complicated situations. Although not treated herein, design problems lurk in the background. *I.e.*, how to choose r and β to fulfill specified kinematic performance? This mechanism admits two design variables. What opportunities does this present to the designer?

REFERENCES

[1] Klein, F. (1939): *Elementary Mathematics from an Advanced Standpoint: Geometry*, Dover Publications, Inc., New York, N.Y., U.S.A.

[2] Klein, F. (1932): *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, and Analysis*, Dover Publications, Inc., New York, N.Y., U.S.A.