Summary Notes of Dynamic Force Analysis

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This document will help you understand and kinematic analysis for linkages. Recall Newton's Law:

$$\sum \mathbf{F} = m\mathbf{a} \quad (1)$$

$$\sum \mathbf{M} = I\alpha \quad (2)$$

Remember that all of these equation can be applied in the x, y, z directions, such that:

$$F_x = F\cos(\theta) \quad (3)$$

$$F_y = F\sin(\theta) \quad (4)$$

The decomposition of a vector can be graphically seen in figure 1.

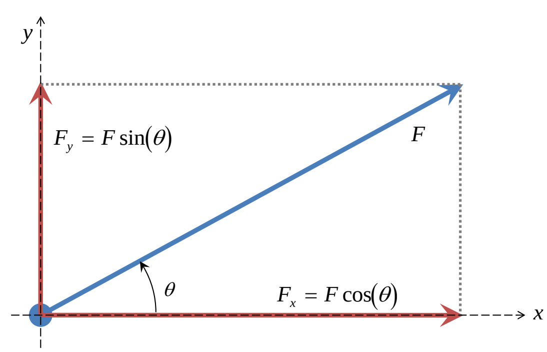

Figure 1: Decomposition of a vector into its associated x and y components

1 Single Link

Let us consider a single link rotating about a fixed point as seen in figure 2. We want to find the forces acting on the pin and torque on the member given the kinematic inputs. If you have performed the position, velocity and acceleration analysis for the motion you want, you can now determine the necessary input or driving torque to preform the desired motion/operation.

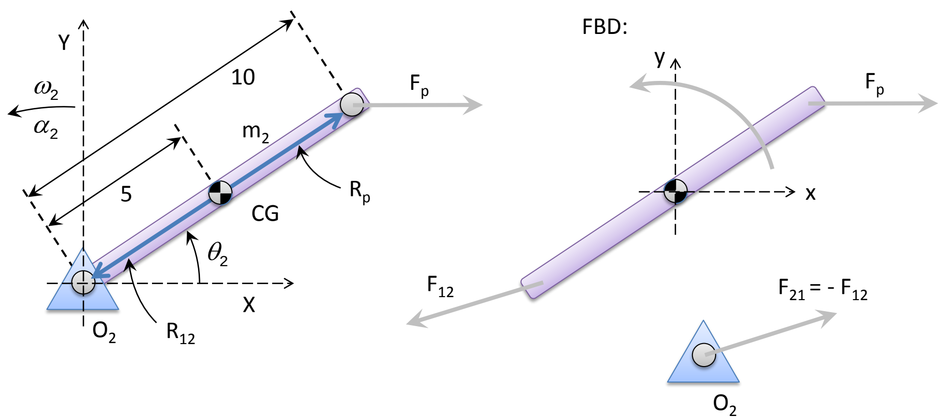

Figure 2: Kinematic Analysis of a single link in pure rotation

Summing the forces on the link we will have:

$$\sum \mathbf{F} = \mathbf{F}_P + \mathbf{F}_{12} = m_2 \mathbf{a}_G \tag{5}$$

Note the signs or directions will come from the actual definition of $\mathbf{F}_P$ and $\mathbf{F}_{12}$. The sum of the moments or torques will be:

$$\sum \mathbf{T} = \mathbf{T}_{12} + \Big(\mathbf{R}_{12} \times \mathbf{F}_{12}\Big) + \Big(\mathbf{R}_p \times \mathbf{F}_p\Big) = I_G \alpha \tag{6}$$

Decomposing equations 5 and 6 into the XY reference frame they become:

$$F_{P_x} + F_{12_x} = m_2 a_{G_x} \tag{7}$$

$$F_{P_y} + F_{12_y} = m_2 a_{G_y} \tag{8}$$

$$\mathbf{T}_{12} + \Big(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\Big) + \Big(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\Big) = I_G \alpha \tag{9}$$

These equations can written in Matrix Form as:

$$[\mathbf{A}] \quad \times \quad \{\mathbf{X}\} \quad = \quad [\mathbf{C}]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R_{12_y} & R_{12_x} & 1 \end{bmatrix} \times \begin{Bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} m_2 a_{G_x} - F_{P_x} \\ m_2 a_{G_y} - F_{P_y} \\ I_G \alpha - \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x} \right) \end{bmatrix} \tag{10}$$

Note that the these equations have been defined in the general sense to that they do not pertain to a specific instant in time.

1.1 Single Bar Example

The link in figure 2 is 10 inches long and weighs 4lbs. The centre of gravity is on the line of centres located 5 inches from either end of link. Its mass moment of inertia about the centre of gravity is 0.08lb-in-sec^2. The kinematic data is given as:

θ_2 deg	ω_2 rad/sec	α_2 rad/sec^2	a_{G_2} in/sec^2
30	20	15	2001 at 208^o

The external force of 40lb is applied at 0^o. Find the force acting at the pin and the driving torque needed to maintain the motion with the given acceleration for this instant in time.

Solution

Notice the units in this question, we must find m, the mass of the link:

$$m = \frac{weight}{g} = \frac{4lb}{386in/sec^2} = 0.01blobs$$

Using equations 3 and 4 we will decompose the $\mathbf{R}_{12}$ and $\mathbf{P}$ into x and y components based on it the link's orientation at this moment in time:

$$\begin{aligned} R_{12_x} &= -4.33 \\ R_{12_y} &= -2.50 \\ R_{P_x} &= +4.33 \\ R_{P_y} &= +2.50 \end{aligned}$$

Notice here is where the $\pm$ become very important. We will also decompose a_G into x and y components:

$$\begin{aligned} a_{G_x} &= -1766.78 \\ a_{G_y} &= -939.41 \end{aligned}$$

The general solution also has the applied force broken into components:

$$\begin{aligned} F_{P_x} &= 40 \\ F_{P_y} &= 0 \end{aligned}$$

At this instant in time there is no y component to the applied force. To solve this equation we substitute the values into matrix equation we derived above, Equation 10:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2.50 & -4.33 & 1 \end{bmatrix} \times \begin{Bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} \left(0.01 \times -1766.78\right) - 40 \\ \left(0.01 \times -939.41\right) - 0 \\ 0.08 \times 15 - \left(4.33 \times 0 - 2.5 \times 40\right) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2.50 & -4.33 & 1 \end{bmatrix} \times \begin{Bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} -57.67 \\ -9.39 \\ 101.2 \end{bmatrix}$$

Solve this equation by inverting **A** and pre-multiplying that inverse by **B** to obtain the unknowns contained in vector **X**:

$$\{\mathbf{X}\} = [\mathbf{A}]^{-1}[\mathbf{B}]$$

In MATLAB the inverting the **A** matrix can be done by: `inv(A)`. Also, in matrix math order does matter thus `X = inv(A) * B` will result in the correct solution of:

$$\begin{aligned} F_{12_x} &= -57.67 lb \\ F_{12_y} &= -9.39 lb \\ T_{12} &= 204.72 lb \cdot in \end{aligned}$$

2 Fourbar System

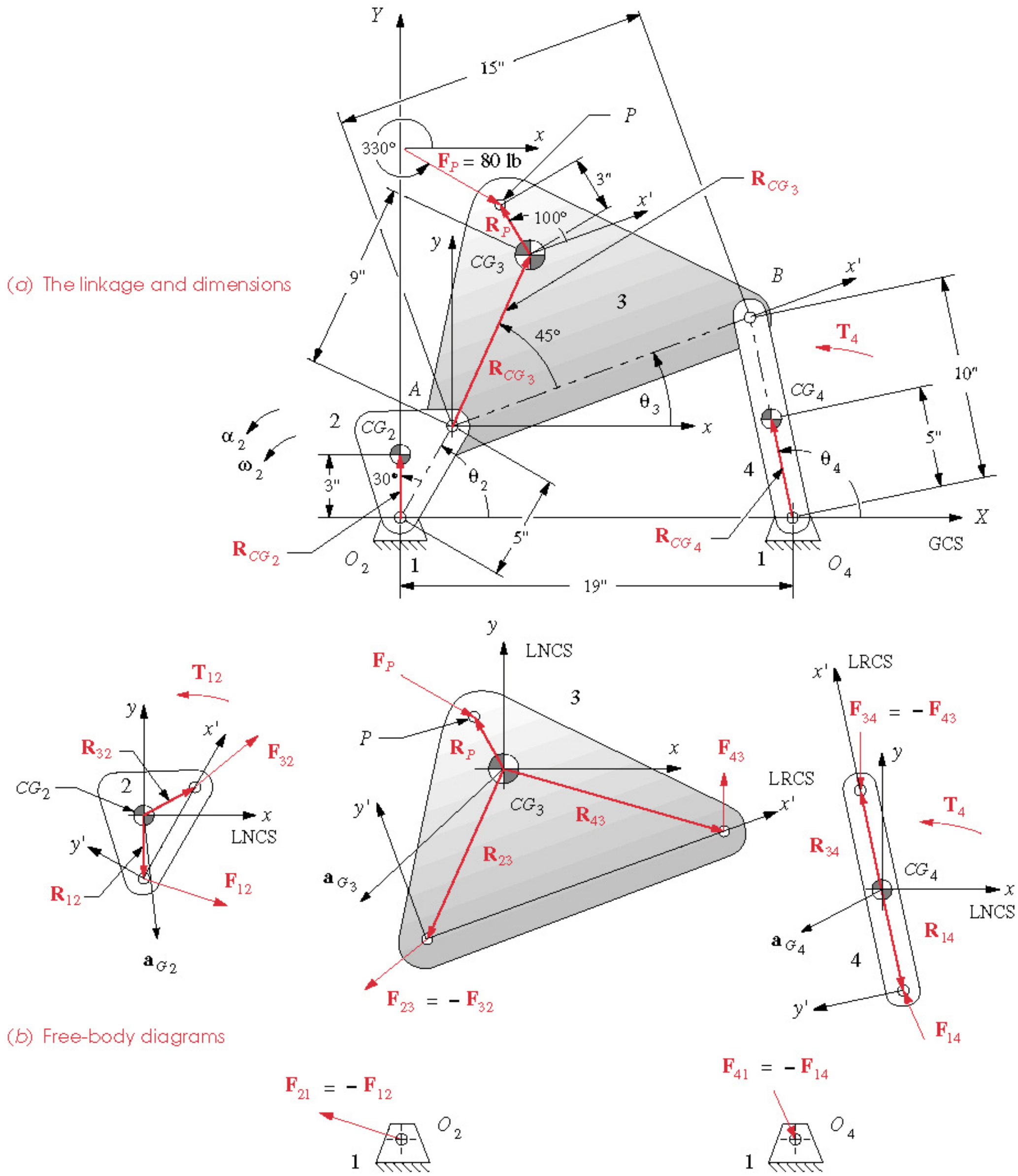

Figure 3: Kinematic Analysis of a four bar linkage [from Norton, Design of Machinery]

Figure 3 shows a four linkage. All dimensions of the links, is know and a complete kinematic analysis has been performed. We are now interested in the forces acting in the system. A single link has three unknowns, F_x, F_y and the applied torque, T. In a fourbar system there are 3 moving links and and 4 joints. Each joint will have 2 unknown forces (x and y components) for a total of 8 unknown forces which need to be solved for. In addition to the forces a driving torque must also be determined too, thus a total of 9 unknowns exist for a fourbar linkage. For each link we can develop 3 equations by taking applying Newton's law for the forces in the x and y directions and taking a moment about the CG. The end result is 9 equations with 9 unknowns. We will use matrix algebra to solve this large system of equations.

Looking figure 3 and examining link 2, typically the input link, we obtain the following equations:

$$F_{12_x} + F_{32_x} = m_2 a_{G_{2_x}} \tag{11}$$

$$F_{12_y} + F_{32_y} = m_2 a_{G_{2_y}} \tag{12}$$

$$T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{32_x} F_{32_y} - R_{32_y} F_{32_x}\right) = I_{G_2} \alpha_2 \tag{13}$$

Note that, $\mathbf{F}_{12}$ is the the force of 1 on 2 and $\mathbf{F}_{23}$ is the the force of 2 on 3. One will also notice that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Examining link 3, the coupler link which has an external load, F_P acting on it, we obtain the following equations:

$$F_{43_x} - F_{32_x} + F_{P_x} = m_3 a_{G_{3_x}} \tag{14}$$

$$F_{43_y} - F_{32_y} + F_{P_y} = m_3 a_{G_{3_y}} \tag{15}$$

$$\left(R_{43_x} F_{43_y} - R_{43_y} F_{43_x}\right) - \left(R_{23_x} F_{32_y} - R_{23_y} F_{32_x}\right) + \ldots$$
$$\left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) = I_{G_3} \alpha_3 \tag{16}$$

Examining link 4, which has an external torque load, T_4 acting on it, we obtain the following equations:

$$F_{14_x} - F_{43_x} = m_4 a_{G_{4_x}} \tag{17}$$

$$F_{14_y} - F_{43_y} = m_4 a_{G_{4_y}} \tag{18}$$

$$T_4 + \left(R_{14_x} F_{14_y} - R_{14_y} F_{14_x}\right) - \left(R_{34_x} F_{43_y} - R_{34_y} F_{43_x}\right) = I_{G_4} \alpha_4 \tag{19}$$

The general form of the Newton's equation is:

$$F_{ij} + F_{jk} + \sum F_{ext_j} = m_j a_{G_j} \tag{20}$$

$$(R_{ij} \times F_{ij}) + (R_{jk} \times F_{jk}) + \sum T_j + \left(R_{ext} \times \sum F_{ext_j}\right) = I_{G_j} \alpha_j \tag{21}$$

where $j = 2, 3, \ldots, n$ and $i = j - 1$ and $k = j + 1, j \neq n$ if $j = n, k = 1$ and $F_{ji} = -F_{ij}$ and $F_{kj} = -F_{jk}$. This notation means that j is the current link you are examining and i is the previous link while k is the next link.

Assembling the equations into a matrix we get the following:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & -R_{43_y} & R_{43_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & R_{34_y} & -R_{34_x} & -R_{14_y} & R_{14_x} & 0 \end{bmatrix} \times$$

$$\begin{Bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2} \alpha_2 \\ m_3 a_{G_{3x}} - F_{P_x} \\ m_3 a_{G_{3y}} - F_{P_y} \\ I_{G_3} \alpha_3 - R_{P_x} F_{P_y} + R_{P_y} F_{P_x} \\ m_4 a_{G_{4x}} \\ m_4 a_{G_{4y}} \\ I_{G_4} \alpha_4 - T_4 \end{bmatrix} \tag{22}$$

2.1 FourBar Example

In figure 3 the 5in long crank, link 2, weighs 1.5lb. Its CG is at 3in and 30^o from the line of centres. Its mass moment of inertia about its CG is 0.4lb-in-sec^2. Its kinematic data is:

θ_2 deg	ω_2 rad/sec	α_2 rad/sec^2	a_{G_2} in/sec^2
60	25	-40	1878.84 at 273.66^o

The 15in long coupler, link 2, weighs 7.7lb. Its CG is at 9in and 45^o from the line of centres. Its mass moment of inertia about its CG is 1.5lb-in-sec^2. Its kinematic data is:

θ_3 deg	ω_3 rad/sec	α_3 rad/sec^2	a_{G_3} in/sec^2
20.92	-5.87	120.9	3646.1 at 226.5^o

The ground link is 19in long and the rocker, link 4 is 10in while weighing 5.8lb. Its CG is at 5in and 0^o from the line of centres. Its mass moment of inertia about its CG is 0.8lb-in-sec^2. Its kinematic data is:

θ_4 deg	ω_4 rad/sec	α_4 rad/sec^2	a_{G_4} in/sec^2
104.41	7.93	276.29	1416.8 at 207.2^o

There is an external torque of 120lb-in acting on the rocker, link 4. There is also a external load of 80lb at 330^o acting on the coupler, link 3, at point P which is located at 3in and 100^o from the CG.

Find $\mathbf{F}_{12}, \mathbf{F}_{32}, \mathbf{F}_{43}, \mathbf{F}_{14}$ and $\mathbf{T}_{12}$ needed to maintain the given acceleration for this instantaneous position.

Solution

We must find the m, the mass of each link. Notice the units in this question:

$$m_2 = \frac{weight}{g} = \frac{1.5lb}{386in/sec^2} = .004blobs$$
$$m_3 = \frac{weight}{g} = \frac{7.7lb}{386in/sec^2} = .020blobs$$
$$m_4 = \frac{weight}{g} = \frac{5.8lb}{386in/sec^2} = .015blobs$$

Next we will decompose the $\mathbf{R}_{12}, \mathbf{R}_{32}, \mathbf{R}_{23}, \mathbf{R}_{43}, \mathbf{R}_{34}, \mathbf{R}_{14}$ and $\mathbf{P}$ into x and y components based on it the link's orientation:

$$R_{12_x} = 0.000; \quad R_{12_y} = -3.00$$
$$R_{32_x} = +2.50; \quad R_{32_y} = +1.33$$
$$R_{23_x} = -3.67; \quad R_{23_y} = -8.22$$
$$R_{43_x} = +10.33; \quad R_{43_y} = -2.86$$
$$R_{34_x} = -1.24; \quad R_{34_y} = -4.84$$
$$R_{14_x} = +1.24; \quad R_{14_y} = -4.84$$
$$R_{P_x} = +1.54; \quad R_{P_y} = +2.57$$

Calculate the x and y components of the acceleration of the CGs for the links:

$$a_{G_{2x}} = +119.94; \quad a_{G_{2y}} = -1875.01$$
$$a_{G_{3x}} = -2509.35; \quad a_{G_{3y}} = -2645.23$$
$$a_{G_{4x}} = -1259.67; \quad a_{G_{4y}} = -648.50$$

Calculate the x and y components of the applied force on point P:

$$F_{P2_x} = +69.28; \quad F_{P2_x} = -40.00$$

Subbing these values into equation we get:

$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -1.33 & 2.5 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -8.217 & 3.673 & 2.861 & 10.339 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4.843 & 1.244 & 4.843 & 1.244 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0.004 \times 119.94 \\ 0.004 \times -1875.01 \\ 0.4 \times -40.00 \\ 0.02 \times -2509.35 - (69.28) \\ 0.02 \times -2509.35 - (-40.00) \\ 1.5 \times 120.9 - \big(-1.542 \times -40 - 2.574 \times 69.28\big) \\ 0.015 \times -1259.67 \\ 0.015 \times -648.50 \\ 0.8 \times 276.29 - (120) \end{bmatrix} = \begin{bmatrix} 0.480 \\ -7.500 \\ -16.000 \\ -119.47 \\ -12.91 \\ 298.00 \\ -18.90 \\ -9.727 \\ 101.03 \end{bmatrix}$$

$$\{X\} = \begin{Bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{Bmatrix}$$

Solve this equation by inverting **A** and pre-multiplying that inverse by **B** to obtain the unknowns, **X**:

$$\{\mathbf{X}\} = [\mathbf{A}]^{-1}[\mathbf{B}]$$

In MATLAB the inverting the **A** matrix can be done by: `inv(A)`. Also, in matrix math order does matter thus `X = inv(A) * B` will result in the correct solution of:

$$\left\{\begin{array}{c} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{array}\right\} = \left\{\begin{array}{c} -117.65 \\ 107.84 \\ 118.13 \\ 100.34 \\ -1.34 \\ 87.43 \\ -20.23 \\ 77.71 \\ 243.23 \end{array}\right\}$$