Input-output Equation for Planar Four-bar Linkages

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Abstract. In this paper the generalised input-output (I-O) equation for planar 4*R* function generators is derived in a new way, leading to the algebraic form of the well known Freudenstein equation. The long term goal is to develop a generalised method to derive constraint based algebraic I-O equations that can be used for continuous approximate synthesis, where the synthesis equations are integrated between minimum and maximum input angle values resulting in a linkage whose objective function has been optimised over every output angle. In this paper we use a planar projection of Study's soma and the Cartesian displacement constraints for the dyads. These are mapped to the image space leading to four constraint equations in terms of the image space coordinates and the sines and cosines of the input and output angles. Using the tangent of the half angle substitution the trigonometric equations are converted to algebraic ones. Algebraic methods are used to eliminate the image space coordinates, then the polynomial resultants are found to obtain common roots leading to the desired equations.

Keywords: Function generators, continuous approximate synthesis, kinematic mapping, polynomial resultants.

1 Introduction

A planar 4*R* function generator correlates driver and follower angles in a functional relationship. The mechanism essentially generates the function $\varphi = f(\psi)$, or vice versa, see Fig. 1. Design methods typically employ the Freudenstein synthesis equations to identify link lengths required to generate the function [2,4]. For over-determined sets of prescribed input-output (I-O) angle pairs, these equations are linear in the three unknown Freudenstein parameters, which are ratios of the link lengths, and can be solved for using any least squares method to minimise a specified performance error. To the best of the authors knowledge, there are no alternative algebraic models of the function generator displacement equations in the accessible literature.

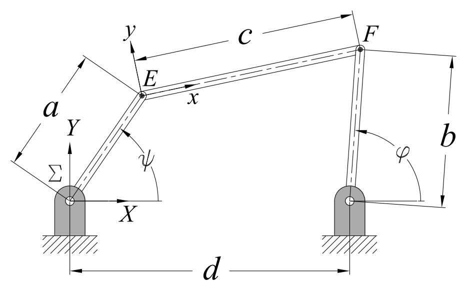

Fig. 1. 4*R* function generator.

It has been observed [7,5] that as the cardinality of the prescribed discrete I-O data set increases, the corresponding four-bar linkages that minimise the Euclidean norm of the design and structural errors tend to converge to the same linkage. The important implication is that minimising the Euclidean norm, or any p-norm, of the structural error can be accomplished indirectly by minimising the same norm of the design error. In [5] the trigonometric Freudenstein synthesis equations are integrated in the range between minimum and maximum input values, thereby reposing the discrete approximate synthesis problem as a continuous one whereby the objective function is optimised over the entire I-O range. Hence, our long-term goal is to determine a general method to derive motion constraint based algebraic I-O equations that may be used together with the method of continuous approximate synthesis [5] to obtain the very best linkage that can generate an arbitrary function. The goal of this paper is to develop one in the hope of providing new insight into the continuous approximate synthesis of function generators, while mitigating numerical integration error. Of course, the same equation will be obtained by making the tangent half-angle substitutions directly into the Freudenstein equation then collecting terms after factoring, normalising, and eliminating non-zero factors. But that must be the case since the geometric relations require that outcome, however this is irrelevant because the goal is to generalise a method to develop constraint based algebraic I-O equations for continuous approximate synthesis of planar, spherical, and spatial linkages. This paper represents the first step in that long journey.

2 Geometric and Algebraic Approach

The Freudenstein equation relating the input to the output angles of a planar $4R$ four-bar mechanism, with link lengths as in Fig. 1, was first put forward in [3]. In the equation the angle ψ is traditionally selected to be the input while φ is the output angle:

$$k_1 + k_2\cos(\varphi_i) - k_3\cos(\psi_i) = \cos(\psi_i - \varphi_i). \tag{1}$$

Equation (1) is linear in the k_i Freudenstein parameters, which are defined in terms of the link length ratios as

$$\left.\begin{aligned} k_1 &\equiv \frac{(a^2+b^2+d^2-c^2)}{2ab}, \\ k_2 &\equiv \frac{d}{a}, \\ k_3 &\equiv \frac{d}{b}. \end{aligned}\right\} \Leftrightarrow \begin{cases} d = 1, \\ a = \dfrac{1}{k_2}, \\ b = \dfrac{1}{k_3}, \\ c = (a^2+b^2+d^2-2abk_1)^{1/2}. \end{cases}$$

The new idea starts the same as with the Freudenstein method, writing the displacement constraints in terms of the I-O angles. Continuing with tradition, we select ψ to be the input angle and φ to be the output angle. Let Σ be a non moving Cartesian coordinate system with coordinates X and Y whose origin is located at the centre of the ground fixed link R-pair with length a. Let E be a coordinate system that moves with the coupler of length c whose origin is at the centre of the distal R-pair of link a, having basis directions x and y.

The displacement constraints for the origin of E can be expressed as

$$\begin{aligned} X - a\cos\psi &= 0, \\ Y - a\sin\psi &= 0, \end{aligned} \tag{2}$$

while those for point F, located at the centre of the distal R-pair on the output link with length b are

$$\begin{aligned} X - d - b\cos\varphi &= 0, \\ Y - b\sin\varphi \quad &= 0. \end{aligned} \tag{3}$$

Next, we use a planar projection of Study's soma coordinates [8] to establish the I-O equation. Any displacement in Euclidean space, E_3, can be mapped in terms of the coordinates of a 7-dimensional projective image space using the transformation [1]

$$\mathbf{T} = \begin{bmatrix} x_0^2+x_1^2+x_2^2+x_3^2 & 0 & 0 & 0 \\ 2(-x_0y_1+x_1y_0-x_2y_3+x_3y_2) & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ 2(-x_0y_2+x_1y_3+x_2y_0-x_3y_1) & 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ 2(-x_0y_3-x_1y_2+x_2y_1+x_3y_0) & 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix}. \tag{4}$$

This transforms the coordinates of any point described in a moving $3D$ coordinate system E to the coordinates of the same point in a relatively fixed $3D$ coordinate system Σ (assuming that the two frames are initially coincident) after a given displacement of E relative to Σ in terms of the coordinates of a point on the Study quadric, S_6^2. In order for a point in the image space to represent a real displacement, and therefore to be located on S_6^2, the non-zero condition of $x_0^2+x_1^2+x_2^2+x_3^2 \neq 0$ must be satisfied.

The transformation matrix $\mathbf{T}$ simplifies considerably when we consider displacements that are restricted to the plane. Three degrees of freedom are lost and hence four Study parameters vanish. The displacements may be restricted to any plane. Without loss in generality, we may select one of the principal planes in Σ. Thus, we arbitrarily select the plane $Z = 0$. Since E and Σ are assumed to be initially coincident, this means

$$\begin{bmatrix} W \\ X \\ Y \\ 0 \end{bmatrix} = \mathbf{T} \begin{bmatrix} w \\ x \\ y \\ 0 \end{bmatrix}, \tag{5}$$

leaving us with the four soma coordinates

$$(x_0 : x_3 : y_1 : y_2). \tag{6}$$

The non-zero condition is now $x_0^2 + x_3^2 \neq 0$, and the fourth row and column of the reduced $\mathbf{T}$ contains only this condition as the last element with zeros elsewhere leading to the trivial equation $Z = z = 0$. We can therefore eliminate the fourth row and column and normalise the coordinates with the nonzero condition giving the planar mapping equation

$$\mathbf{T} = \frac{1}{x_0^2+x_3^2}\begin{bmatrix} x_0^2+x_3^2 & 0 & 0 \\ 2(-x_0y_1+x_3y_2) & x_0^2-x_3^2 & -2x_0x_3 \\ -2(x_0y_2+x_3y_1) & 2x_0x_3 & x_0^2-x_3^2 \end{bmatrix}. \tag{7}$$

We can now express a point in Σ in terms of the soma coordinates and the corresponding point coordinates in E as

$$\begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \mathbf{T}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \frac{1}{x_0^2+x_3^2}\begin{bmatrix} x_0^2+x_3^2 \\ 2(-x_0y_1+x_3y_2)+(x_0^2-x_3^2)x-(2x_0x_3)y \\ -2(x_0y_2+x_3y_1)+(2x_0x_3)x+(x_0^2-x_3^2)y \end{bmatrix}. \tag{8}$$

The novelty of the approach begins with creating two Cartesian vector constraint equations containing the nonhomogeneous coordinates in Equations (2) and (3), but substituting the values in Equation (8) for (X,Y). These two vector equations are $\mathbf{F}_1 = \mathbf{0}$ and $\mathbf{F}_2 = \mathbf{0}$:

$$\mathbf{F}_1 = \frac{1}{x_0^2+x_3^2}\begin{bmatrix} 2(-x_0y_1+x_3y_2)+(x_0^2-x_3^2)x-2x_0x_3y-(a\cos\psi)(x_0^2+x_3^2) \\ -2(x_0y_2+x_3y_1)+2x_0x_3x+(x_0^2-x_3^2)y-(a\sin\psi)(x_0^2+x_3^2) \end{bmatrix} = \mathbf{0};$$

$$\mathbf{F}_2 = \frac{1}{x_0^2+x_3^2}\begin{bmatrix} 2(-x_0y_1+x_3y_2)+(x_0^2-x_3^2)x-2x_0x_3y-(b\cos\varphi+d)(x_0^2+x_3^2) \\ -2(x_0y_2+x_3y_1)+2x_0x_3x+(x_0^2-x_3^2)y-(b\sin\varphi)(x_0^2+x_3^2) \end{bmatrix} = \mathbf{0}.$$

Now we determine equations for the coupler. The coordinate system that moves with the coupler has its origin, point E, on the centre of the R-pair, as in Fig. 1, having coordinates $(x,y) = (0,0)$, while point F is on the R-pair centre on the other end having coordinates $(x,y) = (c,0)$. One more vector equation, $\mathbf{H}_1$ is obtained by substituting $(x,y) = (0,0)$ in $\mathbf{F}_1$, and another, $\mathbf{H}_2$ is obtained by substituting $(x,y) = (c,0)$ in $\mathbf{F}_2$. Next $\mathbf{H}_1$ and $\mathbf{H}_2$, two rational expressions, are converted to factored normal form. This is the form where the numerator and denominator are relatively prime polynomials with integer coefficients. The denominators for both $\mathbf{H}_1$ and $\mathbf{H}_2$ are the nonzero condition $x_0^2+x_3^2$, which can safely be factored out of each equation leaving the following two vector equations with polynomial elements:

$$\mathbf{H}_1 = \begin{bmatrix} -a\cos\psi(x_0^2+x_3^2)+2(-x_0y_1+x_3y_2) \\ -a\sin\psi(x_0^2+x_3^2)-2(x_0y_1+x_3y_2) \end{bmatrix} = \mathbf{0}; \tag{9}$$

$$\mathbf{H}_2 = \begin{bmatrix} -(b\cos\varphi+d)(x_0^2+x_3^2)+c(x_0^2-x_3^2)+2(-x_0y_1+x_3y_2) \\ -b\sin\varphi(x_0^2+x_3^2)+2c(x_0x_3)-2(x_0y_2+x_3y_1) \end{bmatrix} = \mathbf{0}. \tag{10}$$

The system of four displacement constraints on the I-O equations are $\mathbf{H}_1 = \mathbf{0}$ and $\mathbf{H}_2 = \mathbf{0}$. However, these are trigonometric equations. We convert them to algebraic ones using the tangent of the half-angle substitutions

$$u = \tan\frac{\psi}{2}, \;\; v \;=\; \tan\frac{\varphi}{2},$$

and

$$\cos\psi = \frac{1-u^2}{1+u^2}, \quad \sin\psi = \frac{2u}{1+u^2},$$
$$\cos\varphi = \frac{1-v^2}{1+v^2}, \quad \sin\varphi = \frac{2v}{1+v^2}.$$

The usual constraint equations in the kinematic mapping image space are obtained by considering $\mathbf{H}_1$ and $\mathbf{H}_2$ with the tangent of the half-angles, giving four new algebraic polynomials when considering the individual elements converted to factored normal form. The denominators are u^2+1 and v^2+1 which can safely be factored out because they are always non-vanishing. The resulting four algebraic equations are expressed in terms of the elements of $\mathbf{K}_1 = \mathbf{0}$ and $\mathbf{K}_2 = \mathbf{0}$:

$$\mathbf{K}_1 = \begin{bmatrix} (au^2-a)(x_0^2+x_3^2)+2u^2(-x_0y_1+x_3y_2)+2(-x_0y_2+x_3y_1) \\ -2au(x_0^2+x_3^2)-2(1+u^2)(-x_0y_2+x_3y_1) \end{bmatrix} = \mathbf{0}; \quad (11)$$

$$\mathbf{K}_2 = \begin{bmatrix} (v^2(b-d)+b-d)(x_0^2+x_3^2)+(cv^2+c)(x_0^2-x_3^2)+ \\ 2(1+v^2)(-x_0y_1+x_3y_2) \\ 2(v^2+1)(cx_0x_3-x_0y_2-x_3y_1)-2bv(x_0^2+x_3^2) \end{bmatrix} = \mathbf{0}. \quad (12)$$

Factoring the resultant of the first and second elements of $\mathbf{K}_1 = \mathbf{0}$ with respect to u, as well as the first and second elements of $\mathbf{K}_2 = \mathbf{0}$ with respect to v yields the two displacement constraint equations in the image space:

$$a^2(x_0^2+x_3^2)-4(y_1^2+y_2^2) = 0,$$
$$(b^2-c^2-d^2)(x_0^2+x_3^2)+2cd(x_0^2-x_3^2)+4c(x_0y_1+x_3y_2)+$$
$$4d(-x_0y_1+x_3y_2)-4(y_1^2+y_2^2) = 0.$$

Inspection of the quadratic forms of these two equations reveals that they are two hyperboloids of one sheet, which is exactly what is expected for two *RR* dyads [6]. But these are not the constraints we are looking for. We want to eliminate the image space coordinates using $\mathbf{K}_1 = \mathbf{0}$ and $\mathbf{K}_2 = \mathbf{0}$ to obtain an algebraic polynomial with the tangent half angles u and v as variables and link lengths as coefficients.

To obtain this algebraic polynomial we start by setting the homogenising coordinate $x_0 = 1$, which can safely be done since we are only concerned with real finite displacements. Next, observe that the two equations represented by the components of $\mathbf{K}_1 = \mathbf{0}$ (Equation (11)) have a simpler form than those of $\mathbf{K}_2 = \mathbf{0}$ (Equation (12)), and are linear in y_1 and y_2. Solving these two equations for y_1 and y_2 reveals that

$$y_1 = \frac{1}{2}\frac{a(u^2-2ux_3-1)}{u^2+1}, \quad (13)$$

$$y_2 = \frac{1}{2}\frac{a(u^2x_3+2u-x_3)}{u^2+1}. \quad (14)$$

Equations (13) and (14) reveal the common denominator of u^2+1, which can never be less than 1, and hence may be factored out. Now we back-substitute these expressions for y_1 and y_2 into the array components of Equation (12), thereby eliminating these image space coordinates, and factor the resultant with respect to x_3 which yields four factors. The first three are

$$4c^2, \ (u^2+1)^3, \ (v^2+1)^3.$$

None of these three factors can ever be zero and at the same time represent a real displacement constraint, hence they are eliminated. The remaining factor is a polynomial with only u and v as variables and link lengths a, b, c, and d, as coefficients. This is exactly the constraint equation we desire. It is factored, and the terms collected then distributed over u and v revealing

$$\begin{gathered}(a-b+c+d)(a-b-c+d)u^2v^2+(a+b-c+d)(a+b+c+d)u^2+\\(a+b-c-d)(a+b+c-d)v^2-8abuv+(a-b+c-d)(a-b-c-d)=0.\end{gathered} \tag{15}$$

Equation (15) is an algebraic polynomial of degree four which represents the I-O equation for any planar 4R mechanism. It has two singular points at infinity, namely those of the X- and Y-axes. These two singular points are either double points, or acnodes, i.e. isolated, or hermit points in the solution set of a polynomial equation in two real variables. When both are double points the mechanism is a double crank, when both are acnodes the mechanism is a double rocker. In the event the mechanism is a folding four-bar then the degree of Equation (15) is less than four.

Freudenstein's equation [4] is linear in the ratios of the link lengths and therefore is ideally suited to identifying link lengths that minimise some mechanism performance error in a least squares sense for approximate synthesis. The corresponding algebraic form of Freudenstein's equation is Equation (15), which is quadratic in the link lengths a, b, c, and d, but still lends itself to linear least squares error minimisation subject to quadratic constraints, and the method presented in [5]. However, in the following example we shall use exact synthesis, using only three of the prescribed sets on I-O pairs, and leave the approximate case to future work.

3 Example

This example serves to demonstrate that Equation (15) can be used to identify link lengths to create a 4R mechanism to generate an arbitrary function. Here, the function is specified in terms of the tangent of the half angle parameters $v = f(u)$ as

$$v = 2+\tan\left(\frac{u}{u^2+1}\right). \tag{16}$$

Eight I-O pairs $[u,v]$ were specified as, using Fig. 1 for reference,

$$[0,2], \left[\frac{1}{4},\frac{30055}{13419}\right], \left[\frac{1}{2},\frac{49597}{20471}\right], \left[\frac{3}{4},\frac{48857}{19383}\right], \left[1,\frac{64699}{25409}\right], \left[\frac{5}{4},\frac{25536}{10091}\right], \left[\frac{3}{2},\frac{110471}{44235}\right], \left[2,\frac{49597}{20471}\right].$$

Since a function generator is scalable, and hence only the ratios of the link lengths are needed, we set $d = 1$ and solve for the remaining three using the first, fourth, and eighth I-O pairs, giving, in generic units of length, $a = -0.23$, $b = 1.43$, and $c = 1.20$. Another linkage was identified using the second, fourth, and seventh I-O pairs giving nearly identical link lengths. Note that it is not uncommon in computational methods to obtain negative link lengths. These lengths are directed distances, and $a = -0.23$ means that the distance is directed from the distal R-pair in link a to the origin, instead of the other way around, as in [9].

In Fig. 2 the prescribed I-O function is plotted as the dashed curve, and the I-O function generated by the two identified linkages are plotted as the solid curves. The generated function shows good fidelity relative to the prescribed function over a reasonable range of I-O angles.

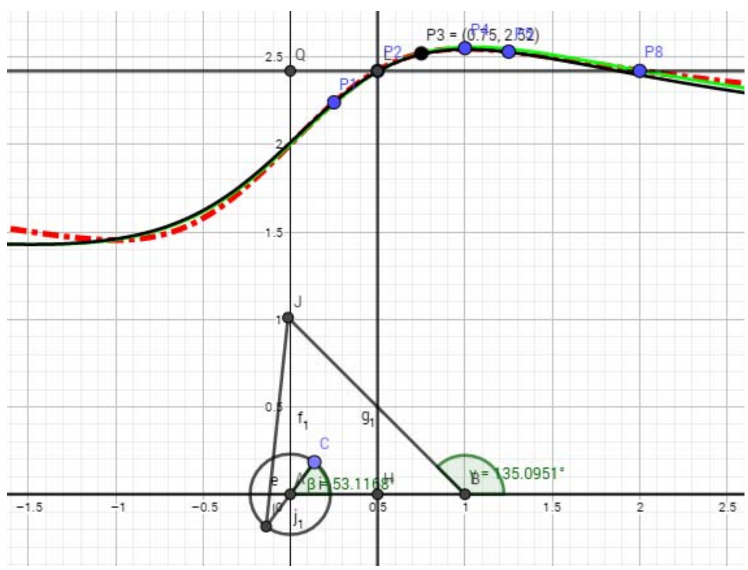

Fig. 2. Graphical representation of results.

4 Conclusions

In this paper a new method for deriving the I-O equations of planar $4R$ function generators was presented. The Cartesian displacement constraints of the two dyads comprising a planar $4R$ mechanism are expressed in terms of lengths and angles. This set of general constraint equations is mapped to a planar projection of Study's soma coordinates. The reason for using this unconventional form of planar kinematic mapping is to be able to apply these methods to spherical and spatial function generators in future work, where all eight soma coordinates will be needed. The result of this step is two arrays in terms of the I-O angles, link lengths, and soma coordinates. The equations are converted from trigonometric expressions to algebraic ones with the tangent of the half angle substitutions. These equations are used to eliminate the soma coordinates. What remains is factored using resultants, and all non-zero factors are eliminated, ultimately leaving an algebraic polynomial that is of degree four in the tangent of the half angle parameters, and quadratic in the link lengths.

The I-O equation derived in this paper is an algebraic, however nonlinear, polynomial in terms of the link lengths. Regardless, this algebraic formulation will significantly mitigate the effect of round-off errors observed in the numerical integration of the trigonometric Freudenstein synthesis equations [5]. While the very same I-O equation is, necessarily, obtained starting from the Freudenstein equation, the point of the presented material is to generalise the derivation of function generator I-O equations. The ultimate goal is to use continuous I-O data sets to synthesise the very best linkage to generate an arbitrary planar, spherical, or spatial function. Derivation of the planar algebraic I-O equation is one of the first steps towards this goal.

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