# Largest Ellipse Inscribing an Arbitrary Polygon

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## 1. INTRODUCTION

In this paper classical analytic projective geometry is used to provide an alternate approach to characterizing the velocity performance of parallel mechanisms in the presence of actuation redundancy as reported in [1]. Therein the aim is to determine the ellipse with the largest area that inscribes an arbitrary polygon. In this context, the area of the ellipse is proportional to the kinematic isotropy of the mechanism, while the polygon is defined by the reachable workspace of the mechanism, as discussed in [1]. There, the approach is a numerical maximization problem, essentially fitting the largest area inscribing ellipse starting with a unit circle.

A projective collineation is a transformation that maps collinear points onto collinear points in the projective plane. We propose to determine the general planar projective collineation that maps the unit circle inscribing a symmetric convex polygon onto an ellipse that inscribes the given convex polygon. The polygon containing the unit circle is constructed such that it has the same number of vertices, and hence edges, as the generally non-symmetric, but convex, polygon representing the workspace constraints of the mechanism. We shall call this the *boundary polygon.* Given that the coordinates of the vertices of both polygons are known, it is a simple matter to compute the transformation that maps the vertices of the symmetric polygon onto the boundary polygon. The same transformation is used to map the homogeneous parametric equation of the inscribing unit circle onto the corresponding ellipse that inscribes the boundary polygon. The unit circle that inscribes the symmetric polygon is, clearly, the largest inscribing ellipse. However, the transformed ellipse that inscribes the boundary polygon is generally not the one possessing the largest area. An additional step is required. In this paper we describe a simple construction for convex quadrilaterals that leads to this last step. Future work will begin with the last step, and aim towards a generalization for arbitrary convex polygons.

## 2. BOUNDARY QUADRILATERALS

We start with the case where the boundary polygon is a convex quadrilateral. When the boundary polygon has more than four edges a numerical approach must be employed. However, the numerical approach can be based on the following. Two distinct sets of four points in the projective plane $P_2$ uniquely determine a projective collineation if the points in the two sets are distinct, and if no three points are on the same line. Let the first set of four points have the coordinates $W(W_0 : W_1 : W_2)$, $X(X_0 : X_1 : X_2)$, $Y(Y_0 : Y_1 : Y_2)$, and $Z(Z_0 : Z_1 : Z_2)$. Let the second set of four points have the coordinates $w(w_0 : w_1 : w_2)$, $x(x_0 : x_1 : x_2)$, $y(y_0 : y_1 : y_2)$, and $z(z_0 : z_1 : z_2)$.

When expressed as a vector, the ratios implied by the homogeneous coordinates can be expressed by an arbitrary scaling factor:

$$[w_0 : w_1 : w_2]^T = \mu[w_0 : w_1 : w_2]^T. \quad (1)$$

The corresponding affine coordinates are

$$x_w = \frac{\mu w_1}{\mu w_0}; \qquad y_w = \frac{\mu w_2}{\mu w_0}. \quad (2)$$

This is why different scalar multiples of a set of homogeneous coordinates represent the same point in the projective plane.

The projective collineation may be viewed as a linear transformation that maps the coordinates of a point described in a particular coordinate system onto the coordinates of a different point in the same coordinate system. The geometry can be represented by the vector-algebraic relationship

$$\lambda \begin{bmatrix} W_0 \\ W_1 \\ W_2 \end{bmatrix} = \mu \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}. \quad (3)$$

Without loss in generality, we can set $\rho = \lambda/\mu$ and express Equation (3) more compactly as

$$\rho \mathbf{W} = \mathbf{T}\mathbf{w}. \quad (4)$$

The elements of the linear transformation matrix depend on the details of the mapping. As it represents a general projective collineation there are no orthogonality conditions on the rows or columns of $\mathbf{T}$. This means that the elements can take on any numerical value. Thus the mapping between two points in an arbitrary collineation consists of 9 variables. If we wish to determine the mapping given a point and its image then $\mathbf{T}$ represents 9 unknowns, but, because of the use of homogenous coordinates, at most 8 are independent. Still, to remain general the scaling factor $\rho$ must be counted among the unknowns because the given points come from a Cartesian coordinate system while the mapping is projective. The result is that the coordinates of four points, along with those of their images, are enough to uniquely define the eight independent elements of the transformation matrix and the four independent scaling factors, $\rho_i$, $i \in \{1, 2, 3, 4\}$.

The vertices of an arbitrary quadrilateral represent four points $W$, $X$, $Y$, and $Z$. We consider the image of these four points $w$, $x$, $y$, and $z$, to be the vertices of the square, containing the unit circle, centred on the origin of the coordinate system in which the quadrilateral is defined. Now a set of equations must be written so that the elements of $\mathbf{T}$ can be computed in terms of the point and image coordinates:

$$\begin{array}{c} t_{11}w_0 + t_{12}w_1 + t_{13}w_2 - \rho_1 W_0, \\ t_{21}w_0 + t_{22}w_1 + t_{23}w_2 - \rho_1 W_1, \\ t_{31}w_0 + t_{32}w_1 + t_{33}w_2 - \rho_1 W_2, \\ t_{11}x_0 + t_{12}x_1 + t_{13}x_2 - \rho_2 X_0, \\ \vdots \\ t_{31}z_0 + t_{32}z_1 + t_{33}z_2 - \rho_4 Z_2. \end{array} \qquad (5)$$

Equations (5) represent 12 equations in 13 unknowns, however we can set $t_{11} = 1$. It is a simple matter to solve for the 12 unknowns, however we only require the eight elements of $\mathbf{T}$.

The ellipse possessing the largest area inscribing the unit square is clearly the unit circle. This inscribing ellipse can be transformed into an ellipse that inscribes the boundary quadrilateral by mapping the parametric equation for the unit circle using $\mathbf{T}$. The matrix $\mathbf{T}$ is a general projective transformation. It preserves the properties of being a line and of being a conic (that is points on lines are mapped onto points on lines, and points on conics are mapped onto points on conics), but it possesses no metric invariants. Thus, the property of being the largest inscribing ellipse is annihilated.

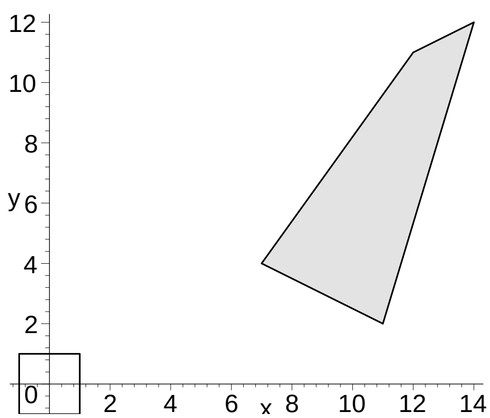


Figure 1. UNIT SQUARE AND BOUNDARY QUADRILATERAL.

## 3. EXAMPLE

Consider the unit square and general quadrilateral shown in Figure 1. The homogeneous coordinates of the vertices of the boundary quadrilateral are $W(1 : 14 : 12)$, $X(1 : 11 : 2)$, $Y(1 : 7 : 4)$, and $Z(1 : 12 : 11)$, while the image points, the square vertices, have homogeneous coordinates $w(1 : 1 : 1)$, $x(1 : 1 : -1)$, $y(1 : -1 : -1)$, and $z(1 : -1 : 1)$. The projective collineation defined by the vertices of the two quadrilaterals is

$$T = \frac{1}{6445}\begin{bmatrix} 6445 & 1027 & -1442 \\ -29123 & 4447 & -2468 \\ -11035 & -721 & 2054 \end{bmatrix}. \qquad (6)$$

The unit circle, centred on the origin, inscribing the unit square is the largest area inscribing ellipse. The matrix $\mathbf{T}$ is used to transform its parametric equation. But, because of how the problem has been posed, the circle represents the image of the ellipse that inscribes the boundary quadrilateral. To obtain the parametric equation of the desired ellipse, the inverse of $\mathbf{T}$ premultiplies the unit circle parametric equation:

$$\mathbf{e} = \mathbf{T}^{-1}\mathbf{c}, \qquad (7)$$

where $\mathbf{c} = [1 : \cos(\theta) : \sin(\theta)]$, and the resulting parametric ellipse equation is

$$\mathbf{e} = \begin{bmatrix} 8\cos(\theta) - 29\sin(\theta) - 55 \\ 20\cos(\theta) - 433\sin(\theta) - 651 \\ 50\cos(\theta) - 438\sin(\theta) - 524 \end{bmatrix}. \qquad (8)$$

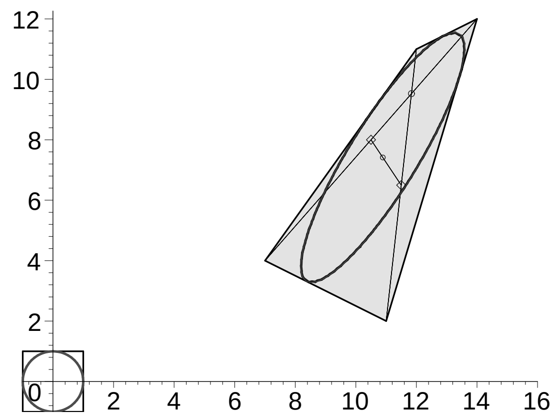


Figure 2. TRANSFORMED INSCRIBING ELLIPSES.

## 4. CONCLUSIONS AND FUTURE WORK

It is immediately apparent that the projection in the Cartesian plane of the collineation translates, rotates, and dilatates the quadrilateral and its inscribing ellipse such that it becomes the square centred on the origin inscribed by the unit circle. The ellipse centre is on the line defined by the midpoints of the quadrilateral diagonals, see Figure 2. In fact, all inscribing conics of the quadrilateral have centres which lie on this line. The conic is an ellipse if, and only if, the centre point lies between the diagonal midpoints. Hence, the largest area inscribing ellipse centre point lies on this line between the diagonals.

The general implicit equation of a conic section is $ax^2 + by^2 + cxy + dx + ey + f = 0$. The coefficients of the implicit equation of the family of inscribing ellipses can be determined in terms of the parameter $t$ in the parametric space of the diagonal midpoint line. Finally, determine the area function for the ellipse, which depends on $t$: $A(t) = A(a(t), b(t), \ldots, f(t))$. The local extremum of $A(t)$ yields the inscribing ellipse possessing the largest area. The $t$ that satisfies $dA/dt = 0$ yields the coordinates of the ellipse centre. This parameter, together with the four boundary quadrilateral edges uniquely determines the elllipse.

After the analytical details of this last step are established we shall proceed to the problem of approximating $\mathbf{T}$ in a least squares sense given boundary polygons possessing greater than four vertices.

## REFERENCES


[1] Krut, S., Company, O., Pierrot, F., 2002, "Velocity Performance Indexes for Parallel Mechanisms with Actuaion Redundancy", *Proc. Wkshp. Fund. Issues & Future Reas. Dir. for Parallel Mech. & Manip*, Oct. 3-4, 2002, C.M. Gosselin & I. Ebert-Uphoff, eds. Québec City, QC., Canada.