# Singular Configurations of Wrist-Partitioned 6R Serial Robots: a Geometric Perspective for Users

M.J.D. HAYES[1,2], M.L. HUSTY[3] and P.J. ZSOMBOR-MURRAY[4]

[1]Mechanical & Aerospace Engineering, Carleton University, Canada
[2]Institut für Automation, Montanuniversität Leoben, Austria
[3]Institut für Technische Mathematik, Geometrie und Bauinformatik, Universität Innsbruck, Austria
[4]Centre for Intelligent Machines, McGill University, Canada

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**Abstract**

In this paper the singular configurations of wrist-partitioned 6R serial robots in general, and the KUKA KR-15/2 industrial robot in particular, are analytically described and classified. While the results are not new, the insight provided by the geometric analysis for users of such robots is. Examining the problem in the joint axis parameter space, it is shown that when the end-effector reference point is taken to be the wrist centre the determinant of the associated Jacobian matrix splits into four factors, three of which can vanish. Two of the three potentially vanishing factors give a complete description of the positioning singularities and the remaining one a complete description of the orientation singularities, in turn providing a classification scheme.

# Configurations Singulières des Robots Sériels à Poignet Sphérique et Six Couples Rotöides: une Perspective Géométrique pour les Utilisateurs

**Résumé**

Dans cet article les configurations singuliéres des robots sériels à poignet sphérique et six couples rotöides en général, et celle du robot industriel KUKA KR-15/2 en particulier, sont analytiquement décrites et classifiées. Bien que les résultats ne soient pas nouveaux, la perspective fournie par l'analyse géométrique pour des utilisateurs de tels robots l'est. En examinant le problème dans l'espace commun de paramètre d'axe, on montre que quand le point de référence terminal est pris comme étant le centre du poignet le déterminant de la matrice associée du Jacobian se divise en quatre facteurs, dont trois peuvent disparaître. Deux des trois facteurs qui peuvent potentiellement disparaître donnent une description complète des singularités de positionnement et les autres une description complète des singularités d'orientation, fournissant ainsi une méthode de classification.

# 1 Introduction

The singularities of wrist-partitioned six-revolute (6R) axis serial robots have already been thoroughly investigated and classified. The aim of this paper is to describe certain results in a readily accessible way using elementary concepts from linear algebra and geometry. 6R serial robot singularities occur at configurations corresponding to singularities of the $6 \times 6$ Jacobian matrix relating the robot's joint rates to the end-effector (EE) Cartesian velocities. The formulation of the Jacobian found in this paper summarises and builds upon material found in text books such as [1, 2, 3].

Singular configurations of three degree-of-freedom (DOF) positioning robots, also called *regional* manipulators, are analysed in terms of Cartesian *singularity surfaces* in [4]. 3R positioning manipulator singularities are enumerated and classified using the notion of *generic manipulator singularities* in [5]. The orientation of the EE in a wrist-partitioned manipulator is usually adjusted by its *spherical* wrist. The nature of wrist orienting singularities are explained in the context of the *hyperbolic normal form* in [6].

The concept of *removable* singularities has been a major research area. This notion states that if a singularity occurs on the path between two configurations, it may, under certain conditons, be removed by slightly altering the path. Removable singularities occur in robots possessing three, four, and five DOF, whereas singularities associated with six DOF robots are nonremovable. A representative method for removing singularities from the three positioning joints of a six DOF PUMA is investigated in [7]. Classification and enumeration of 5R robots possessing nonremovable singularities is presented in [8].

Singularities in redundantly actuated manipulators can lead to self motions. These are, perhaps, the most insidious because the EE can move when the manipulator *should* be stationary. A recent investigation into particular parallel platform architectures, one intended for use as a flight simulator, have self motions in every configuration in its workspace [9]. Additional work has been directed towards determining conditions for avoidability and unavoidability in serial redundant robots, see [10].

Hunt [11] revived a geometric entity called the *cylindroid*, which fell from use during the middle part of the twentieth century. The cylindroid is a ruled surface generated by the resultant wrench of two conjugate screws of equal pitch. It is well known that the columns of the Jacobian are Plücker line, or screw coordinates of the six revolute axes, see [12]. Certain properties of the cylindroid regarded in the context of screw theory provide an algebraic-geometric framework of great use in the rank analysis of robot manipulator Jacobians [13]. For instance, the set of all singular configurations of serial six DOF manipulators is described using Lie algebra properties of the screw space in [14].

There are many other important contributions regarding general 6R singularities contained in the literature. Some representative examples are [15, 16, 17, 18, 19]. The list is by no means exhaustive, but is included to serve as a guideline for further study by the reader so inclined. The literature is well known among kinematicians, geometricians, and robotics researchers. Despite this fact, operating manuals for industrial models give either an insignificant treatment of the subject, or none at all, see for instance [20, 21, 22]. What is truly unfortunate is that it appears most users of 6R robots are generally not well acquainted

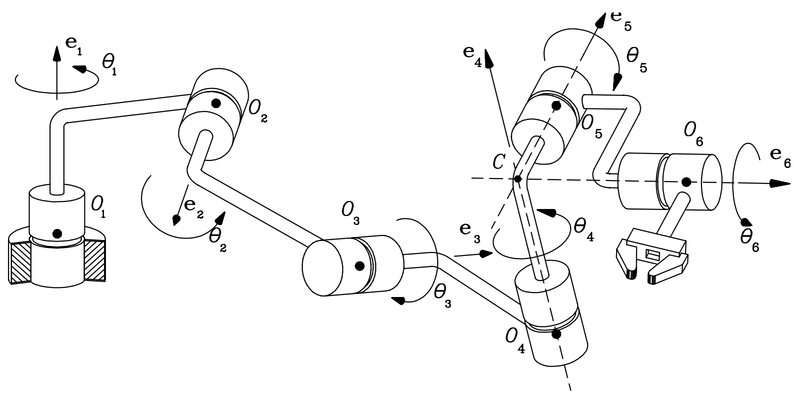


Figure 1: General $6R$ wrist-partitioned architecture.

with the literature, rather only the operating manuals of their specific robot.

When the robot comes to a sudden, crashing halt the operator is often left mystified as to why. Courses offered by robot manufacturers are a partial remedy in that cautionary lessons are given: warning the operator not to programme positions too near the work space boundary; avoid alignment of some axes; exercise caution in regions close to singularities and monitor for rapid increases in joint rates. However, the learning curve to the understanding of the causes is frustratingly long and steep. Additionally, while the English-language literature examined gives a complete and exact classification, it requires a very advanced level of mathematical and geometric knowledge and experience. What is required, from a robot users point of view, is a focused geometric interpretation of how the singularities arise, given the structure of the associated Jacobian. This gives the motivation to present the following analysis.

## 1.1 Kuka KR-15/2 Description

A wrist-partitioned, or decoupled manipulator is defined as one whose *wrist* axes (the last three axes) intersect in a common point, see Figure 1. The wrist is also *spherical* because when the intersection point, $C$, of the axes is fixed then all points on the wrist move on spheres centred at $C$. It is also said to be *partitioned*, or *decoupled* because the positioning and orienting problems can be considered separately. That is, when point $C$ is the end-effector (EE) reference point, arbitrary displacements can be thought of as the translation of point $C$ combined with the orientation of the EE reference frame, whose origin is $C$.

The KUKA KR-15/2 is illustrated in Figure 2, showing its six axes together with its base, $\{B\}$, and EE, $\{E\}$, reference frames. Coordinate reference frames are attached to each link using the Denavit-Hartenberg procedure [23]. Thus, the EE can be brought to any desired position and orientation, within the workspace of the robot, by changing the joint angles $\theta_i$ about their respective joint rotation axes $\mathbf{e}_i$, where $i \in \{1, 2, \ldots, 6\}$ (see Figure 1). The point of intersection of axes 4, 5, and 6, point $C$ in Figures 1 and 2, is considered as the EE reference point.

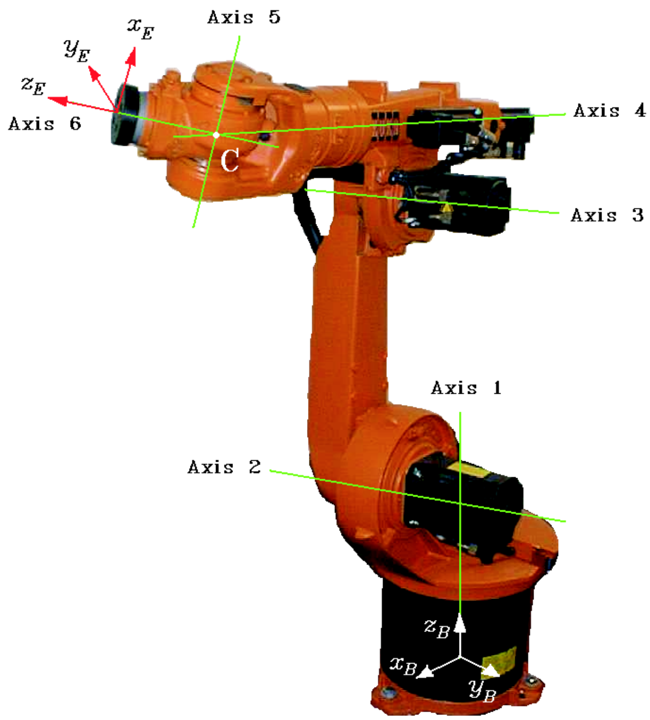


Figure 2: The six axes, base, $\{B\}$, and EE, $\{E\}$, reference frames of a KR-15/2.

# 2 The Jacobian

The Jacobian is a time-varying linear transformation that relates the Cartesian velocities of the EE to the time rate of change of the joint angles. It allows for the relationship between the two vectors to be expressed as

$$\mathbf{v} = \mathbf{J}\dot{\boldsymbol{\theta}}, \tag{1}$$

where the six joint rates are given by

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \ldots & \dot{\theta}_6 \end{bmatrix}^T. \tag{2}$$

The velocity vector is

$$\mathbf{v} = \begin{bmatrix} \mathbf{w} \\ \dot{\mathbf{c}} \end{bmatrix}, \tag{3}$$

with

$$\mathbf{w} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \dot{\mathbf{c}} = \begin{bmatrix} \dot{c}_x \\ \dot{c}_y \\ \dot{c}_z \end{bmatrix}, \tag{4}$$

where $\mathbf{w}$ is the angular velocity vector of the EE reference frame and $\dot{\mathbf{c}}$ the linear velocity vector of $C$ all relative to the fixed base frame.

It is well known that the determinant of the Jacobian of a six-axis robot is invariant under a change of the EE reference point [2, 3]. It will prove useful in the analysis to consider $C$ as this point. To determine the Jacobian matrix, we require the direction vectors of the joint axes, $\mathbf{e}_i$. We additionally need the position vectors, $\mathbf{r}_i$, of point $C$ with respect to the $i$th joint axis coordinate frame origin. Both $\mathbf{e}_i$ and $\mathbf{r}_i$ must be expressed in terms of the base coordinate reference system.

Moreover, we will require the *moments* of the lines of the joint axes relative to the origin of the base frame, shown in Figure 2. The moment vector of a line about point $C$, indicated by $\mathbf{m}_C$, is defined as the cross-product of the position vector of a point on the line, emanating from point $C$, with the line-bound direction vector of the line itself. If the position vector is $\mathbf{r}$ and the line-bound vector is $\mathbf{e}$ then the moment vector with respect to $C$ is defined as

$$\mathbf{m}_C \quad = \quad \mathbf{r} \times \mathbf{e}. \tag{5}$$

It is convenient to instead define the $\mathbf{r}_i$ as the vector from the origin of the $i$th axis reference frame, indicated by $O_i$, to point $C$, rather than as pointing from $C$ to $O_i$. In the definitions below, the order of the cross-product is reversed to agree with the change of sense of the $\mathbf{r}_i$. It is important to note that, although we will focus our attention in particular on the KUKA KR-15/2, the only condition on the following analysis is that the wrist be partitioned.

## 2.1 Positioning Problem

For a general wrist-partitioned 6R manipulator, the motion of the EE can be decoupled into distinct components: the *positioning* of point $C$ and the *orientation* of the EE [4, 15]. The location of point $C$ in the base reference frame is clearly independent of joint angles $\theta_4$, $\theta_5$ and $\theta_6$. In this sense, the EE reference point can be brought to different positions only by changing the angles about axes 1, 2 and 3. Therefore, the linear velocity of point $C$ depends only on the time rate of change of the joint angles $\theta_1$, $\theta_2$ and $\theta_3$. The linear velocity components contributed by each angular joint velocity must be perpendicular to the planes spanned by corresponding pairs of vectors of angular joint velocities and the $\mathbf{r}_i$, which may be expressed as $\dot{\theta}_i\mathbf{e}_i \times \mathbf{r}_i = \boldsymbol{\omega}_i \times \mathbf{r}_i$. We can write:

$$\dot{\mathbf{c}} \quad = \quad \dot{\theta}_1\mathbf{e}_1 \times \mathbf{r}_1 + \dot{\theta}_2\mathbf{e}_2 \times \mathbf{r}_2 + \dot{\theta}_3\mathbf{e}_3 \times \mathbf{r}_3, \tag{6}$$

again, $\mathbf{r}_i$ being the position vector of $C$ with respect to $O_i$, and $\mathbf{e}_i$ the direction vector of the axes, both expressed in coordinates of the base frame.

## 2.2 Orienting Problem

The angular velocity vector, $\mathbf{w}$, of the EE reference frame whose origin is on $C$ can be written as the vector sum of the contributions of the angular velocities of the individual joints:

$$\mathbf{w} \quad = \quad \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots + \boldsymbol{\omega}_6 \quad = \quad \dot{\theta}_1\mathbf{e}_1 + \dot{\theta}_2\mathbf{e}_2 + \ldots + \dot{\theta}_6\mathbf{e}_6. \tag{7}$$

### 2.3 The Jacobian Matrix

Given the relations in Equations (6) and (7) we see immediately that the Jacobian of Equation (1) has the form:

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix}, \\ &= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{e}_6 \\ \mathbf{e}_1 \times \mathbf{r}_1 & \mathbf{e}_2 \times \mathbf{r}_2 & \mathbf{e}_3 \times \mathbf{r}_3 & & \mathbf{0}_{3\times 3} & \end{bmatrix}. \end{aligned} \tag{8}$$

The six columns of $\mathbf{J}$ are the Plücker line ray-coordinates of the six axes. They are distinguished from Plücker line axis-coordinates by virtue of the order of their elements [24]. As ray-coordinates, the first three elements in each column are the direction cosines of the corresponding axis; the last three are the components of the moment of the axis with respect to $C$. Because axes 4, 5 and 6 all pass through $C$ their moment components with respect to $C$ are zero, hence $\mathbf{J}_{22} = \mathbf{0}_{3\times 3}$.

# 3 Singularities

When the Jacobian becomes rank deficient, the system of equations relating the joint rates to the EE velocity contain linear dependencies (the degree of the deficiency equals the number of dependencies among the equations). From a computational point of view this implies that the system cannot be solved for $\dot{\boldsymbol{\theta}}$. Either no solutions exist, or an infinite number do. This means that no unique set of joint rates map to the desired EE velocity and control of the robot becomes problematic.

The conditions on loss of full rank are exactly the conditions on the relative positions and orientations of the six axes leading to singular configurations of the robot. When in a singular configuration there is some direction along, or surface contained in the workspace upon which it is impossible to move, or apply forces and moments, regardless of the joint rates, or joint torques. This is a consequence of the design of the robot reflected in the structure of Equation (1).

In general, the determinant of a large square matrix, $\mathbf{J}$, that can be sub-divided into four distinct square sub-matrices, $\mathbf{J}_{11}$, $\mathbf{J}_{12}$, $\mathbf{J}_{21}$ and $\mathbf{J}_{22}$, is not equal to the products of the determinants of its four square sub-matrices. However, it it can be shown [25] that if the lower-right sub-matrix, $\mathbf{J}_{22}$, contains only zeroes then the determinant of $\mathbf{J}$ is the negative product of the upper-right and lower-left sub-determinants.

Clearly, $\mathbf{J}_{22}$ in Equation (10) is all zeroes. Therefore, $\det(\mathbf{J}) = -(\det(\mathbf{J}_{12}))(\det(\mathbf{J}_{21}))$. Here $\det(\mathbf{J}_{12})$ contributes the first three factors in Equation (11) while $\det(\mathbf{J}_{21})$ contributes the fourth. The factors from the upper-left sub-determinant have no effect on $\det(\mathbf{J})$, nor on conditions for singular configurations. Hence, the analysis that follows in not restricted to the particular architecture of the KUKA KR-15/2. It can be tailored to any 6R wrist-partitioned manipulator.

Because $\mathbf{J}_{21}$ concerns only the linear velocity of $C$ while $\mathbf{J}_{12}$ concerns only the angular velocity of the wrist, the singular configurations associated with the vanishing of the factors of the determinant of $\mathbf{J}$ can be classified according as the configuration is position, or orientation singular.

### 3.1 The KR-15/2 Jacobian

For the KR-15/2 the first axis always points along the $z$-axis of the base frame, see Figure 2. Moreover, axes 2 and 3 are parallel to each other and to the $xy$-plane of the base frame, this in turn means that $\mathbf{e}_2 = \mathbf{e}_3$. Hence, $\mathbf{J}_{11}$ simplifies to

$$\mathbf{J}_{11} = \begin{bmatrix} 0 & e_{3_x} & e_{3_x} \\ 0 & e_{3_y} & e_{3_y} \\ e_{1_z} & 0 & 0 \end{bmatrix}. \tag{9}$$

These simplifications, in turn, simplify the expressions for the cross-products in $\mathbf{J}_{21}$. The Jacobian reduces to

$$\mathbf{J} = \begin{bmatrix} 0 & e_{3_x} & e_{3_x} & e_{4_x} & e_{5_x} & e_{6_x} \\ 0 & e_{3_y} & e_{3_y} & e_{4_y} & e_{5_y} & e_{6_y} \\ e_{1_z} & 0 & 0 & e_{4_z} & e_{5_z} & e_{6_z} \\ -e_{1_z} r_{1_y} & e_{3_y} r_{2_z} & e_{3_y} r_{3_z} & 0 & 0 & 0 \\ e_{1_z} r_{1_x} & -e_{3_x} r_{2_z} & -e_{3_x} r_{3_z} & 0 & 0 & 0 \\ 0 & e_{3_x} r_{2_y} - e_{3_y} r_{2_x} & e_{3_x} r_{3_y} - e_{3_y} r_{3_x} & 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

These simplifications are valid for any 6R wrist-partitioned robot whose second and third axes are parallel, both perpendicular to the first axis. The determinant of $\mathbf{J}$, which is the negative product of the determinants of $\mathbf{J_{21}}$ and $\mathbf{J_{12}}$, can be factored into the following 4 products:

$$\begin{aligned} \det(\mathbf{J}) = {} & e_{1_z}(r_{2_z} e_{3_y} r_{3_x} - e_{3_x} r_{2_z} r_{3_y} + e_{3_x} r_{3_z} r_{2_y} - r_{3_z} e_{3_y} r_{2_x})(r_{1_y} e_{3_x} - e_{3_y} r_{1_x}) \\ & (e_{4_x} e_{5_z} e_{6_y} - e_{4_x} e_{6_z} e_{5_y} + e_{4_y} e_{5_x} e_{6_z} - e_{4_y} e_{6_x} e_{5_z} + e_{4_z} e_{6_x} e_{5_y} - e_{4_z} e_{5_x} e_{6_y}). \end{aligned} \tag{11}$$

## 4 Classification of Singularities

All conditions which cause a configuration of the KR-15/2, or an architecturally similar 6R robot, to be singular are represented algebraically by Equation (11). The determinant vanishes whenever any, or any combination, of the factors vanishes. Since this causes the Jacobian to become rank deficient, these conditions represent general singularities, not simply those for position or orientation, of every wrist-partitioned 6R robot. From the standpoint of vectors, it is clear that the second factor will vanish when the EE reference point $C$ lies in the plane spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$, see Figure 3. The third factor vanishes when $C$ lies on axis 1. The fourth factor is clearly the crossproduct of the three wrist axes, and vanishes whenever any pair becomes collinear. Still, we believe insight is gained by considering the singularities from the perspective of the basis of the vectorspace comprising the components of the $\mathbf{e}_i$ and $\mathbf{r}_i$. We will now discuss the geometric implications.

## 4.1 Case 1

The first factor is $e_{1_z}$. Since this quantity represents the direction of the first joint axis it can never be identically zero. Indeed, the components of the direction vectors of the joint axes are direction cosines and thus *unit vectors*. We may safely set $e_{1_z} = 1$ and rewrite the determinant as the product of just three factors:

$$\begin{aligned} \det(\mathbf{J}) = &(r_{2_z} e_{3_y} r_{3_x} - e_{3_x} r_{2_z} r_{3_y} + e_{3_x} r_{3_z} r_{2_y} - r_{3_z} e_{3_y} r_{2_x})(r_{1_y} e_{3_x} - e_{3_y} r_{1_x}) \\ &(e_{4_x} e_{5_z} e_{6_y} - e_{4_x} e_{6_z} e_{5_y} + e_{4_y} e_{5_x} e_{6_z} - e_{4_y} e_{6_x} e_{5_z} + e_{4_z} e_{6_x} e_{5_y} - e_{4_z} e_{5_x} e_{6_y}). \end{aligned} \tag{12}$$

Should any of these factors vanish the associated configuration of the robot is singular, i.e. at least one degree of freedom is lost. In the following three subsections the conditions that cause each of the three factors to vanish are examined from the perspective of the parameter space geometry. The corresponding conditions in the Cartesian space of the base coordinate reference system are also discussed.

## 4.2 Case 2: Elbow Singularity

If the first factor in Equation (12) is equal to 0 the corresponding singular configuration is called an *elbow singularity*. Equation (13) involves products of the elements of $\mathbf{e}_3$, $\mathbf{r}_2$ and $\mathbf{r}_3$.

$$r_{2_z} e_{3_y} r_{3_x} - e_{3_x} r_{2_z} r_{3_y} + e_{3_x} r_{3_z} r_{2_y} - r_{3_z} e_{3_y} r_{2_x} \quad = \quad 0. \tag{13}$$

In the coordinate space with these parameters as basis vectors, Equation (13) is an eight-dimensional third-order surface; $(e_{3_x}, e_{3_y}, r_{2_x}, r_{2_y}, r_{2_z}, r_{3_x}, r_{3_y}, r_{3_z})$ being the parameters.

Neither the relative layout of the three vectors nor their magnitudes are affected by rotations about axis 1, so we may consider axes 2 and 3 only when they are parallel to the $yz$-plane. We then consider the intersection of the surface with the plane $e_{3_x} = 0$. In this plane the $x$-components of the axis direction vectors are 0. Thus, terms containing $e_{3_x}$ vanish and the factor reduces to a five-dimensional cubic curve (the curve of intersection of the plane and the parameter singularity surface):

$$r_{2_z} e_{3_y} r_{3_x} - r_{3_z} e_{3_y} r_{2_x} \quad = \quad 0. \tag{14}$$

The remaining two terms contain only $e_{3_y}$ because axes 2 and 3 are always parallel to the $xy$-plane and never have a $z$-direction component. Since the direction vector can never vanish, $\mathbf{e}_3$ can be normalized, which leaves the condition:

$$r_{2_z} r_{3_x} = r_{3_z} r_{2_x}, \text{ or } \quad r_{2_z} : r_{3_z} = r_{2_x} : r_{3_x}. \tag{15}$$

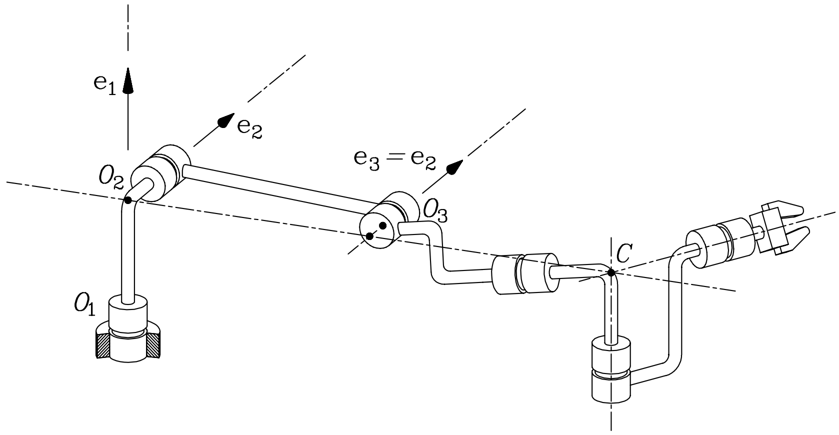


Figure 3: Stäubli RX series *elbow singular* configuration.

After inspecting Equation (15) it is clear the condition is satisfied whenever $\mathbf{r}_2$ and $\mathbf{r}_3$ are aligned on the $z$- or $x$-axes, respectively. But, in general it is satisfied whenever $\mathbf{r}_2$ and $\mathbf{r}_3$ are aligned. Recall that $\mathbf{r}_i = \mathbf{c} - \mathbf{O}_i$, which means we can write Equation (15) in terms of the angle between the $z$- and $y$-components for each of $\mathbf{r}_2$ and $\mathbf{r}_3$, i.e., $\phi_1$ and $\phi_2$:

$$\|\mathbf{r}_3\| \cos\phi_3 : \|\mathbf{r}_2\| \cos\phi_2 \quad = \quad \|\mathbf{r}_3\| \sin\phi_3 : \|\mathbf{r}_2\| \sin\phi_2. \tag{16}$$

Equation (16) can only be satisfied when $\phi_3 = \pm\phi_2 \pmod{\pi}$, or when the magnitude of either $\mathbf{r}_2$ or $\mathbf{r}_3$ vanishes. The latter requires that point $C$ be located on the origin of the coordinate reference frame of either joint 2 or 3, but this is physically impossible for the KR-15/2, and most other 6R wrist-partitioned robots, due to link interference. Additionally, the case where $\phi_3 = \phi_2 \pm \pi$ is precluded by joint limits and interference. This type of positional singularity is therefore restricted to the condition that $\phi_3 = \pm\phi_2$.

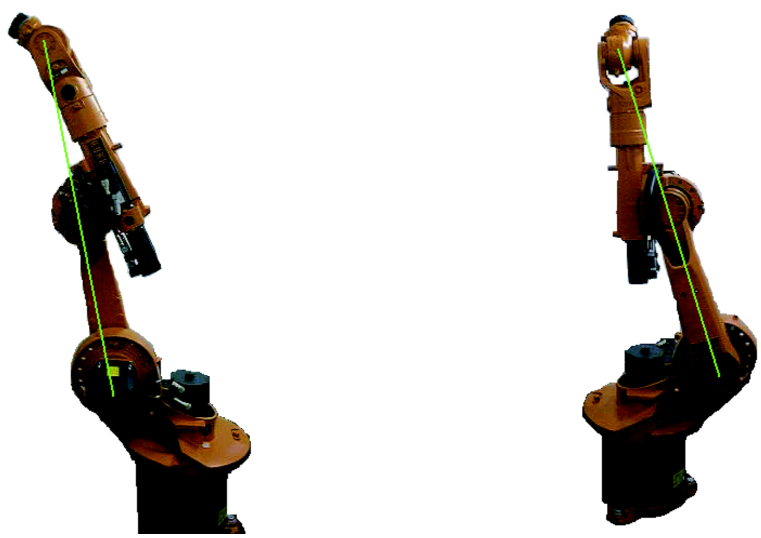

Figure 4: Two KUKA KR-15/2 *elbow singular* configurations.

In summary, the elbow singular configurations occur whenever point $C$ lies in the plane spanned by axes 2 and 3. For the KUKA KR-15/2, this can only occur when $O_2$, $O_3$ and $C$ are distinct and collinear. When the robot is in an elbow singular configuration changes in the angles of joints 2 and 3, $\Delta\theta_2$ and $\Delta\theta_3$, can produce motions only in the direction normal to the plane $\Pi_{23}$ containing $\mathbf{e}_2$ and $\mathbf{e}_3$. No motions of $C$ in $\Pi_{23}$ parallel to $\mathbf{e}_2$ and $\mathbf{e}_3$ are possible. Two elbow singular configurations are shown in Figure 4.

Owing to the construction of the KR-15/2 the elbow singular sub-space comprises a portion of the surface of a fixed torus centred at the origin of the base reference frame. In general, this is true for all wrist-partitioned robots. The torus shape parameters are dependent on the link lengths and joint offset between axis 1 and 2. For robots designed such that axes 1 and 2 intersect, such as the Stäubli RX series, and the Kawasaki JS-10, the torus degenerates to a double sphere.

Clearly, the elbow singularity surface represents the limits of the robot workspace. As the working envelope is easily visualised, elbow singular configurations should be easily anticipated. They may be avoided by keeping the EE at a *safe* distance from its limits.

## 4.3 Case 3: Shoulder Singularity

If the second factor in Equation (12) vanishes the configuration is said to be *shoulder singular*, see Figure 5. The following arguments illustrate why.

$$r_{1_y} e_{3_x} - e_{3_y} r_{1_x} \quad = \quad 0. \tag{17}$$

Equation (17) may be viewed as a four-dimensional quadric surface in the coordinate space whose basis are the parameters $(e_{3_x}, e_{3_y}, r_{1_x}, r_{1_y})$. It can vanish under three circumstances.

1. If $e_{3_x} = 0$ then either $e_{3_y} = 0$, or $r_{1_x} = 0$. Since $\mathbf{e}_3$ is a direction vector which remains parallel to the $xy$-plane then both $x$- and $y$-components cannot simultaneously vanish. Hence, the factor will vanish only if $r_{1_x} = 0$. This means that point $C$ lies in the $yz$-plane of the base. Because of the construction of the KR-15/2, $C$ is additionally constrained to be on the $z$-axis in this plane.
2. If $e_{3_y} = 0$ then either $e_{3_x} = 0$, or $r_{1_y} = 0$. In this circumstance $r_{1_y} = 0$ because, as mentioned above, both $x$- and $y$-components cannot simultaneously vanish. Now, $C$ will lie in the $zx$-plane of the base frame. Again, in the case of the KR-15/2, $C$ is additionally constrained to lie on the $z$-axis.
3. If neither $e_{3_x} = 0$, or $e_{3_y} = 0$ then $r_{1_x} = r_{1_y} = 0$. This condition also means that $C$ is on the $z$-axis of the base frame. While it is possible for a KUKA KR-15/2 to attain this configuration, it is not possible for the Stäubli RX architecture.

Shoulder singular configurations for the KUKA KR-15/2 occur when point $C$ lies anywhere on the $z$-axis of the base frame, which is the axis of joint 1, see Figure 6. But, in general these singularities occur when $C$ lies on an architecture specific line in the plane $\Pi_{12}$, containing axes 1 and 2. Plane $\Pi_{12}$, whose orientation is set by $\vartheta_1$, is covered by lines parallel to $\mathbf{e}_2$ and $\mathbf{e}_3$ which all intersect $\mathbf{e}_1$. When $C$ lies in $\Pi_{12}$, no linear, or angular velocities are

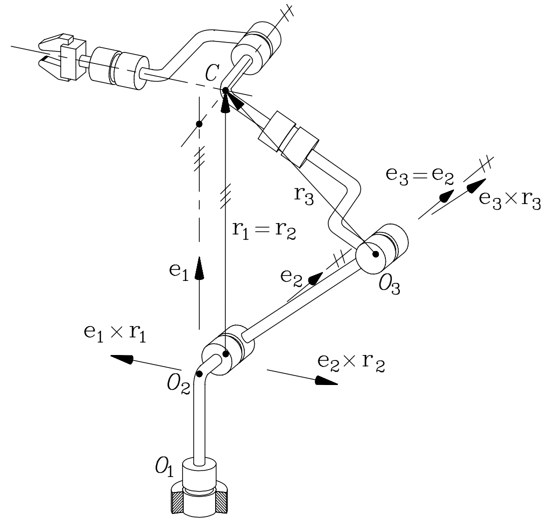


Figure 5: A Stäubli RX series *shoulder singular* configuration.

possible in the directions parallel to $\mathbf{e}_2$ and $\mathbf{e}_3$, although it can move on a line parallel to $\mathbf{e}_1$, the axis of the first joint. Point $C$ can have any location along this singularity line, and the line itself can be rotated about axis 1. Therefore, the shoulder singularities occupy the surface of a right-circular cylinder of radius $r$, whose central axis lies on axis 1. For the KR-15/2, $r = 0$. Because the shoulder singularities are contained within the reachable workspace, they are more difficult to anticipate than elbow singularities.


Figure 6: Two KUKA KR-15/2 *shoulder singular* configurations.

## 4.4 Case 4: Wrist Singularity

The last case concerns the third factor in Equation (12). This condition depends only on the direction cosines of the three wrist axes 4, 5 and 6. For this reason they are termed *wrist singularities*, and are pure *orienting singulatities* in contrast to the *positioning singularities* of the elbow and shoulder, which depend on the position of $C$.

Equation (18) represents the determinant of $\mathbf{J}_{21}$. It may be thought of as a nine-dimensional third-order surface in the coordinate space of the nine components of $\mathbf{e}_4$, $\mathbf{e}_5$ and $\mathbf{e}_6$.

$$e_{4_x}e_{5_z}e_{6_y} - e_{4_x}e_{6_z}e_{5_y} + e_{4_y}e_{5_x}e_{6_z} - e_{4_y}e_{6_x}e_{5_z} + e_{4_z}e_{6_x}e_{5_y} - e_{4_z}e_{5_x}e_{6_y} \quad = \quad 0. \tag{18}$$

For orientations of the wrist axes the location of the wrist centre, $C$, is arbitrary. Let it be located on the origin of the fixed base frame. Moreover, this singularity condition depends on the relative orientation of axes 4, 5 and 6, so we can consider one to be fixed relative to the others and the base frame. In this way we will consider the intersections of the cubic surface with the nine individual coordinate planes. Here we will examine $\mathbf{e}_5$:

1. $\mathbf{e}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: Equation (18) reduces to

$$e_{4_y}e_{6_z} - e_{4_z}e_{6_y} = 0. \tag{19}$$

2. $\mathbf{e}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: Equation (18) reduces to

$$e_{4_z}e_{6_x} - e_{4_x}e_{6_z} = 0. \tag{20}$$

3. $\mathbf{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: Equation (18) reduces to

$$e_{4_x}e_{6_y} - e_{4_y}e_{6_x} = 0. \tag{21}$$

The above three intersections leave the condition that the cross-product of the projections of two direction vectors in the respective coordinate planes must vanish. This means $\mathbf{e}_4$ must be parallel to $\mathbf{e}_6$. This situation arises whenever axes 4 and 6 are aligned.

Similar analysis can be made by examining both $\mathbf{e}_4$ and $\mathbf{e}_6$ in the same way as we just examined $\mathbf{e}_5$. The results show that the singular conditions require $\mathbf{e}_5$ to be parallel to $\mathbf{e}_6$, and $\mathbf{e}_4$ to be parallel to $\mathbf{e}_5$. However, because of the construction of the wrist: axes 5 and 6 as well as axer 4 and 5 are always perpendicular. Thus, these conditions are not physically possible and the corresponding wrist singular configurations are never attainable.

Figure 7: Two KUKA KR-15/2 *wrist singular* configurations.

To summarise, the condition for wrist singular configurations is only satisfied when axes 4 and 6 are parallel. Thus, the reachable wrist singular configurations comprise only a portion of the third-order wrist parameter singularity surface: a quadratic curve. Figure 7 shows two wrist singular configurations. Because this condition can be satisfied independently of the position of $C$, the following discomforting fact applies to all wrist-partitioned 6R robots: the entire reachable workspace is potentially wrist singular.

# 5 Conclusions

We have presented an analytical description and classification of the complete set of singular configurations of the KUKA KR-15/2 six-axis serial robot in particular, and all wrist-partitioned 6R robots in general. The analysis shows that all general singular positions are either shoulder, elbow, or wrist singularities, or any combination thereof, no others exist. The shoulder singular positions of $C$ comprise an architecture specific right circular cylinder whose central axis is the $z$-axis of the base frame. For the KUKA KR-15/2 the cylinder radius is zero and the cylinder degenerates to the $z$-axis itself. Elbow singularities of point $C$ occupy a portion of a torus, or double sphere, however, this surface is the boundary of the workspace. For the KUKA KR-15/2 the surface is a torus segment. Wrist singularities can occupy the entire reachable workspace of $C$. It is interesting to note that it has never been explicitly stated in the literature, or so it appears, that the entire orientable workspace is potentially singular, even in papers dealing specifically with orientation singularities, such as [10].

While these results are not entirely new, the geometric analysis of the conditions that cause the Jacobian to become rank deficient are. When the robot suddenly jerks to a halt because the controller has anticipated motion through a singularity, the operator is often left mystified, despite the clarity of the error message that a singular configuration has been reached. We believe that presenting the conditions in this way allows the users of wrist-partitioned industrial robots to better understand their singular configurations.

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