Classification of Singular Configurations of the KUKA KR-15/2 Six-axis Serial Robot

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Abstract

In this paper the singular configurations of the KUKA KR-15 wrist-partitioned serial industrial robot are analytically described and classified. While the results are not new, the insight provided for users of wrist-partitioned by the geometric analysis is. It is shown that the determinant of the associated Jacobian matrix splits into four factors, three of which can vanish. When the end-effector (EE) reference point is taken to be the wrist centre, which can be done without loss of generality for wrist-partitioned robots, two of these three factors give a complete description of the positioning singularities and the remaining one a complete description of the orientation singularities. Moreover, it is shown that for robots possessed of similar architecture, the determinant of the Jacobian always factors into exactly the determinant of a 3×3 sub-Jacobian concerned exclusively with wrist-centre linear velocities, and another sub-Jacobian concerned exclusively with the angular velocities of the EE reference frame. This in turn means that the associated singularities can be classified according as the configuration is position, or orientation singular.

1 Introduction

The singularities of wrist-partitioned six-axis serial robots have already been thoroughly investigated and classified [1, 2, 3, 4, 5, 6, 7]. Despite this fact, operating manuals for industrial models give either an insignificant treatment of the subject, or none at all, see for instance [8]. Additionally, the English-language literature examined does not give a clear geometric interpretation of how the singularities arise, given the structure of the associated Jacobian. This gives the motivation to present the following analysis.

A wrist-partitioned, or decoupled manipulator is defined as one whose *wrist* axes, the last three axes, intersect in a common point, see Figure 1. The wrist is also *spherical* because when the intersection point, C, of the axes is fixed then all points on the wrist move on spheres centred at C. It is also said to be *partitioned*, or *decoupled* because the positioning and orienting problems can be considered separately. That is, when point C

is the end-effector (EE) reference point, arbitrary displacements can be thought of as the translation of point C combined with the orientation of the EE reference frame, whose origin is C.

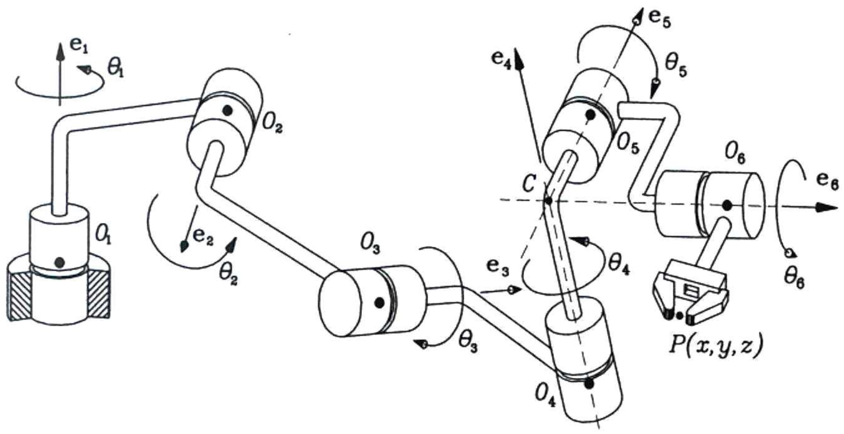

Figure 1: A general $6R$ robot with wrist-partitioned architecture (courtesy of Prof. J. Angeles).

Figure 2: A KUKA KR-15 in a general, non-singular configuration.

2 Manipulator Description

The KUKA KR-15 is illustrated in Figure 2. Its six axes together with its base and EE reference frames are depicted in Figure 3. Coordinate reference frames are attached to each link using the Denavit-Hartenberg procedure [9]. Thus, the EE can be brought to any desired position and orientation, within the workspace of the robot, by changing the joint angles θ_i about their respective joint rotation axes, $\mathbf{e}_i$, where $i \in \{1, 2, \ldots, 6\}$ (see Figure 1). The point of intersection of axes 4, 5, and 6, point C in Figure 1, is considered as the EE reference point. It is well known that the determinant of the Jacobian of a six-axis robot is invariant under a change of the EE reference point. So, without loss in generality, we can consider C as this point, even for the KR-15. To determine the Jacobian matrix, we require the direction vectors of the joint axes, $\mathbf{e}_i$. We additionally need the position vectors, $\mathbf{r}_i$, of point C with respect to the ith joint axis coordinate frame origin. The Plücker ray-coordinates of each joint axis are the columns of the Jacobian [10].

The first three elements in the Plücker array are the three direction cosines of the line. The last three are the components of the moment of the line with respect to a reference point. In our case the reference point is C. The moment vector of a line about point C, indicated by $\mathbf{m}_C$, is defined as the cross-product of the position vector of a point on the line, emanating from point C, with the line-bound direction vector of the line itself. If the position vector is $\mathbf{r}$ and the line-bound vector is $\mathbf{e}$ then the moment vector with respect to C is defined as

$$\mathbf{m}_C = \mathbf{r} \times \mathbf{e}. \tag{1}$$

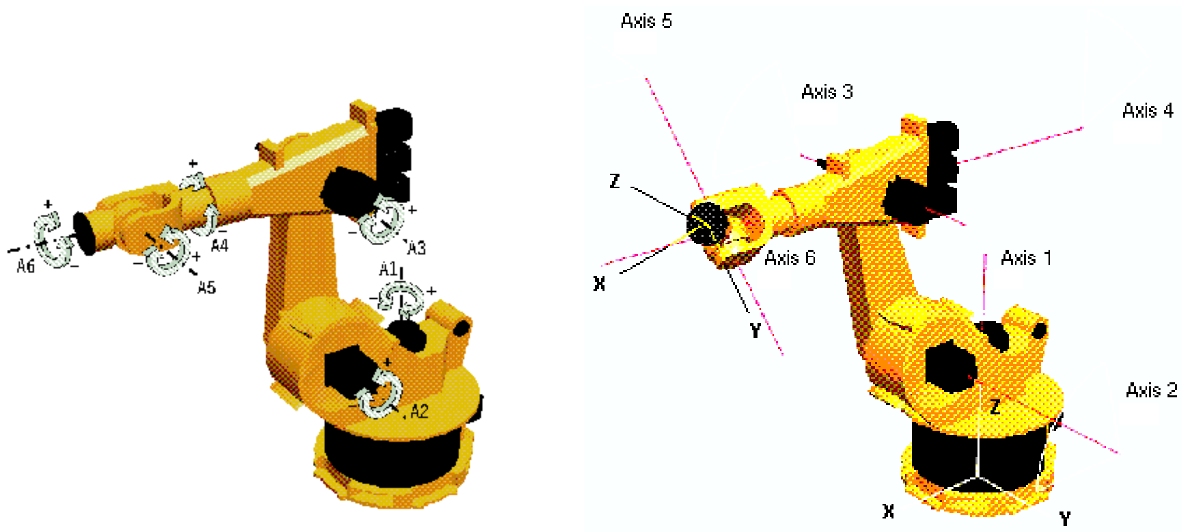

Figure 3: The six axes, A1-A6, and the base and EE reference frames of a KR-15 (note, the configuration on the left is *wrist singular*.

However, it will be more useful for us to have an expression in terms of the base coordinate reference frame, shown in Figure 3. To do this we express the $\mathbf{e}_i$ in terms of the coordinates of the base frame. We also write the $\mathbf{r}_i$ as the vector from the origin of the ith axis reference frame, indicated by O_i, to point C. These $\mathbf{r}_i$ vectors must also be transformed to the base reference frame. The order of the cross-product is then reversed to agree with the change of sense of the $\mathbf{r}_i$, i.e., they now point from O_i to C instead of the other way around.

It is important to note that, although we will focus our attention in particular on the KUKA KR-15, the only conditions are that the wrist be partitioned and that axes 2 and 3 be parallel. The offsets of the first four axes do not change the analysis.

2.1 Positioning Problem

As mentioned earlier, the motion of the EE can be decoupled into distinct problems: the *positioning* of point C and the *orientation* of the EE. The location of point C in the base reference frame is clearly independent of joint angles θ_4, θ_5 and θ_6. In this sense, the EE reference point can be brought to different positions only by changing the angles about axes 1, 2 and 3. Therefore, the linear velocity of point C depends only on the time rate of change of the joint angles θ_1, θ_2 and θ_3. The linear velocity components contributed by each angular joint velocity must be perpendicular to the planes spanned by corresponding pairs of the vectors of the angular joint velocities and the $\mathbf{r}_i$, which may be expressed as $\dot{\theta}_i\mathbf{e}_i \times \mathbf{r}_i = \boldsymbol{\omega}_i \times \mathbf{r}_i$. We can write:

$$\dot{\mathbf{c}} \;=\; \dot{\theta}_1\mathbf{e}_1 \times \mathbf{r}_1 + \dot{\theta}_2\mathbf{e}_2 \times \mathbf{r}_2 + \dot{\theta}_3\mathbf{e}_3 \times \mathbf{r}_3, \tag{2}$$

again, $\mathbf{r}_i$ being the position vector of C with respect to O_i, but expressed in the base frame.

2.2 Orienting Problem

The angular velocity vector, $\mathbf{w}$, of the EE reference frame whose origin is on C can be written as the vector sum of the contributions of the angular velocities of the individual joints:

$$\mathbf{w} \;=\; \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots + \boldsymbol{\omega}_6 \;=\; \dot{\theta}_1\mathbf{e}_1 + \dot{\theta}_2\mathbf{e}_2 + \ldots + \dot{\theta}_6\mathbf{e}_6. \tag{3}$$

2.3 The Jacobian Matrix

The Jacobian is a time-varying linear transformation that relates the Cartesian velocities of the EE, i.e., $\dot{\mathbf{c}}$ and $\mathbf{w}$, to the time rate of change of the joint angles, or joint rates. It allows for the relationship between the two vectors to be expressed as

$$\mathbf{v} \;=\; \mathbf{J}\dot{\boldsymbol{\theta}}, \tag{4}$$

where

$$\dot{\boldsymbol{\theta}} \;=\; \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \ldots & \dot{\theta}_6 \end{bmatrix}^T, \tag{5}$$

and the velocity vector is

$$\mathbf{v} \;=\; \begin{bmatrix} \mathbf{w} \\ \dot{\mathbf{c}} \end{bmatrix}, \tag{6}$$

with

$$\mathbf{w} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \dot{\mathbf{c}} = \begin{bmatrix} \dot{c}_x \\ \dot{c}_y \\ \dot{c}_z \end{bmatrix}, \tag{7}$$

where $\mathbf{w}$ is the angular velocity vector of the EE reference frame and $\dot{\mathbf{c}}$ the linear velocity vector of C all relative to the fixed base frame.

Given the relations in Equations (2) and (3) we see immediately that the Jacobian of Equation (4) has the form:

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{12} \end{bmatrix}, \\ &= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{e}_6 \\ \mathbf{e}_1 \times \mathbf{r}_1 & \mathbf{e}_2 \times \mathbf{r}_2 & \mathbf{e}_3 \times \mathbf{r}_3 & & \mathbf{0}_{3\times 3} & \end{bmatrix}. \end{aligned} \tag{8}$$

Thus it is to be seen that

$$\mathbf{J}_{11} = \begin{bmatrix} e_{1_x} & e_{2_x} & e_{3_x} \\ e_{1_y} & e_{2_y} & e_{3_y} \\ e_{1_z} & e_{2_z} & e_{3_z} \end{bmatrix}, \tag{9}$$

$$\mathbf{J}_{12} = \begin{bmatrix} e_{4_x} & e_{5_x} & e_{6_x} \\ e_{4_y} & e_{5_y} & e_{6_y} \\ e_{4_z} & e_{5_z} & e_{6_z} \end{bmatrix}, \tag{10}$$

$$\mathbf{J}_{21} = \left[\begin{bmatrix} e_{1_x} \\ e_{1_y} \\ e_{1_z} \end{bmatrix} \times \begin{bmatrix} r_{1_x} \\ r_{1_y} \\ r_{1_z} \end{bmatrix} \quad \begin{bmatrix} e_{2_x} \\ e_{2_y} \\ e_{2_z} \end{bmatrix} \times \begin{bmatrix} r_{2_x} \\ r_{2_y} \\ r_{2_z} \end{bmatrix} \quad \begin{bmatrix} e_{3_x} \\ e_{3_y} \\ e_{3_z} \end{bmatrix} \times \begin{bmatrix} r_{3_x} \\ r_{3_y} \\ r_{3_z} \end{bmatrix} \right], \tag{11}$$

$$\mathbf{J}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

3 Singularities

The above Jacobian formulation is valid for every six-axis wrist-partitioned robot jointed exclusively with R-pairs. However, it only represents robots whose EE reference point is the intersection of the wrist axes. The Jacobian depends on the relative locations and directions of the lines of the six joint axes. The lines are independent of the choice of EE reference point. It can be shown that the determinant of the Jacobian is invariant with regard to the reference point. Without loss in generality we can use the special form as illustrated above.

When the Jacobian loses full rank its determinant vanishes. The conditions on loss of full rank are exactly the conditions on the relative positions and orientations of the six lines leading to singular configurations of the robot. When in a singular configuration there is some direction along, or surface contained in the workspace upon which it is impossible to move, or apply forces and moments, regardless of the joint rates, or joint torques. This is a consequence of the structure of Equation (4).

For the KR-15 the first axis always points along the z-axis of the base frame, see Figure 3. Moreover, axes 2 and 3 are parallel to each other and to the xy-plane of the base frame, this in turn means that $\mathbf{e}_2 = \mathbf{e}_3$. Hence, $\mathbf{J}_{11}$ simplifies to

$$\mathbf{J}_{11} = \begin{bmatrix} 0 & e_{3_x} & e_{3_x} \\ 0 & e_{3_y} & e_{3_y} \\ e_{1_z} & 0 & 0 \end{bmatrix}. \tag{13}$$

These simplifications, in turn, simplify the expressions for the cross-products in $\mathbf{J}_{21}$. The Jacobian reduces to

$$\mathbf{J} = \begin{bmatrix} 0 & e_{3_x} & e_{3_x} & e_{4_x} & e_{5_x} & e_{6_x} \\ 0 & e_{3_y} & e_{3_y} & e_{4_y} & e_{5_y} & e_{6_y} \\ e_{1_z} & 0 & 0 & e_{4_z} & e_{5_z} & e_{6_z} \\ -e_{1_z} r_{1_y} & e_{3_y} r_{2_z} & e_{3_y} r_{3_z} & 0 & 0 & 0 \\ e_{1_z} r_{1_x} & -e_{3_x} r_{2_z} & -e_{3_x} r_{3_z} & 0 & 0 & 0 \\ 0 & e_{3_x} r_{2_y} - e_{3_y} r_{2_x} & e_{3_x} r_{3_y} - e_{3_y} r_{3_x} & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

The determinant of $\mathbf{J}$ can be factored into the following 4 products:

$$\begin{aligned} \det(\mathbf{J}) = {} & e_{1_z}(r_{1_y} e_{3_x} - e_{3_y} r_{1_x})(r_{2_z} e_{3_y} r_{3_x} - e_{3_x} r_{2_z} r_{3_y} + e_{3_x} r_{3_z} r_{2_y} - r_{3_z} e_{3_y} r_{2_x}) \\ & (e_{4_x} e_{5_z} e_{6_y} - e_{4_x} e_{6_z} e_{5_y} + e_{4_y} e_{5_x} e_{6_z} - e_{4_y} e_{6_x} e_{5_z} + e_{4_z} e_{6_x} e_{5_y} - e_{4_z} e_{5_x} e_{6_y}). \end{aligned} \tag{15}$$

In general, the determinant of a large square matrix that can be sub-divided into four distinct square sub-matrices, $\mathbf{J}_{11}$, $\mathbf{J}_{12}$, $\mathbf{J}_{21}$ and $\mathbf{J}_{22}$, is not equal to the products of the determinants of its four square sub-matrices. However, it is easy to show that if the upper-left square sub-matrix, $\mathbf{J}_{11}$ is singular and the lower-right square sub-matrix, $\mathbf{J}_{22}$, contains only zeros, then the determinant of the matrix is equivalent to the product of the two non-vanishing sub-determinants.

Clearly, the sub-matrices $\mathbf{J}_{11}$ and $\mathbf{J}_{22}$ in Equation (14) are always singular. Therefore, $\det \mathbf{J} = (\det \mathbf{J}_{12})(\det \mathbf{J}_{21})$. Here $\det \mathbf{J}_{12}$ contributes the first three factors in Equation (15) while $\det \mathbf{J}_{21}$ contributes the fourth. Because $\mathbf{J}_{12}$ concerns only the linear velocity of C while $\mathbf{J}_{12}$ concerns only the angular velocity of the wrist, the singular configurations associated with the vanishing of the factors of the determinant of $\mathbf{J}$ can be classified according as the configuration is position, or orientation singular.

4 Classification of Singularities

All conditions which cause a configuration of the KR-15 to be singular are represented algebraically by Equation (15). The determinant vanishes whenever any, or any combination, of the factors vanishes. Since this causes the Jacobian to become rank deficient, these conditions represent general singularities, not simply those for position or orientation, of every wrist-partitioned 6R robot. We will now discuss the geometric implications.

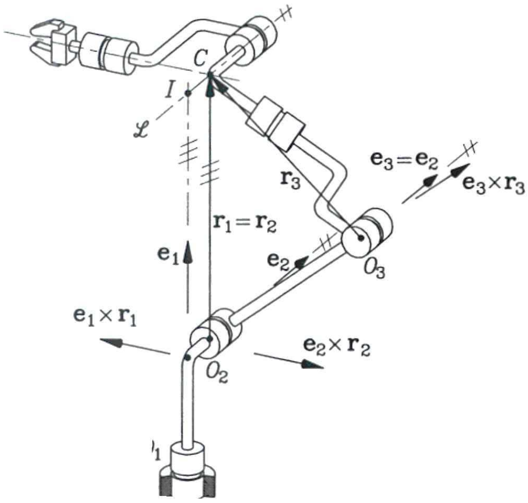

Figure 4: A PUMA 560 *shoulder singular* configuration singularity (courtesy of Prof. J. Angeles).

4.1 Case 1

The first factor is e_{1_z}. Since this quantity represents the direction of the first joint axis it can never go to zero. Indeed, the direction vectors of the joint axes may be considered as the direction cosines and thus *unit vectors*. We may safely set its value to $e_{1_z} = 1$ and rewrite the determinant as the product of just three factors:

$$\det(\mathbf{J}) = (r_{1_y}e_{3_x} - e_{3_y}r_{1_x})(r_{2_z}e_{3_y}r_{3_x} - e_{3_x}r_{2_z}r_{3_y} + e_{3_x}r_{3_z}r_{2_y} - r_{3_z}e_{3_y}r_{2_x})$$
$$(e_{4_x}e_{5_z}e_{6_y} - e_{4_x}e_{6_z}e_{5_y} + e_{4_y}e_{5_x}e_{6_z} - e_{4_y}e_{6_x}e_{5_z} + e_{4_z}e_{6_x}e_{5_y} - e_{4_z}e_{5_x}e_{6_y}). \quad (16)$$

Should any of these factors vanish the associated configuration of the robot is singular, i.e. at least one degree of freedom is lost. In the following, the conditions that cause each of the three factors to vanish are examined from a geometric perspective.

4.2 Case 2: Shoulder Singularity

If the first factor in Equation (16) vanishes then the EE reference point C lies somewhere on axis 1, the "shoulder axis" [10], the configuration is said to be *shoulder singular*. Two shoulder singular configurations are depicted in Figure 5. The following arguments illustrate why this is so.

$$r_{1_y}e_{3_x} - e_{3_y}r_{1_x} = 0. \quad (17)$$

Equation (16) may be viewed as a four-dimensional quadric surface in the coordinate space whose basis are the parameters $(e_{3_x}, e_{3_y}, r_{1_x}, r_{1_y})$. This factor can vanish under three circumstances.

1. If $e_{3_x} = 0$ then either $e_{3_y} = 0$, or $r_{1_x} = 0$. Since $\mathbf{e}$ is a direction vector which remains parallel to the xy-plane then both x- and y-components cannot simultaneously vanish. Hence, the factor will vanish only if $r_{1_x} = 0$. This means that the EE reference

point C lies in the yz-plane of the base. Because of the construction of the KR-15, C is additionally constrained to be on the z-axis in this plane.

2. If $e_{3_y} = 0$ then either $e_{3_x} = 0$, or $r_{1_y} = 0$. In this circumstance $r_{1_y} = 0$ because, as mentioned above, both x- and y-components cannot simultaneously vanish. Now, C will lie in the zx-plane of the base frame. Again, in the case of the KR-15, C is additionally constrained to lie on the z-axis.

3. If neither $e_{3_x} = 0$ nor $e_{3_y} = 0$ then $r_{1_x} = r_{1_y} = 0$. This condition also means that C is on the z-axis of the base frame.

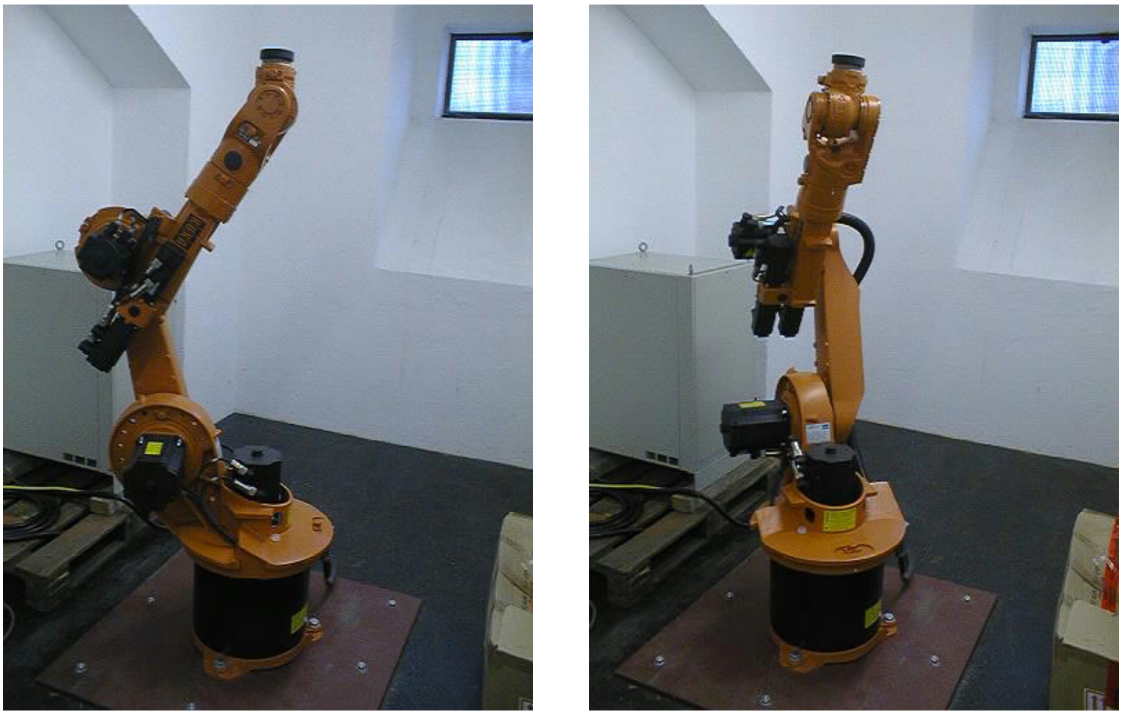

Figure 5: Two KUKA KR-15 *shoulder singular* configurations.

We can conclude that shoulder singular configurations for the KUKA KR-15 occur when point C lies anywhere on the z-axis of the base frame. Thus, the shoulder singularity quadric surface degenerates to a double line: the axis of joint 1, see Figure 5. But, in general these singularities occur when C lies in the plane, Π, containing axis 1 which is parallel to the plane defined by $\mathbf{e}_2$ and $\mathbf{e}_3$. The plane Π is covered by parallel lines which all intersect $\mathbf{e}_1$. Additionally, these lines all intersect $\mathbf{e}_2$ and $\mathbf{e}_3$ in the same point at infinity. Thus all lines through $\mathbf{e}_1$ that are parallel to $\mathbf{e}_2$ and $\mathbf{e}_3$ have one point in common, although it is not finite. When C lies in the plane Π no linear, or angular velocities are possible in the directions parallel to $\mathbf{e}_2$ and $\mathbf{e}_3$. However, point C can move on a line parallel to $\mathbf{e}_1$ in Π. The distance of this line from $\mathbf{e}_1$ depends on a particular design parameter, let's call it r. Point C can have any location along this singularity line, and the line itself can be rotated about axis 1. Hence, in general the shoulder singularities occupy the surface of a right-circular cylinder of radius r, whose central axis lies on axis 1. For the KR-15, $r = 0$.

4.3 Case 3: Elbow Singularity

If the second factor in Equation (16) is equal to 0 the corresponding singular configuration is called an *elbow singularity.* They occur whenever C lies in the plane spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$, see Figure 7. The following arguments illustrate how they arise. Equation (18) involves products of the elements of $\mathbf{e}_3$, $\mathbf{r}_2$ and $\mathbf{r}_3$.

$$r_{2_z} e_{3_y} r_{3_x} - e_{3_x} r_{2_z} r_{3_y} + e_{3_x} r_{3_z} r_{2_y} - r_{3_z} e_{3_y} r_{2_x} \;\; = \;\; 0. \tag{18}$$

In the coordinate space with these parameters as basis vectors, Equation (18) is an eight-dimensional third-order surface; $(e_{3_x}, e_{3_y}, r_{2_x}, r_{2_y}, r_{2_z}, r_{3_x}, r_{3_y}, r_{3_z})$ being the parameters.

Neither the relative layout of the three vectors nor their magnitudes are affected by rotations about axis 1. Without loss in generality we may consider axes 2 and 3 only when they are parallel to the yz-plane. We then consider the intersection of the surface with the plane $e_{3_x} = 0$. In this plane the x-components of the axis direction vectors are 0. Thus terms containing e_{3_x} vanish and the factor reduces to a five-dimensional cubic curve (the curve of intersection of the plane and the singularity surface):

$$r_{2_z} e_{3_y} r_{3_x} - r_{3_z} e_{3_y} r_{2_x} \;\; = \;\; 0. \tag{19}$$

The remaining two terms contain only e_{3_y} because axes 2 and 3 are always parallel to the xy-plane and never have a z-direction component. Since the direction vector can never vanish, $\mathbf{e}_3$ can be normalized, which leaves the condition:

$$r_{2_z} r_{3_x} = r_{3_z} r_{2_x}, \text{ or } r_{2_z} : r_{3_z} = r_{2_x} : r_{3_x}. \tag{20}$$

After inspecting Equation (20) it is clear the condition is satisfied whenever $\mathbf{r}_2$ and $\mathbf{r}_3$ are aligned on the z- or x-axes, respectively. But, in general it is satisfied whenever $\mathbf{r}_2$ and $\mathbf{r}_3$ are aligned. Recall that $\mathbf{r}_i = \mathbf{c} - \mathbf{O}_i$, which means we can write Equation (20) in terms of the angle between the z- and y-components for each of $\mathbf{r}_2$ and $\mathbf{r}_3$, i.e., ϕ_1 and ϕ_2:

$$\|\mathbf{r}_3\| \cos\phi_3 : \|\mathbf{r}_2\| \cos\phi_2 \;\; = \;\; \|\mathbf{r}_3\| \sin\phi_3 : \|\mathbf{r}_2\| \sin\phi_2. \tag{21}$$

Equation (21) can only be satisfied when $\phi_3 = \pm\phi_2 \pmod{\pi}$, or when the magnitude of either $\mathbf{r}_2$ or $\mathbf{r}_3$ vanishes. The latter requires that point C be located on the origin of the coordinate reference frame of either joint 2 or 3, but this is physically impossible for the KR-15 due to link interference. Additionally, the case where $\phi_3 = \phi_2 \pm \pi$ is precluded by joint limits and interference. This type of positional singularity is therefore restricted to the condition that $\phi_3 = \pm\phi_2$.

In summary, the elbow singular configurations occur whenever point C lies in the plane spanned by axes 2 and 3. For the KUKA KR-15, this can only occur when O_2, O_3 and C are distinct and collinear. Two elbow singular configurations are shown in Figure 7. Owing to the construction of the KR-15 the shoulder singular sub-space comprises a portion of the surface of a fixed torus centred at the origin of the base reference frame. In general, this is true for all wrist-partitioned robots. The torus shape parameters are dependent on the link lengths and joint offset between axis 1 and 2. For robots with no offset between axes 1 and 2, such as the Puma 560, and the Kawasaki JS-10, the torus degenerates to a double sphere.

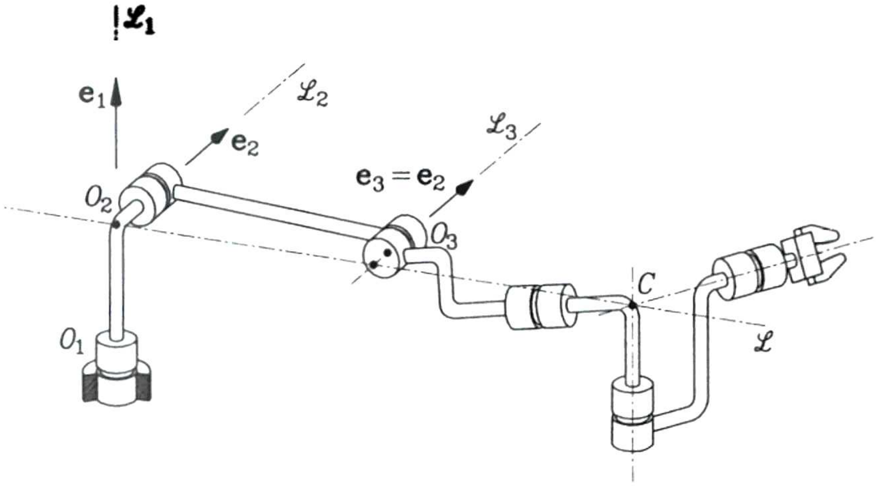

Figure 6: Puma architecture elbow singularity.

When the robot is in an elbow singular configuration changes in the angles of joints 2 and 3, θ_2 and θ_3 can produce motions only in the direction normal to the plane Π_{23} containing $\mathbf{e}_2$ and $\mathbf{e}_3$. Motions of C in Π_{23} normal to $\mathbf{e}_1 \times \mathbf{r}_1$ are not possible.

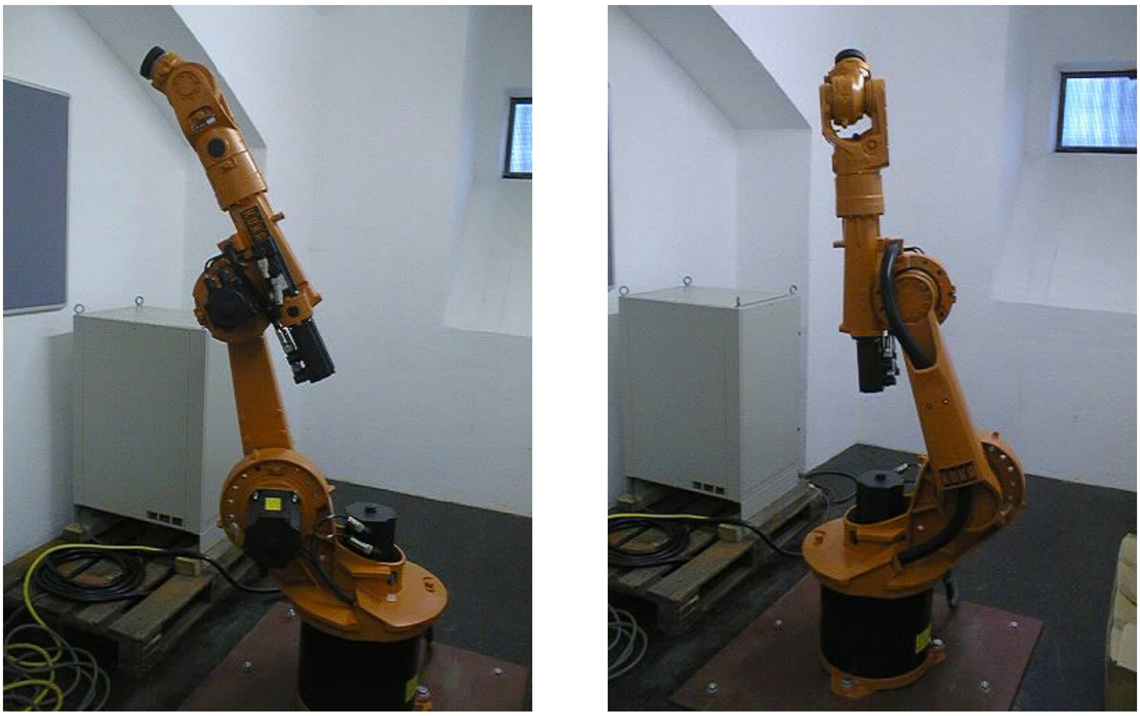

Figure 7: Two KUKA KR-15 elbow singular configurations.

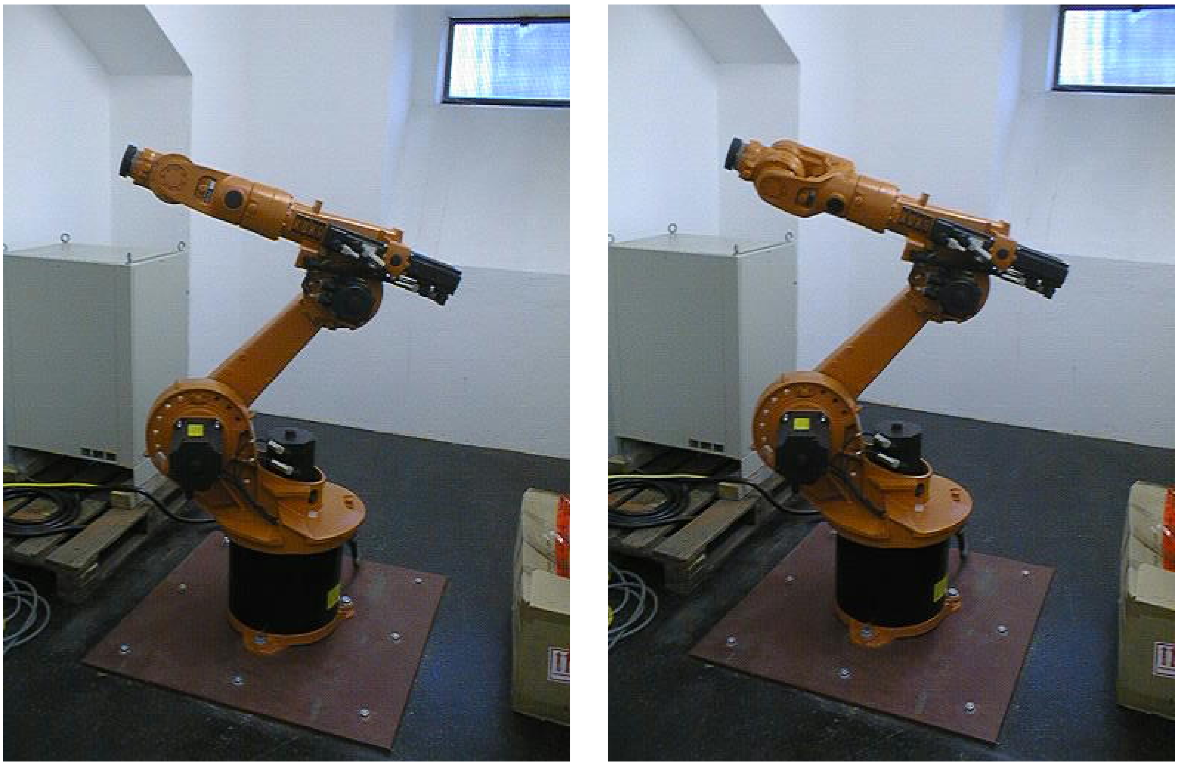

Figure 8: Two KUKA KR-15 *wrist singular* configurations.

4.4 Case 4: Wrist Singularity

The last case concerns the third factor in Equation (16). This condition depends only on the direction cosines of the three wrist axes 4, 5 and 6. For this reason they are termed wrist singularities. It is important to note that these are purely *orienting singulatities*: at least one rotational degree of freedom is lost. This is in contrast to the elbow and shoulder singulatities, which depend on the position of C, making them *positioning singularities*. Wrist singularities arise whenever axes 4 and 6 become collinear, see Figures 3 and 8. Why this is so is illustrated with the following arguements.

Equation (22) may be thought of as a nine-dimensional third-order surface in the coordinate space of the nine components of $\mathbf{e}_4$, $\mathbf{e}_5$ and $\mathbf{e}_6$.

$$e_{4_x}e_{5_z}e_{6_y} - e_{4_x}e_{6_z}e_{5_y} + e_{4_y}e_{5_x}e_{6_z} - e_{4_y}e_{6_x}e_{5_z} + e_{4_z}e_{6_x}e_{5_y} - e_{4_z}e_{5_x}e_{6_y} = 0. \qquad (22)$$

For orientations of the wrist axes we can consider the wrist centre, C, to be located on the origin of the fixed base frame. Moreover, this singularity condition depends on the relative orientation of axes 4, 5 and 6, so we can consider one to be fixed relative to the others and the base frame. In this way we will consider the intersections of the cubic surface with the nine individual coordinate planes.

1. Here we will examine $\mathbf{e}_4$:

(a) $\mathbf{e}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: Equation (22) reduces to

$$e_{5_z} e_{6_y} - e_{5_y} e_{6_z} = 0. \tag{23}$$

(b) $\mathbf{e}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: Equation (22) reduces to

$$e_{5_x} e_{6_z} - e_{5_z} e_{6_x} = 0. \tag{24}$$

(c) $\mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: Equation (22) reduces to

$$e_{5_y} e_{6_x} - e_{5_x} e_{6_y} = 0. \tag{25}$$

The above three intersections leave the condition that the cross-product of the projections of two direction vectors in the respective coordinate planes must vanish. This condition means that $\mathbf{e}_5$ must be parallel to $\mathbf{e}_6$. However, because of the construction of the wrist, axes 5 and 6 are always perpendicular. The conclusion is that none of the above conditions are physically possible and these wrist singular configurations are never attainable.

2. Next, consider $\mathbf{e}_5$:

(a) $\mathbf{e}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: Equation (22) reduces to

$$e_{4_y} e_{6_z} - e_{4_z} e_{6_y} = 0. \tag{26}$$

(b) $\mathbf{e}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: Equation (22) reduces to

$$e_{4_z} e_{6_x} - e_{4_x} e_{6_z} = 0. \tag{27}$$

(c) $\mathbf{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: Equation (22) reduces to

$$e_{4_x} e_{6_y} - e_{4_y} e_{6_x} = 0. \tag{28}$$

In this case $\mathbf{e}_4$ must be parallel to $\mathbf{e}_6$. Unlike above, this situation can arise whenever axes 4 and 6 are aligned.

3. Finally, we will examine $\mathbf{e}_6$:

   (a) $\mathbf{e}_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: Equation (22) reduces to

$$e_{4_z} e_{5_y} - e_{4_y} e_{5_z} = 0. \tag{29}$$

   (b) $\mathbf{e}_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: Equation (22) reduces to

$$e_{4_x} e_{5_z} - e_{4_z} e_{5_x} = 0. \tag{30}$$

   (c) $\mathbf{e}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: Equation (22) reduces to

$$e_{4_y} e_{5_x} - e_{4_x} e_{5_y} = 0. \tag{31}$$

   This condition is similar to the first in that it means $\mathbf{e}_4$ must be parallel to $\mathbf{e}_5$. In this case the construction of the wrist is such that axes 4 and 5 are always perpendicular. Hence, these three wrist singular configurations are never reachable.

To summarize, the condition for wrist singular configurations is only satisfied when axes 4 and 6 are parallel. Thus, the reachable wrist singular configurations comprise only a portion of the third-order wrist singularity surface: a quadratic curve. Figure 8 shows two wrist singular configurations. Because this condition can be satisfied independently of the position of C, the following discomforting fact applies to all wrist-partitioned 6R robots: the entire reachable workspace is potentially wrist singular.

5 Conclusions

We have presented an analytical description and classification of the complete set of singular configurations of the KUKA KR-15 six-axis serial robot in particular, and all wrist-partitioned 6r robots in general. The analysis shows that all general singular positions are either shoulder, elbow, or wrist singularities, or any combination thereof, no others exist. The shoulder singular positions of C comprise a line: the z-axis; elbow singularities of point C occupy a portion of a torus, however, this surface is the boundary of the workspace; wrist singularities can occupy the entire reachable workspace of C. While these results are not new, the geometric analysis of the conditions that cause the Jacobian to become rank deficient are. When the robot suddenly jerks to a halt because the controller has anticipated motion through a singularity, the operator is often left mystified, despite the clarity of the error message that a singular configuration has been reached. We believe that presenting the conditions in this way allows the users of wrist-partitioned industrial robots to better understand the scope of the issue.

References

[1] A. Karger. "Classification of Serial Robot-Manipulators with Nonremovable Singularities". *ASME, Journal of Mechanical Design*, vol. 118, no. 2: pages 202–208, 1996.

[2] A. Karger. "Singularity Analysis of Serial Robot-Manipulators". *ASME, Journal of Mechanical Design*, vol. 118, no. 4: pages 520–525, 1996.

[3] J.W. Burdick. "A Classification of 3R Reginal Manipulator Singularities and Geometries". *Mechanism and Machine Theory*, vol. 30, no. 1: pages 71–89, 1995.

[4] F.T. Cheng, T.L. Hour, Y.Y. Sun, and T.H. Chen. "Study and Resolution of Singularities for a 6-DOF PUMA Manipulator". *IEEE Trans. on Sys., Man, and Cyb., Part B: Cybernetics*, vol. 27, no. 2: pages 332–343, 1997.

[5] K. Tchoń. "Singularities of the Euler Wrist". *Mechanism and Machine Theory*, vol. 35: pages 505–515, 2000.

[6] K. Tchoń and R. Muszynski. "Singularities of Nonredundant Robot Kinematics". *International Journal of Robotics Research*, vol. 16, no. 1: pages 60–76, 1997.

[7] K.L. Doty. Jacobians and Singularities. Technical Report MIL050992kld, Machine Intelligence Laboratory, University of Florida, May 9, 1992.

[8] KUKA Documentation CD. KUKA Roboter GmbH, 1998.

[9] J. Denavit and R.S. Hartenberg. "A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices". *J. of Applied Mechanics*, pages 215–221, 1955.

[10] J. Angeles. *Fundamentals of Robotic Mechanical Systems: Theory, Methods, and Algorithms*. Springer-Verlag, New York, N.Y., U.S.A., 1997.