Largest Area Ellipse Inscribing an Arbitrary Convex Quadrangle

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Abstract. A novel algorithm is presented which employs a projective extension of the Euclidean plane to identify the entire one-parameter family of inscribing ellipses, subject to a set of four linear constraints in the plane of the pencil, and directly identifies the area maximising one given any convex quadrangle. In the algorithm, four specified bounding vertices, no three collinear, determine four line equations describing a convex quadrangle. Considering the quadrangle edges as four polar lines enveloping an ellipse, together with one of the corresponding pole points on the ellipse, we define five bounding constraints on the second order equation revealing a description of the pencil of inscribing line conics. This envelope of line conics is then transformed to its point conic dual for visualisation and area maximisation. The ellipse area is optimised with respect to the single pole point and the maximum area inscribing ellipse emerges.

Keywords: convex quadrangle; point and line ellipses; pole point and polar line.

1 Introduction

Planar algebraic curves have long been the focus of algebraic and geometric investigation, see [12, 14, 15] for example. Still, for some reason, the problem of determining the ellipse possessing the largest area inscribing an arbitrary convex quadrangle has evaded attention in the published literature. Despite this there is genuine need for this knowledge in a variety of engineering applications.

Consider systems of design, or measurement variables in an electrical, or mechanical system. Covariance is a measure of how changes within one variable are related to changes in a second; the covariance between two variables, therefore, becomes a measure of to what degree each variable is dependent upon the other. Currently, covariance ellipses are generated in many fields of study in order to analyse data sets in an effort to understand the physical processes or relations

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which are present within a given system. In statistical analysis the covariance ellipse of n separate variables, given distinct data points, can be generated as an $n \times n$ matrix [13]. The diagonal of the matrix represents the variance of each variable within the data set, while each non-diagonal element represents the covariance of each variable with another. The indices of the matrix element indicates which two variables are involved. For a two variable system the matrix is 2×2 and symmetric, possessing a form identical to that of the quadratic form of an ellipse. The largest area ellipse indicates the maximum covariance between the variables.

Performance indices for machine design are used to compare specific elements of capability. Redundantly actuated parallel mechanisms have operational force outputs that are not unique; these forces do not correspond to a unique set of joint forces, which can help reduce the effect of singularities [10, 17]. Analysis of kinematic isotropy, or the capacity of a mechanism to change position, orientation, and velocity given its pose in the workspace yields insight regarding velocity performance [9]. In this context, the area of the ellipse inscribing the arbitrary polygon defined by the reachable workspace of the redundantly actuated parallel mechanism is proportional to the kinematic isotropy of the mechanism. In [9, 10] the approach to identifying the maximum area inscribing ellipse is a numerical problem, essentially fitting the ellipse inscribing the linear constraints defining the velocity profile of the mechanism by starting with the unit circle.

To the best of the author's knowledge, there are only a handful of papers that report investigations into determining maximum area ellipses inscribing arbitrary polygons. The dual problem, that is the problem of determining the polygons of greatest area inscribed in an ellipse is reported in [11]. While interesting, this dual problem is not germane to determining the maximum area ellipse inscribing a polygon. Three papers by the same author [5–7] appear to lead to a solution to the general problem of finding the largest area ellipse inscribing an n-sided convex polygon, however the papers focus on the proof of the existence of a solution rather than an explicit algorithm for computing the ellipse equation, or shape coefficients. Finally, a numerical fitting approach is presented in [1] that uses brute force convex optimisation techniques to fit the largest volume ellipsoid inscribing a polyhedron. But this technique is essentially the iterative fitting approach used in [9, 10]. For maximum area ellipses inscribing convex quadrangles one expects that there should be an elegant closed-form algebraic solution to the problem.

To arrive at a solution consider that for every four lines which comprise a convex quadrangle, there exists a pencil of inscribing ellipses which lies tangent to all four of these lines [5]. One, and only one of the pencil of ellipses possesses maximal area [6]. The most general point form of the equation of the second degree using homogeneous coordinates is [15]

$$a_{00}x_0^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0. \tag{1}$$

This equation expresses all conic sections in terms of coordinates of points, where the signs of the coefficients determine the conic type [2]. Five relations between the six a_{ij} point conic coefficients are sufficient to determine any conic. Thus, the coordinates of five distinct finite points, no three collinear, are required to determine an ellipse. All points on the circumference of the ellipse satisfy the equation

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 0, \tag{2}$$

where $\mathbf{x}$ is a homogeneous point triple, while $\mathbf{A}$ is the 3×3 symmetric, positive definite coefficient matrix composed of the a_{ij} point conic shape coefficients, where $a_{ij} = a_{ji}$. The same conic can be described by its envelope of tangents using the line coordinates of the tangents. We term the line form of the shape coefficient matrix $\mathbf{A}_L$, and the homogeneous line coordinates are X_i. Because $\mathbf{A}_L$ is symmetric it's elements obey the equality $A_{ij} = A_{ji}$, and the line form of the general equation of the second degree can be expressed as

$$\sum A_{ij} X_i X_j = 0. \tag{3}$$

All tangents enveloping the ellipse satisfy the equation

$$\mathbf{X}^T \mathbf{A}_L \mathbf{X} = 0, \tag{4}$$

where $\mathbf{X}$ is a triple of line coordinates and matrix $\mathbf{A}_L$ is symmetric and positive definite containing the A_{ij} line conic shape coefficients. The line and point triples are dual to one another, as are the point shape matrix and line shape matrix. Moreover, it can be shown that

$$\mathbf{A} = \mathbf{A}_L^{-1} = \frac{\mathrm{adj}\mathbf{A}_\mathrm{L}}{\det\mathbf{A}_\mathrm{L}}. \tag{5}$$

Five constraints are required to uniquely identify an ellipse, but the lines of the edges of a convex quadrangle provide only four. One additional condition is required. The pole and polar are respectively a point and a line that have a unique reciprocal relationship with respect to a given conic section [3]. If the pole point lies on the conic section, its polar is the tangent line to the conic section at that point [15]. Hence, given a conic section and a line tangent to the conic, the corresponding pole point is the tangent point of the polar line with respect to the conic. For an ellipse that inscribes a convex quadrangle, the edges of the quadrangle are polar lines to the points on the ellipse, and the pole points are the tangent points of the edges and the ellipse. We obtain the fifth constraint as the pole point with respect to the polar line comprising one of the quadrangle edges. As the location of this pole point is varied, the entire pencil of inscribing ellipses is generated with the pair of internal diagonals of the complete quadrangle being the bounding, zero-area, degenerate ellipses.

To establish the maximum area inscribing ellipse, we use the area formula [2] for an arbitrary ellipse in general position. This formula is expressed as a ratio of the determinant of the point shape coefficient matrix, $\det\mathbf{A}$ to the determinant of the quadratic form of the point equation of the ellipse, which we call $\det\mathbf{A}_0$

where

$$\det\mathbf{A}_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2. \tag{6}$$

Using these determinants, the general area formula for ellipse k is

$$\text{Area}(k) = \left(\frac{\det\mathbf{A}}{(\det\mathbf{A}_0)^{3/2}} \right) \pi. \tag{7}$$

In order to simplify the computations, we attach a homogeneous reference coordinate system with $(x_0 : x_1 : x_2)$ coordinates to one of the quadrangle's vertices and place the origin at the left most vertex of an edge, and direct the x_1-axis along the length of that edge. The coordinates of the pole point along that edge consists of only the homogenising coordinate and a coordinate on the x_1-axis, hence we arbitrarily label the pole point a_x. The location of a_x is restricted by the vertices of that edge. The degenerate ellipses corresponding the pole points located at either vertex are the respective internal diagonals of the quadrangle.

The area is maximised by parameterising the point conic shape matrix $\mathbf{A}$ in terms of the pole point a_x by computing the zeros of the first partial derivative of $\mathbf{A}$ with respect to a_x

$$\frac{\partial \text{Area}(k)}{\partial a_x} = 0. \tag{8}$$

Of course this equation has multiple zeros, but only one, corresponding to the maximum area inscribing ellipse, lies between the two vertices of the quadrangle edge on the x_1-axis [4].

2 Solution Procedure

Consider an arbitrary convex quadrangle. Select an arbitrary vertex and place the origin of a reference coordinate system possessing homogeneous coordinates $(x_0 : x_1 : x_2)$ on that vertex. Select x_0 to be the homogenising and the x_1 coordinate axis to be pointing towards the terminal vertex of the associated edge.

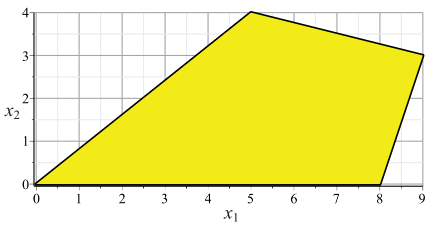

Fig. 1. Coordinate system placement.

See Fig. 1 for example. The four vertices considered are, in counter-clockwise order, $(1 : 0 : 0)$, $(1 : 8 : 0)$, $(1 : 9 : 3)$, and $(1 : 5 : 4)$.

Let the polar line g_x containing the pole point a_x be on the edge along the x_1-axis. The line coordinates of any line g are $[G_0 : G_1 : G_2]$ and can be computed as a Grassmannian expansion of the point coordinates of two points on the line [8]. For g_x in particular we use the two vertices along the x_1-axis:

$$\begin{vmatrix} G_0 & G_1 & G_2 \\ 1 & 0 & 0 \\ 1 & 8 & 0 \end{vmatrix} = [0:0:8] \; = \; [0:0:1]. \tag{9}$$

The vector whose elements are the pole point coordinates of the tangent point p is in general obtained by multiplying the line conic shape coefficient matrix by the vector of line coordinates of g [15]:

$$\mathbf{p} = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{02} & A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} G_0 \\ G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} A_{00}G_0 + A_{01}G_1 + A_{02}G_2 \\ A_{01}G_0 + A_{11}G_1 + A_{12}G_2 \\ A_{02}G_0 + A_{12}G_1 + A_{22}G_2 \end{bmatrix}. \tag{10}$$

The de-homogenised coordinates in the x_1 direction are, in general, termed p_x and can be determined from Eq. (10) as

$$p_x = \frac{A_{01}G_0 + A_{11}G_1 + A_{12}G_2}{A_{00}G_0 + A_{01}G_1 + A_{02}G_2}. \tag{11}$$

Along the x_1-axis the x_2-coordinate is always identically zero, and hence the line coordinates of the x_1-axis are $g_x = [0:0:1]$. Give the components of g_x, it is to be seen that Eq. (11) reduces to

$$a_x = \frac{A_{12}}{A_{02}}. \tag{12}$$

Eq. (12) yields an independent line conic constraint equation parametrised in terms of a_x:

$$a_x A_{02} - A_{12} = 0. \tag{13}$$

Additionally, when the line coordinates of the x_1-axis, $g_x = [0 : 0 : 1]$ are substituted into Eq (3) yields the constant line conic constraint equation which is independent of Eq. (12):

$$A_{22} = 0. \tag{14}$$

The remaining three quadrangle edges yield three triples of line coordinates for the edges labelled g_1, g_2, g_3, which produce three more line conic constraint equations. Thus, the system of four polar lines and one pole point is equivalent five linearly independent conditions. This means that the line conic shape coefficients from Eq. (3) can be identified for any value of a_x on the open interval along the x_1-axis between the vertex points on that axis. This reveals the one-parameter pencil of ellipses inscribing a given convex quadrangle.

The pencil of inscribing ellipses can be computed with a Grassmannian expansion of the matrix line conic shape coefficient constraints:

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{11} & A_{12} & A_{22} \\ 0 & 0 & a_x & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ G_{01}^2 & 2G_{01}G_{11} & 2G_{01}G_{21} & G_{11}^2 & 2G_{11}G_{21} & G_{21}^2 \\ G_{02}^2 & 2G_{02}G_{12} & 2G_{02}G_{22} & G_{12}^2 & 2G_{12}G_{22} & G_{22}^2 \\ G_{03}^2 & 2G_{03}G_{13} & 2G_{03}G_{23} & G_{13}^2 & 2G_{13}G_{23} & G_{23}^2 \end{bmatrix}. \tag{15}$$

The expansion along the top row yields expressions for line conic shape coefficients in terms of a_x and the line coordinates of the four quadrangle edges g_x, g_1, g_2, and g_3. The matrix $\mathbf{A}_L$ is populated with established values and inverted to reveal the matrix of corresponding point conic shape coefficients. The entire family of inscribing ellipses is thus obtained. To obtain the one inscribing ellipse with maximum area, the expression in Eq. (8) is evaluated and solved for a_x. The single value for a_x on the x_1-axis between the vertices of the edge laying on that axis is the pole point of the maximum area ellipse inscribing the quadrangle.

3 Example

To illustrate the algorithm we will proceed with an example using the quadrangle illustrated in Fig. 1. Recall, the vertices are $(1:0:0)$, $(1:8:0)$, $(1:9:3)$, and $(1:5:4)$ in counter-clockwise order. Using appropriate pairs of vertices, the line coordinates of the four edges are computed, in the manner of Eq. (9), to be:

$$\begin{aligned} g_x &= [0:0:1]; \\ g_1 &= [24:-3:1]; \\ g_2 &= [21:-1:-4]; \\ g_3 &= [0:4:-5]. \end{aligned}$$

Using the line coordinates, the line conic shape parameter matrix in Eq. (15) is populated giving

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{11} & A_{12} & A_{22} \\ 0 & 0 & a_x & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 576 & -144 & 48 & 9 & -6 & 1 \\ 441 & -42 & -168 & 1 & 8 & 16 \\ 0 & 0 & 0 & 16 & -40 & 25 \end{bmatrix}. \tag{16}$$

The determinants of the appropriate minors of Eq. (16) are evaluated to reveal that

$$\begin{aligned} A_{00} &= a_x - 32, \\ A_{01} &= -\frac{3}{2}a_x - 144, \\ A_{02} &= -48, \\ A_{11} &= -120a_x, \\ A_{12} &= -48a_x, \\ A_{22} &= 0. \end{aligned}$$

These coefficients are used to populate the line conic shape coefficient matrix $\mathbf{A}_L$ which is inverted to obtain the point conic coefficient matrix $\mathbf{A}$, revealing the point conic coefficients. The general inscribing ellipse point equation, parametrised with a_x, represents the pencil with a pole point on the open interval between the vertices $(1:0:0)$ and $(1:8:0)$ using the coefficients computed in the previous step yields:

$$a_x^2x_0^2 - 2a_xx_0x_1 - \left(\frac{1}{16}a_x^2 - a_x\right)x_0x_2 + x_1^2 - \left(\frac{1}{24}a_x^2 + \frac{61}{48}a_x - 6\right)x_1x_2 + \left(\frac{163}{3072}a_x^2 - \frac{71}{48}a_x + 9\right)x_2^2 = 0.$$

Letting a_x vary on the open line segment between 0 and 8 generates the one parameter pencil of inscribing ellipses illustrated in Fig. 2. Examining this

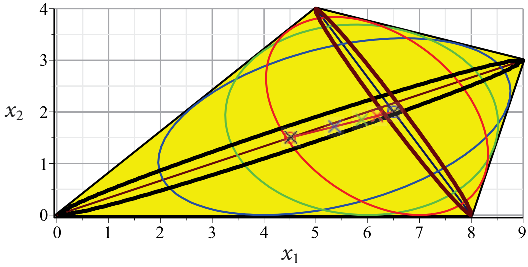

Fig. 2. One parameter family of inscribing ellipses.

figure one immediately sees that centres of the inscribing ellipses are all on a line connecting the midpoints of the two internal quadrangle diagonals, where the midpoints themselves represent the centres of the degenerate ellipses formed by selecting $a_x = 0$ or $a_x = 8$. The midpoints of the internal diagonals are indicated by small circles on the diagonals, while the centres of the illustrated inscribing ellipses are each indicated with a small "x", moreover, the centres of the ellipses are all collinear, all agreeing with well known facts summarised in [15].

Finally, we determine the value of a_x which maximises the inscribing ellipse area using Eqs. (8) and (7). The area of any ellipse k in the pencil is

$$\text{Area}(k) = -48\,\frac{\pi\,\left({a_x}^2 - 29\,a_x + 168\right)a_x}{\sqrt{-a_x\,\left({a_x}^3 - 61\,{a_x}^2 + 1096\,a_x - 5376\right)(a_x - 32)}}. \tag{17}$$

Differentiating Eq. (17) with respect to a_x gives

$$\frac{\partial \text{Area}(k)}{\partial a_x} = 24\,\frac{\left(67\,{a_x}^2 - 1520\,a_x + 5376\right)\pi}{\sqrt{-a_x\,\left({a_x}^3 - 61\,{a_x}^2 + 1096\,a_x - 5376\right)(a_x - 32)}^2}. \tag{18}$$

Equating Eq. (18) to zero and solving for a_x leads to two distinct zeros:

$$a_x = 18.30254191, \text{ and } a_x = 4.384025253.$$

Each value of a_x generates a point conic, but clearly, only one can be an ellipse inscribing the quadrangle. The conics for each value of a_x are, respectively

$$\begin{aligned}334.9830404x_0^2 - 36.60508382x_1x_0 - 39.23898193x_2x_0 +\\ x1^2 + 3.30185366x_2x_1 - .29834468x_2^2 \;=\; 0,\end{aligned} \tag{19}$$

$$\begin{aligned}19.21967742x_0^2 - 8.768050506x_1x_0 - 5.585255092x_2x_0 +\\ x1^2 - 1.229454466x_2x_1 + 3.535090062x_2^2 \;=\; 0.\end{aligned} \tag{20}$$

The conics can be classified using a well known classification method [16] based on four quantities defined by the elements of the matrix of point conic shape coefficients $\mathbf{A}$ which are invariant under the basic Euclidean transformation group. The four invariants are:

$$\Delta = \det\mathbf{A}; \tag{21}$$

$$\Delta_0 = \det\mathbf{A}_0 \;=\; a_{11}a_{22} - a_{12}^2; \tag{22}$$

$$H = a_{11} + a_{22}; \tag{23}$$

$$K = a_{00}H - \left(a_{01}^2 + a_{02}^2\right). \tag{24}$$

For the conic in Eq. (19) corresponding to $a_x = 18.30254191$, which cannot be an inscribing ellipse since this coordinate on the x_1-axis is outside the region limited by the vertices on the x_1-axis, we see that: $\Delta \neq 0$ indicating that the equation represents a regular non-degenerate conic; $\Delta_0 \neq 0$ indicating that the equation represents a conic with a centre; but $\Delta_0 < 0$ indicating Eq. (19) is a regular hyperbola, see Fig. 3.

Since the other value of $a_x = 4.384025253$ does indeed lay in the open interval between the vertices on the x_1-axis, we conclude that this is the pole point on

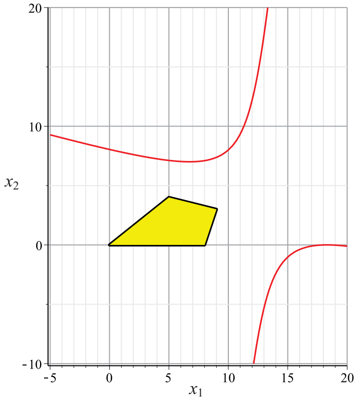

Fig. 3. Hyperbola corresponding to $a_x = 18.30254191$.

the x_1-axis of the largest area inscribing ellipse. Considering Eq. (20), this is confirmed when we see that: $\Delta \neq 0$ indicating that the equation represents a regular non-degenerate conic; $\Delta_0 \neq 0$ indicating that the equation represents a conic with a centre; but that $\Delta_0 > 0$ and the product $\Delta H < 0$ which together indicate that this conic is a regular ellipse, see Fig. 4. Moreover, it is to be observed that the centre of this inscribing ellipse lies on the line connecting the midpoints of the two internal diagonals, as it must [15].

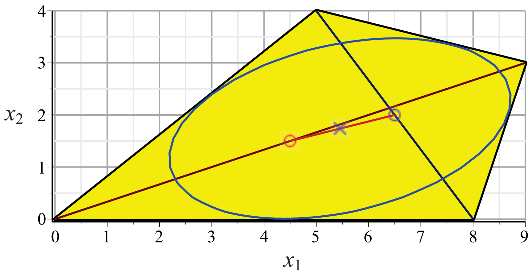

Fig. 4. Maximum area inscribing ellipse corresponding to $a_x = 4.384025253$.

4 Conclusions

In this paper the reciprocal relationship between pole point and polar line was employed to develop an algorithm to determine the largest area ellipse inscribing an arbitrary convex quadrangle. An illustrative example was presented demonstrating use of the algorithm. The quadrangle represents only four linear constraints on the inscribing ellipse; a fifth independent one was required. The fifth constraint turns out to be the pole point corresponding to one of the polar lines forming the convex quadrangle.

This work has applications to determining the upper bound on error ellipses given specific linear constraints, and for determining the maximum area inscribing ellipse given linear constraints that form convex quadrangles which characterize the velocity performance of parallel mechanisms in the presence of actuation redundancy, among a variety of other relevant mechanical engineering applications. Future work will aim to extend the approach to determining the maximum area ellipse inscribing arbitrary n-sided arbitrary convex polygons.

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