

# Extreme Distance to a Spatial Circle

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## Abstract

Determination of shortest distances in the three dimensional task space of robots is pertinent to pick-and-place operations, collision avoidance, and for impact prediction in dynamic simulation. The conventional approach is to find perpendicular distances between planar patches approximating body surfaces. In contrast, this paper treats four variants of shortest distance computations wherein one or both elements are circular edges. These three dimensional cases include circle and point, circle and plane, circle and line and two non coplanar circles. Solutions to these four fundamental problems are developed with elementary geometry. Examples are presented, and the closed form algebraic solutions are verified with descriptive geometric constructions.

## Distance Extrême à un Cercle Spatial

### Résumé

La détermination des distances les plus courtes dans l'espace tridimensionnel de tâche des robots est convenable aux opérations de transfert, action d'éviter de collision, et pour la prévision d'impact dans la simulation dynamique. L'approche conventionnelle est de trouver des distances perpendiculaires entre les pièces rapportées planaires rapprochant des surfaces de corps. En revanche, cet article traite quatre variantes des calculs de distance les plus courts où un ou les deux éléments sont les bords circulaires. Ces cas tridimensionnels incluent le cercle et se dirigent, entourent et surfont, entourent et rayent et deux cercles non coplanaires. Des solutions à ces quatre problèmes fondamentaux sont développées avec la géométrie élémentaire. Des exemples sont présentés, et les solutions algébriques de forme fermée sont vérifiées avec les constructions géométriques descriptives.

# 1 Introduction

Consider some rigid body “approach” scenarios. Insertion of a workpiece feature into a hole, or avoiding collision among machine parts and with surroundings, or even anticipating point of impact in a multibody dynamic simulation are fairly important and routine aspects of robotic manipulation. In these situations it is necessary to track boundary regions on the pairs of bodies in relative motion so as to identify, from instant to instant, the points which are closest to each other. Commonly, the bodies are approximated by polyhedra, spheres, simple solids of revolution, *e.g.*, cylinder, and their planar right truncations. Four situations, involving the latter, will be addressed. The problem posed is to determine the closest point on a given circle in space with respect to a given

1. point,
2. plane,
3. line or
4. second circle.

A moment’s reflection will reveal what sort of “collision course” might relate to any of these four cases. For instance, collision avoidance from a trajectory generation perspective for pick-and-place operations is discussed in detail in [1]. Therefore it is not intended to embark on a descriptive enumeration of such encounters. Rather the efficient computation of these four types of closest distance will be addressed by examining the underlying geometry to formulate a good set of constraint equations to describe, hence solve, the problems at hand.

The issue arose when G. Grabner [2], a doctoral candidate in *Mechanik* at TU-Graz, asked one of us (Husty) if the condition of proximity between spatial circles could be assigned some clear geometric interpretation. After considerable discussion Husty [3] pointed out that a minimal (or maximal) distance to a circle must be measured along a ray which intersects the circle axis; a line on its centre and normal to its plane. Validity of this observation is evident if one imagines a tangent line and notes that only lines on the tangent point and intersecting the axis may be normal to the tangent and coincidentally intersect the curve. The normal condition is the criterion of shortest distance from anywhere to a given line and the differential element of circular arc  $ds$  on the tangent is such a line. Figure 1 shows a generator of a cone of revolution in the pencil of cones that are right-sectioned by the circle and have apices on its axis. The foregoing, possibly tedious, argument is deemed necessary because the circle axis intersection criterion is the constraint on which all four of the mentioned shortest distance cases are based. The first two cases are easy to formulate and compute in various ways. Nevertheless they are included in order to lead up, via the line-to-circle case, to the quite subtle fourth.

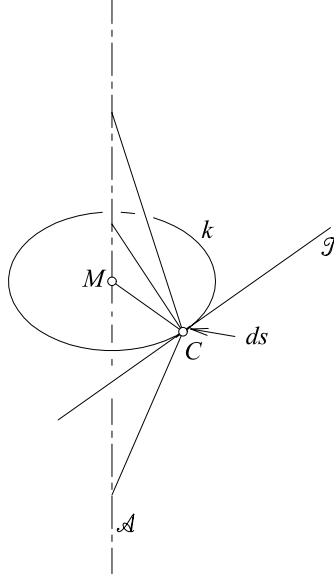


Figure 1: Pencil of cones right-sectioned by circle  $k$  having apices on axis  $\mathcal{A}$ .

## 2 From Point to Circle

Points on a circle in space can be conveniently represented by two simultaneous surface equations; the plane  $m$  on the circle  $k$  and a sphere  $h$  of the same radius  $r$ . Assume point  $P$  is given along with  $k$ , specified by its centre point  $M$  and axis line  $\mathcal{A}$ . A second plane  $p$  on all lines joining  $P$  to  $\mathcal{A}$  provides the third constraining surface necessary to obtain a discrete set of points, *i.e.*, solutions. Describing  $P$  and  $M$  by their homogeneous point coordinates, circle axis  $\mathcal{A}$  by its radial Plücker line coordinates (a thorough discussion of point, line, and plane coordinates is to be found in [4]), and the radius by the scalar  $r$  we have:

$$P\{p_0 : p_1 : p_2 : p_3\}, \quad M\{m_0 : m_1 : m_2 : m_3\},$$

$$\mathcal{A}\{a_{01} : a_{02} : a_{03} : a_{23} : a_{31} : a_{12}\} \text{ and } r.$$

The three constraints which express extreme points  $C\{c_0 : c_1 : c_2 : c_3\}$  on  $k$ , *i.e.*, closest and farthest from  $P$ , can be written after the homogeneous plane coordinates of  $p$ ,  $p\{P_0 : P_1 : P_2 : P_3\}$  are defined.

$$p = \mathcal{A} \cap P, \quad P_i = \sum_{j=0}^3 A_{ij} p_j,$$

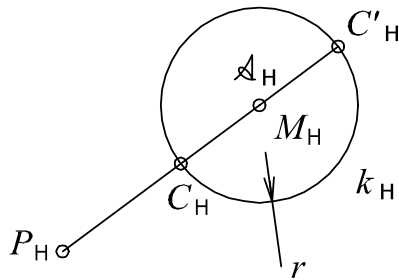
$$\begin{aligned} P_0 &= & A_{01}p_1 & + A_{02}p_2 & + A_{03}p_3 \\ P_1 &= -A_{01}p_0 & & + A_{12}p_2 & - A_{31}p_3 \\ P_2 &= -A_{02}p_0 & -A_{12}p_1 & & + A_{23}p_3 \\ P_3 &= -A_{03}p_0 & + A_{31}p_1 & - A_{23}p_2. \end{aligned}$$

Therefore the constraints are written as follows.

$$h : \begin{aligned} m_0^2(c_1^2 + c_2^2 + c_3^2) - 2m_0c_0(m_1c_1 + m_2c_2 + m_3c_3) \\ + c_0^2(m_1^2 + m_2^2 + m_3^2 - r^2m_0^2) \end{aligned} = 0, \quad (1)$$

$$m : \begin{aligned} -(A_{23}m_1 + A_{31}m_2 + A_{12}m_3)c_0 \\ + m_0A_{23}c_1 + m_0A_{31}c_2 + m_0A_{12}c_3 \end{aligned} = 0, \quad (2)$$

$$p : \quad P_0c_0 + P_1c_1 + P_2c_2 + P_3c_3 = 0. \quad (3)$$



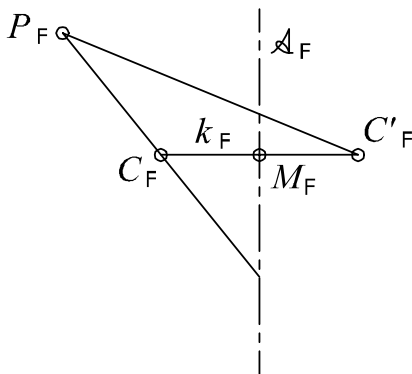
The "shortest distance" from a given point or plane to a given circle is a quadratic problem.

Extreme lines joining plane  $p$  to circle  $k$  intersect axis  $\mathcal{A}$ .

Points  $C$  and  $C'$  on  $k$  are on these lines.

H

F



Extreme lines joining point  $P$  to circle  $k$  intersect axis  $\mathcal{A}$ .  
Points  $C$  and  $C'$  on  $k$  are on these lines.

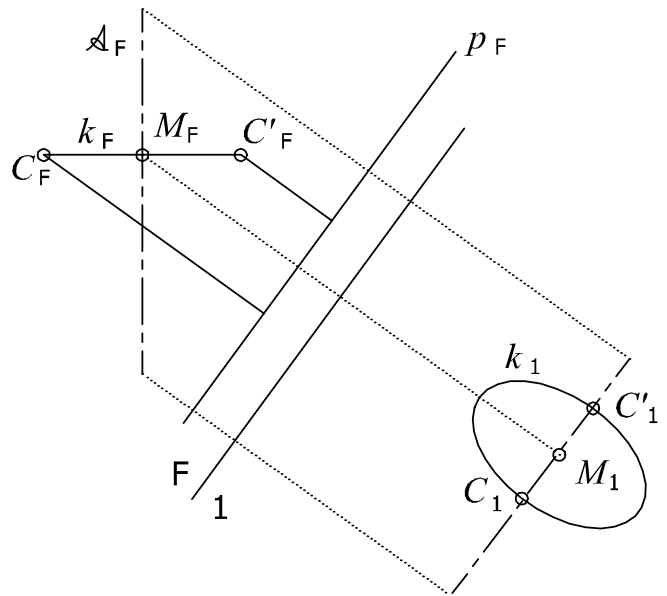


Figure 2: Descriptive geometric solution to the shortest distance problem connecting a circle to a point not in the plane of the circle.

Figure 2 shows a constructive solution to this problem which yields the points  $C$  and  $C'$ ; the nearest and furthest from  $P$ . The lines on  $P$  and  $C$  and  $P$  and  $C'$  are seen to intersect  $\mathcal{A}$ . To reinforce the notion that the solution could be arrived at in various ways,

notice by reexamining Figure 2 that a cone on apex  $P$  and sectioned by  $k$  contains two generators on  $C$  and  $C'$ , where  $p$  cuts  $k$ . Similarly one might express the distance from  $P$  to  $C$  by angularly parameterizing  $C$  around  $k$  and formulating a minimization problem. It is contended however that the set of three explicit constraint equations represents the best way.

### 3 From Plane to Circle

At lower right in Figure 2 one sees the constructive solution to finding the extreme segments connecting a given plane  $p$  to  $k$ . The computational approach proposed is based on the observation that all shortest distances to  $p$  must be measured along lines perpendicular to  $p$ . So Eq. 3 in the set of constraints above is replaced by Eq. 4 below, that of a plane  $a\{A_0 : A_1 : A_2 : A_3\}$  on  $\mathcal{A}$  and normal to given plane  $p\{P_0 : P_1 : P_2 : P_3\}$ .

$$a = \mathcal{A} \cap Q, \quad A_i = \sum_{j=0}^3 A_{ij} q_j,$$

where  $Q$  is a point on the line at infinity,  $Q\{q_0 : q_1 : q_2 : q_3\} \equiv \{0 : P_1 : P_2 : P_3\}$ .

$$\begin{aligned} A_0 &= A_{01}P_1 + A_{02}P_2 + A_{03}P_3, \\ A_1 &= A_{12}P_2 - A_{31}P_3, \\ A_2 &= -A_{12}P_1 + A_{23}P_3, \\ A_3 &= +A_{31}P_1 - A_{23}P_2, \end{aligned}$$

$$a : A_0c_0 + A_1c_1 + A_2c_2 + A_3c_3 = 0. \quad (4)$$

Notice that an elliptical cylinder sectioned by  $k$  and normal to  $p$  contains two generators on  $C$  and  $C'$  where  $a$  cuts  $k$ .

### 4 From Line to Circle

Now the intention is to connect given line  $\mathcal{X}$  to  $k$  via paths of extreme length. The solutions to a specific problem is shown in Figure 3 No constructive solution is available because the surface, which replaces  $p$  or  $a$  in the previous two cases, is no plane but a quadric ruled by all lines intersecting  $\mathcal{A}$  and  $\mathcal{X}$  and normal to  $\mathcal{X}$  as well. This is a three line ruling. The third line is  $Q$  the line at infinity that intersects all lines normal to  $\mathcal{X}$ . *E.g.*, if  $\mathcal{X}\{1 : 0 : 0 : 0 : 0 : 0\}$  then  $Q\{0 : 0 : 0 : 1 : 0 : 0\}$ . The ruling on three lines is  $q$ , a hyperboloid of one sheet which must intersect  $h$  and  $m$  on four not necessarily real points.

#### 4.1 Computing the Quadric

Although the general theory of quadrics and their properties are treated with authority and at considerable length by Sommerville [5] and others, the derivation of the implicit point

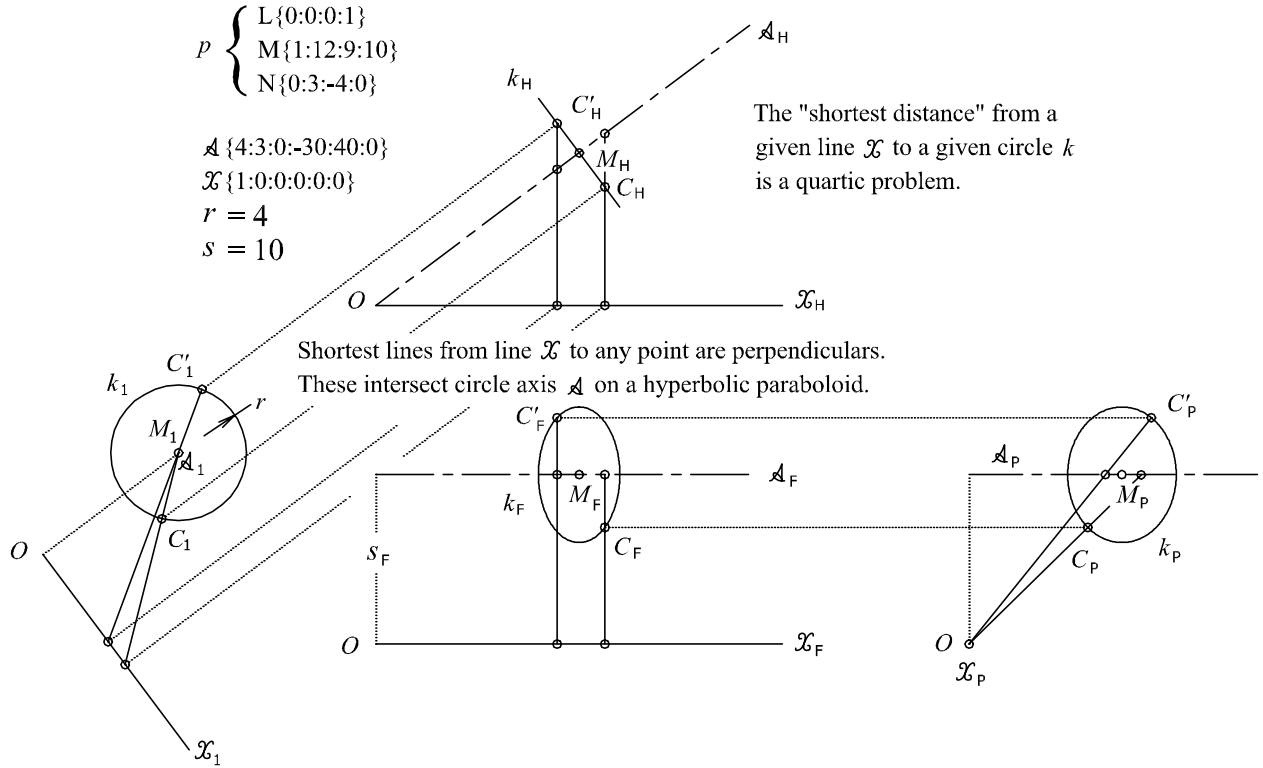


Figure 3: Descriptive geometric solution to the shortest distance problem connecting a circle to a line not in the plane of the circle.

equation of the hyperboloid on three given skew lines in a regulus is not. The implicit equation derivation is alluded to via detailed geometric analysis by Hilbert and Cohn-Vossen in [6], but no algebraic treatment is given. It was however covered in detail in [7] and [8] so only the overall principle of approach will be outlined and the results, in terms of relevant line coordinates, will be presented here. The three given axial lines are  $\mathcal{A}$ ,  $\mathcal{Q}$  and  $\mathcal{X}$  and they must intersect all radial lines  $\mathcal{C}$  in the opposite regulus. *E.g.*,

$$A_{01}c_{01} + A_{02}c_{02} + A_{03}c_{03} + A_{23}c_{23} + A_{31}c_{31} + A_{12}c_{12} = 0.$$

Using the condition that the point  $C$  is on line  $\mathcal{C}$  and eliminating all variable line coordinates  $c_{ij}$  from three line intersection equations like the one above produces the quadric

$$q = q(c_0, c_1, c_2, c_3) = 0.$$

$$\begin{aligned}
& [A_{01}(Q_{31}X_{12} - Q_{12}X_{31}) + Q_{01}(X_{31}A_{12} - X_{12}A_{31}) \\
& \quad + X_{01}(A_{31}Q_{12} - A_{12}Q_{31})] \quad c_1^2 \\
+ & [A_{02}(Q_{12}X_{23} - Q_{23}X_{12}) + Q_{02}(X_{12}A_{23} - X_{23}A_{12}) \\
& \quad + X_{02}(A_{12}Q_{23} - A_{23}Q_{12})] \quad c_2^2 \\
+ & [A_{03}(Q_{23}X_{31} - Q_{31}X_{23}) + Q_{03}(X_{23}A_{31} - X_{31}A_{23}) \\
& \quad + X_{03}(A_{23}Q_{31} - A_{31}Q_{23})] \quad c_3^2 \\
& + [A_{23}(Q_{31}X_{02} - Q_{02}X_{31} + X_{12}Q_{03} - X_{03}Q_{12}) \\
& \quad + Q_{23}(X_{31}A_{02} - X_{02}A_{31} + A_{12}X_{03} - A_{03}X_{12}) \\
& \quad + X_{23}(A_{31}Q_{02} - A_{02}Q_{31} + Q_{12}A_{03} - Q_{03}A_{12})] \quad c_2c_3 \\
& + [A_{31}(Q_{12}X_{03} - Q_{03}X_{12} + X_{23}Q_{01} - X_{01}Q_{23}) \\
& \quad + Q_{31}(X_{12}A_{03} - X_{03}A_{12} + A_{23}X_{01} - A_{01}X_{23}) \\
& \quad + X_{31}(A_{12}Q_{03} - A_{03}Q_{12} + Q_{23}A_{01} - Q_{01}A_{23})] \quad c_3c_1 \\
& + [A_{12}(Q_{23}X_{01} - Q_{01}X_{23} + X_{31}Q_{02} - X_{02}Q_{31}) \\
& \quad + Q_{12}(X_{23}A_{01} - X_{01}A_{23} + A_{31}X_{02} - A_{02}X_{31}) \\
& \quad + X_{12}(A_{23}Q_{01} - A_{01}Q_{23} + Q_{31}A_{02} - Q_{02}A_{31})] \quad c_1c_2 \\
& + [A_{01}(Q_{31}X_{02} - Q_{02}X_{31} - X_{12}Q_{03} + X_{03}Q_{12}) \\
& \quad + Q_{01}(X_{31}A_{02} - X_{02}A_{31} - A_{12}X_{03} + A_{03}X_{12}) \\
& \quad + X_{01}(A_{31}Q_{02} - A_{02}Q_{31} - Q_{12}A_{03} + Q_{03}A_{12})] \quad c_0c_1 \\
& + [A_{02}(Q_{12}X_{03} - Q_{03}X_{12} - X_{23}Q_{01} + X_{01}Q_{23}) \\
& \quad + Q_{02}(X_{12}A_{03} - X_{03}A_{12} - A_{23}X_{01} + A_{01}X_{23}) \\
& \quad + X_{02}(A_{12}Q_{03} - A_{03}Q_{12} - Q_{23}A_{01} + Q_{01}A_{23})] \quad c_0c_2 \\
& + [A_{03}(Q_{23}X_{01} - Q_{01}X_{23} - X_{31}Q_{02} + X_{02}Q_{31}) \\
& \quad + Q_{03}(X_{23}A_{01} - X_{01}A_{23} - A_{31}X_{02} + A_{02}X_{31}) \\
& \quad + X_{03}(A_{23}Q_{01} - A_{01}Q_{23} - Q_{31}A_{02} + Q_{02}A_{31})] \quad c_0c_3 \\
& + [A_{01}(Q_{02}X_{03} - Q_{03}X_{02} + A_{02}(Q_{03}X_{01} - Q_{01}X_{03}) \\
& \quad + A_{03}(Q_{01}X_{02} - Q_{02}X_{01})] \quad c_0^2 = 0. \tag{5}
\end{aligned}$$

## 4.2 A Hyperbolic Paraboloid

In the example of Figure 3 lines  $\mathcal{A}$  and  $\mathcal{X}$  are given by their radial Plücker coordinates, while in Eq. 5 the Plücker coordinates are axial so

$$\begin{aligned}
\mathcal{A}\{A_{01} : A_{02} : A_{03} : A_{23} : A_{31} : A_{12}\} &\equiv \{-30 : 40 : 0 : 4 : 3 : 0\} \\
&\equiv \{-sa_{02} : sa_{01} : 0 : a_{01} : a_{02} : 0\}, \\
\mathcal{Q}\{Q_{01} : Q_{02} : Q_{03} : Q_{23} : Q_{31} : Q_{12}\} &\equiv \{1 : 0 : 0 : 0 : 0 : 0\}, \\
\mathcal{X}\{X_{01} : X_{02} : X_{03} : X_{23} : X_{31} : X_{12}\} &\equiv \{0 : 0 : 0 : 1 : 0 : 0\}.
\end{aligned}$$

Substituting into Eq. 5 produces, with the reference frame attitude chosen in Figure 3, the equation of a nice hyperbolic paraboloid with a principal axis tipped along the space diagonal in the first octant.

$$q : \quad sa_{01}c_0c_2 - a_{02}c_1c_3 \equiv 40c_2 - 3c_1c_3 = 0. \quad (6)$$

Together with the sphere  $h$  this surface intersects the meridial plane  $m$  of the circle in four points. This is illustrated in Figure 4. These could all be real if the plane  $m$  were parallel to  $\mathcal{A}$  and  $\mathcal{X}$  but alas  $m$  is normal to  $\mathcal{A}$  by definition so there are but two extreme lines from  $\mathcal{X}$ , *i.e.*, to  $C$  and  $C'$  shown in Figure 3. For the moment, no method is evident whereby  $q$  might be replaced by a plane so as to make this a true second order problem.

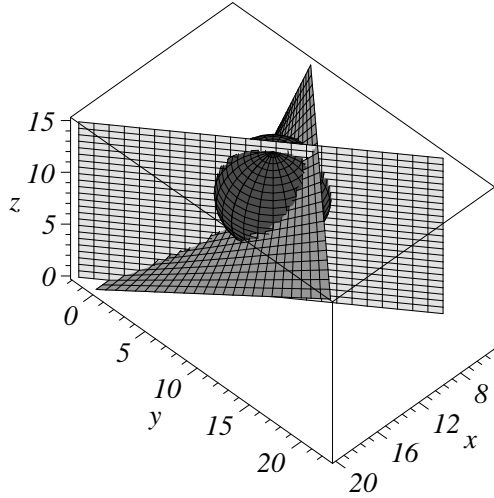


Figure 4: Plane, sphere, and hyperbolic paraboloid.

## 5 From Circle to Circle

What can be done with a formulation based on two given spheres and meridial cutting planes? One may form, on the circles' axis lines so produced and the circles themselves, three surfaces and obtain their points of intersection. What may be the nature of these surfaces? Two lines and each circle taken separately and then both circles taken together with either axis line may all be ruled with straight lines. Such surfaces are called either conoids [9, 10] after Plücker, or cylindroids [11], after Ball; cones wherein the point apex degenerates to a line. The authors could find no literature at hand which might provide convenient method to generate conoid point or line equations of this sort. With the former one would intersect the three surfaces to get points like  $C$  and  $C'$  above. With the latter only two surfaces would be used to seek common lines, thereby fulfilling the key criterion. From



the point equations one concludes, since there are no surfaces of order one, *i.e.*, planes, which contain circles with skew axes, that the solution of extreme circle to circle connections must admit an octic lower bound. The reader's indulgence is asked here because now an approach somewhat different, from that common to the previous three cases, will be adopted.

## 5.1 Layout

Points  $P$  and  $Q$  are on the respective circumference of circles centred on  $C$  and  $M$  and with radii  $R$  and  $r$ , respectively. These points are parameterized according to angles  $\alpha$  and  $\beta$  measured from rays parallel to  $z = 0$  in a reference frame with the  $z$ -axis on the line of intersection on the circle planes  $c$  and  $m$ . These vertical planes are disposed symmetrically, by angle  $\theta$ , on either side of the reference  $x$ -axis. Circle centres  $C$  and  $M$ , on these planes, are an equal distance  $h$  above and below plane  $z = 0$ . The distances from the  $z$ -axis to  $C$  and to  $M$  are  $j$  and  $k$ . The reference frame and circle layout are shown in Figure 5.

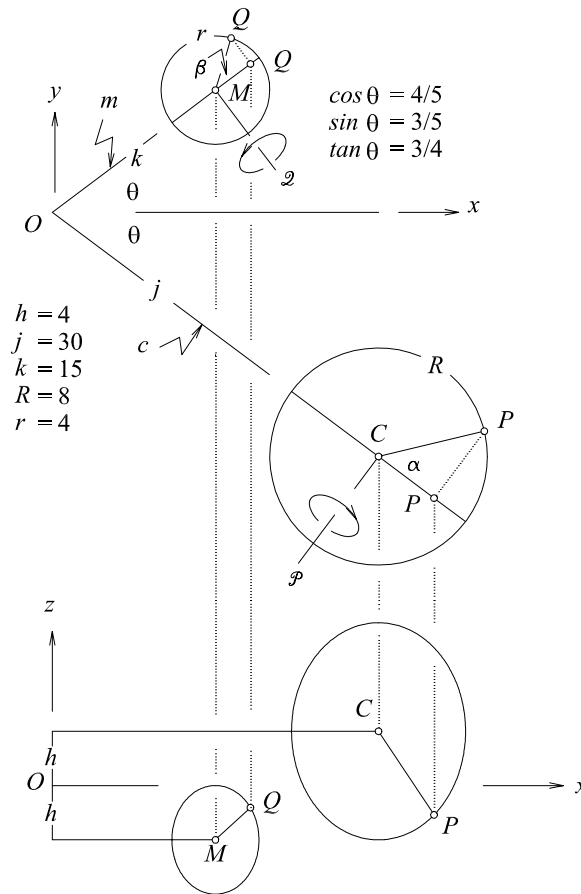


Figure 5: Circle-circle layout.

## 5.2 A Line Geometric Approach

Consider circle axis lines  $\mathcal{P}$  and  $\mathcal{Q}$  on  $C$  and  $M$  and normal to  $c$  and  $m$ , respectively. Some line  $\mathcal{R}$  intersecting  $\mathcal{P}$  and  $\mathcal{Q}$  and on  $P$  and  $Q$  will exhibit minimum span between circle circumferences because all lines in the congruence which intersects circumference and axis are normal to the circle tangent. Only lines in such congruences may qualify as shortest distance paths to an arbitrary point in space. Two constraint equations, based on this principle, will be written and solved for  $\alpha$  and  $\beta$ .

## 5.3 Constraint Equation Specifications

### 5.3.1 The Line $\mathcal{P}$

$$\mathcal{P} = C \cap C_\omega,$$

where  $C_\omega$  is the point that closes  $\mathcal{P}$ .

$$C\{c_0 : c_1 : c_2 : c_3\} \equiv \{1 : j \cos \theta : -j \sin \theta : h\},$$

$$C_\omega\{0 : \sin \theta : \cos \theta : 0\},$$

$$\mathcal{P}\{p_{01} : p_{02} : p_{02} : p_{23} : p_{31} : p_{12}\} \equiv \{\sin \theta : \cos \theta : 0 : -h \cos \theta : h \sin \theta : j\}.$$

### 5.3.2 The Line $\mathcal{Q}$

$$\mathcal{Q} = M \cap M_\omega,$$

where  $M_\omega$  is the point that closes  $\mathcal{Q}$ .

$$M\{m_0 : m_1 : m_2 : m_3\} \equiv \{1 : k \cos \theta : k \sin \theta : -h\},$$

$$M_\omega\{0 : \sin \theta : -\cos \theta : 0\},$$

$$\mathcal{Q}\{q_{01} : q_{02} : q_{02} : q_{23} : q_{31} : q_{12}\} \equiv \{\sin \theta : -\cos \theta : 0 : -h \cos \theta : -h \sin \theta : -k\}.$$

### 5.3.3 The Line $\mathcal{R}$

$$\mathcal{R} = P \cap Q,$$

$$P\{p_0 : p_1 : p_2 : p_3\}$$

$$\equiv \{1 : \cos \theta(j + R \cos \alpha) : -\sin \theta(j + R \cos \alpha) : (h - R \sin \alpha)\}, \quad (7)$$

$$Q\{q_0 : q_1 : q_2 : q_3\}$$

$$\equiv \{1 : \cos \theta(k + r \cos \beta) : \sin \theta(k + r \cos \beta) : -(h - r \sin \beta)\}, \quad (8)$$

$$\begin{aligned}
& \mathcal{R}\{r_{01} : r_{02} : r_{03} : r_{23} : r_{31} : r_{12}\} \\
& \equiv \{-\cos\theta[(j + R\cos\alpha) - (k + r\cos\beta)] \\
& \quad : \sin\theta[(j + R\cos\alpha) + (k + r\cos\beta)] \\
& \quad : -[(h - R\sin\alpha) + (h - r\sin\beta)] \\
& : \sin\theta[(j + R\cos\alpha)(h - r\sin\beta) - (h - R\sin\alpha)(k + r\cos\beta)] \\
& : \cos\theta[(h + R\sin\alpha)(k + r\cos\beta) + (j + R\cos\alpha)(h - r\sin\beta)] \\
& \quad : 2\cos\sin\theta(j + R\cos\alpha)(k + r\cos\beta)\}.
\end{aligned}$$

### 5.3.4 Two Constraint Equations

$$\exists\mathcal{P} \cap \mathcal{R}, \quad \exists\mathcal{Q} \cap \mathcal{R},$$

$$p_{23}r_{01} + p_{31}r_{02} + p_{12}r_{03} + p_{01}r_{23} + p_{02}r_{31} + p_{03}r_{12} = 0, \quad (9)$$

$$q_{23}r_{01} + q_{31}r_{02} + q_{12}r_{03} + q_{01}r_{23} + q_{02}r_{31} + q_{03}r_{12} = 0. \quad (10)$$

Substituting the Plücker coordinates of  $\mathcal{P}$ ,  $\mathcal{Q}$  and  $\mathcal{R}$  given above, removing the common factor  $R$  from Eq. 9 and  $r$  from Eq. 10 produce Eqs. 11 and 12 below.

$$\begin{aligned}
& 2\cos\alpha + \left[\frac{j}{h} + \frac{k}{h}(1 - 2\cos^2\theta)\right]\sin\alpha \\
& + \frac{r}{h}[(1 - 2\cos^2\theta)\sin\alpha\cos\beta - \cos\alpha\sin\beta] = 0, \quad (11)
\end{aligned}$$

$$\begin{aligned}
& 2\cos\beta + \left[\frac{k}{h} + \frac{j}{h}(1 - 2\cos^2\theta)\right]\sin\beta \\
& + \frac{R}{h}[(1 - 2\cos^2\theta)\cos\alpha\sin\beta - \sin\alpha\cos\beta] = 0. \quad (12)
\end{aligned}$$

The following dimensionless ratios are defined thus.

$$\rho_1 \equiv \frac{j}{h}, \quad \rho_2 \equiv \frac{k}{h}, \quad \rho_3 \equiv \frac{r}{h}, \quad \rho_4 \equiv (1 - 2\cos^2\theta),$$

$$\rho_5 \equiv \frac{R}{h}, \quad \rho_6 \equiv \rho_1 + \rho_2\rho_4, \quad \rho_7 \equiv \rho_2 + \rho_1\rho_4, \quad \rho_8 \equiv \rho_3\rho_4, \quad \rho_9 \equiv \rho_5\rho_4.$$

So Eq. 11 and Eq. 12 become Eq. 13 and Eq. 14.

$$2\cos\alpha + \rho_6\sin\alpha + \rho_8\sin\alpha\cos\beta - \rho_3\cos\alpha\sin\beta = 0, \quad (13)$$

$$2\cos\beta + \rho_7\sin\beta + \rho_9\cos\alpha\sin\beta - \rho_5\sin\alpha\cos\beta = 0. \quad (14)$$

Now the following polynomial ratios are introduced.

$$\cos\alpha = \frac{1 - u^2}{1 + u^2}, \quad \sin\alpha = \frac{2u}{1 + u^2}, \quad u \equiv \tan\frac{\alpha}{2},$$

$$\cos \beta = \frac{1 - v^2}{1 + v^2}, \quad \sin \beta = \frac{2v}{1 + v^2}, \quad v \equiv \tan \frac{\beta}{2}.$$

Substituting these into Eq. 13 and Eq. 14 and multiplying out denominators with the product  $(1 + u^2)(1 + v^2)/2$  produce two polynomials in  $u$  and  $v$ .

$$(1 - u^2)(1 + v^2) + \rho_6 u(1 + v^2) + \rho_8 u(1 - v^2)v - \rho_3(1 - u^2)v = 0, \quad (15)$$

$$(1 - v^2)(1 + u^2) + \rho_7 v(1 + u^2) + \rho_9 v(1 - u^2)v - \rho_5(1 - v^2)u = 0. \quad (16)$$

Collecting  $v$  in Eqs. 15 and 16 produces Eqs. 17 and 18.

$$[(\rho_6 - \rho_8)u + (1 - u^2)]v^2 - \rho_3(1 - u^2)v + [(\rho_6 + \rho_8) + (1 - u^2)] = 0, \quad (17)$$

$$[\rho_5 u - (1 + u^2)]v^2 + [\rho_7(1 + u^2) + \rho_9(1 - u^2)]v - [r_5 u - (1 + u^2)] = 0, \quad (18)$$

which may be abridged as

$$j_1 v^2 + j_2 v + j_3 = 0, \quad k_1 v^2 + k_2 v + k_3 = 0. \quad (19)$$

The parameter  $v$  may be eliminated with Eq. 20

$$(j_1 k_2 - k_1 j_2)(j_2 k_3 - k_2 j_3) - (j_3 k_1 - k_3 j_1)^2 = 0, \quad (20)$$

which can be collected on  $u$  to produce an eighth order univariate, Eq. 21, in  $u$ .

$$a_0 u^8 + a_1 u^7 + a_2 u^6 + a_3 u^5 + a_4 u^4 + a_5 u^3 + a_6 u^2 + a_7 u + a_8 = 0, \quad (21)$$

where coefficients are expressed below in terms of the dimensionless ratios  $\rho_i$  and some intermediate combinations thereof which will be tabulated first.

$$\begin{aligned} t_1 &\equiv \rho_7 - \rho_9 & t_2 &\equiv \rho_3 - t_1 & t_3 &\equiv \rho_3 + t_1 \\ t_4 &\equiv \rho_5 + \rho_6 & t_5 &\equiv \rho_3 \rho_5 & t_6 &\equiv \rho_6 - \rho_8 \\ t_7 &\equiv \rho_6 + \rho_8 & t_8 &\equiv \rho_6 - \rho_5 & t_9 &\equiv \rho_7 + \rho_9 \\ t_{10} &\equiv \rho_5 + \rho_6 & t_{11} &\equiv \rho_5 \rho_6 & t_{12} &\equiv -\rho_3 + t_9 \\ t_{13} &\equiv \rho_3 + t_0 & t_{14} &\equiv t_5 - t_1 t_6 & t_{15} &\equiv t_5 + t_1 t_7 \\ t_{16} &\equiv t_5 + t_6 t_9 & t_{17} &\equiv t_5 - t_7 t_9 \end{aligned} \quad (22)$$

$$\begin{aligned} a_8 &\equiv t_2 t_3 - 4 \\ a_7 &\equiv 8t_4 - t_3 t_{14} - t_2 t_{15} \\ a_6 &\equiv -8t_{11} - 4\rho_9 t_1 - 4t_4^2 + t_{14} t_{15} \\ a_5 &\equiv 8t_8 + 4\rho_6 \rho_9 t_1 + 8t_{10} t_{11} + t_2 t_{17} + t_3 t_{16} \\ a_4 &\equiv t_3 t_{12} - 4\rho_9^2 - t_{15} t_{16} - t_{14} t_{17} - t_2 t_{13} + 8 - 8t_8 t_{10} - 4t_{11}^2 \\ a_3 &\equiv -8t_4 + 4\rho_6 \rho_9 t_9 + 8t_8 t_{11} - t_{12} t_{15} + t_{13} t_{14} \\ a_2 &\equiv 8t_{11} + 4\rho_9 t_9 - 4t_8^2 + t_{16} t_{17} \\ a_1 &\equiv -8t_8 + t_{12} t_{17} - t_{13} t_{16} \\ a_0 &\equiv -t_{12} t_{13} - 4. \end{aligned} \quad (23)$$

## 5.4 Solution

Eight values of  $\tan \frac{\alpha}{2}$  are computed with Eq. 21. Then the unique corresponding values of  $\tan \frac{\beta}{2}$  are available, after eliminating  $v^2$  from Eqs. 19, to produce the linear equation in  $v$ , Eq. 24.

$$(j_1 k_2 - k_1 j_2)v - (j_3 k_1 - k_3 j_1) = 0. \quad (24)$$

Finally the coordinates of the desired connecting points  $P$  and  $Q$  are evaluated with Eq. 7 and Eq. 8.

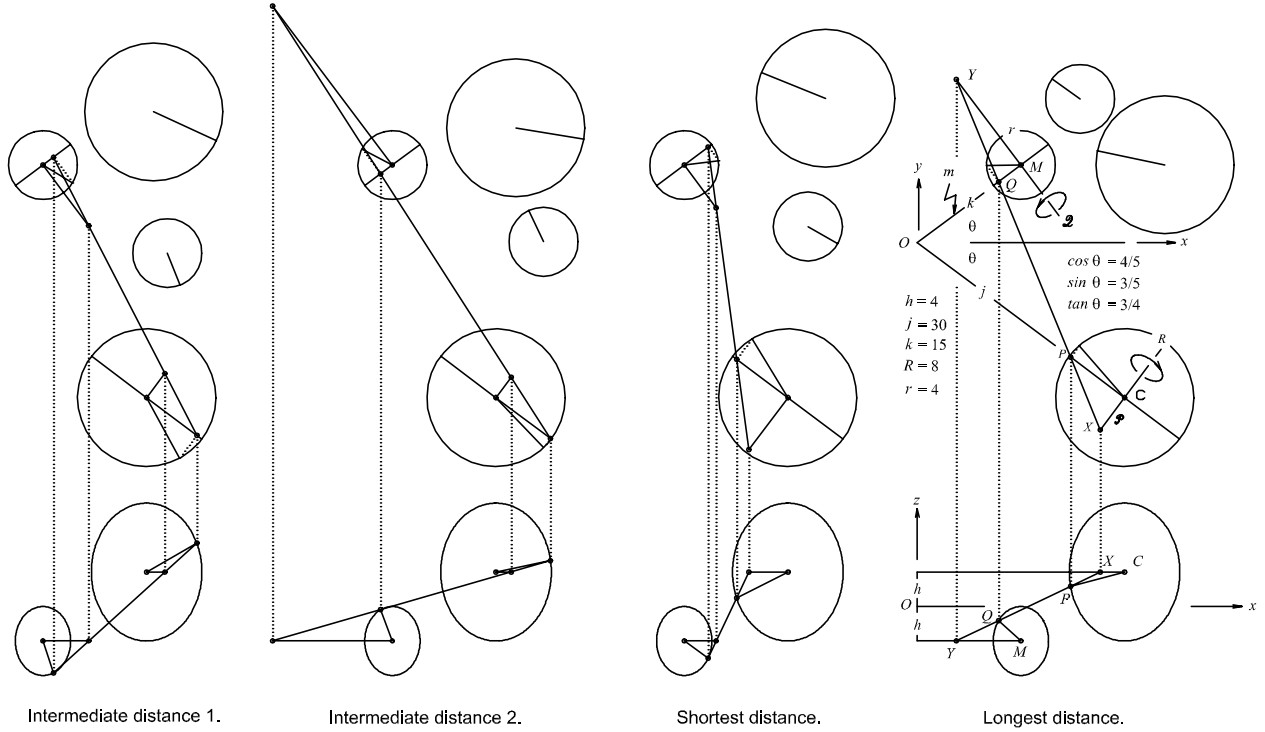


Figure 6: Circle-Circle: checking the four real solutions for circle axis intersection.

## 5.5 An Example

The example shown in Figure 5 was solved using the approach described above and for the parameters shown. Only four of the eight roots are real values of  $\tan \frac{\alpha}{2}$ . These produce the following corresponding values of  $\alpha$  and  $\beta$ .

$\alpha$	$\beta$
$-24.768^\circ$	$-68.088^\circ$
$-9.467^\circ$	$11.234^\circ$
$158.063^\circ$	$-30.001^\circ$
$168.035^\circ$	$144.196^\circ$

Shown in Figure 6 is constructive verification of these results. Note that in all four cases, lines  $PQ$  intersect the circle axes.

## 6 Conclusion

In this paper we have presented closed form algebraic solutions to a set of four spatial shortest distance problems between: a circle and point not in the plane of the circle; a circle and a plane; a circle and line not in the plane of the circle; and two non coplanar circles. The algebraic results have been confirmed with descriptive geometric constructions. Future work requires a careful examination be carried out to identify, with relations among the coefficients of Eq. 23, various geometric conditions pertaining to circle disposition which give rise to:

- a unique shortest distance  $PQ$ ;
- a pair of identical minimum distances;
- algebraic conditions producing a one parameter ruling set of equal minimum distances;

and provide answers to:

- when are minimal distances zero?
- Are there ever more than four real roots (it is most doubtful based on the evidence of the line to circle case)?

Our results have led us to the following question: is there a rapid, sure way to recognize and select true minima among real multiple distinct roots?

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