Introduction to Robotics

MECH 4503

M. John D. Hayes

Carleton University
Mechanical and Aerospace Engineering

January 23, 2008

Contents

Chapter 1

Introduction

In the recent past the exact definition of a *robot* was subject, in some circles, to vigorous debate. A generally accepted definition today comes from the *Robot Institute of America* in the US:

> "A re-programmable manipulator designed to move material, parts, tools, or other specialized devices through variable programmed motions for the performance of a variety of tasks."

This definition distinguishes robots from other automated machines, like NC (numerically controlled) milling machines, by the sophistication of the programmability, and range of applications of the device. For instance, an NC milling machine can not be programmed for pick-and-place operations, or spot welding, while a *robot* can.

A Little History:

- 1948 - Master-slave manipulators developed for handling radioactive materials for nuclear weapons production.
- 1949 - NC machine first developed at MIT.
- 1954 - George Devol: first *industrial robot* (programmable device) used for pick and place for parts transfer. It possessed 4 DOF, tape memory, and point-to-point control.
- 1970 - After purchasing Devol's patents in 1959, Joe Engelberger formed the company Unimation. In 1972, the first PUMA (programmable universal machine for assembly) robot was marketed. As a result, Engelberger has been called the *father of robotics*.

Modern industrial robots have increased in capability and performance with controller and language development, improved mechanisms, sensing and drive systems. In the early 1980's the robot industry grew rapidly thanks to large investments by the automotive industry. However, the leap to "the factory of tomorrow" by other industries turned into a plunge when integration of systems and their economic viability proved disastrous. Only in 1997 did the robotic industry recover to mid 1980's revenue levels.

Spot welding, long the "king" of robot applications has been dethroned by *material handling* (George Devol's original application). This is an indication that the robotics industry has weaned itself from the automotive industry, since material handling cuts across a wide range of industries. Companies that had given up on robotics long ago are now taking a second look and discovering the industry can now provide the solutions they need. Manufacturing executives, who previously assailed the reliability of robots, now give testimonies about the outstanding performance record of robot technology.

<u>Industrial and Service Tasks Which Benefit from Robotisation:</u>

- Material transfer.
- Spray painting.
- Welding.
- Assembly.
- Inspection.
- Manipulation in hazardous environments.
    - Nuclear sector:
        * waste removal;
        * decommission of power plants;
        * accident cleanup.
    - Oceans:
        * maintenance of offshore structures;
        * exploration and research of ocean floor.
    - Space:
        * capture and repair of satellites;
        * structure assembly;
        * inspection and exploration of extraterrestrial objects.
- Mining.
- Forestry.
- Agriculture (planting, harvesting).
- Services (nursing, pharmacy, gas station attendant).

Physical Components of Robotic Manipulators

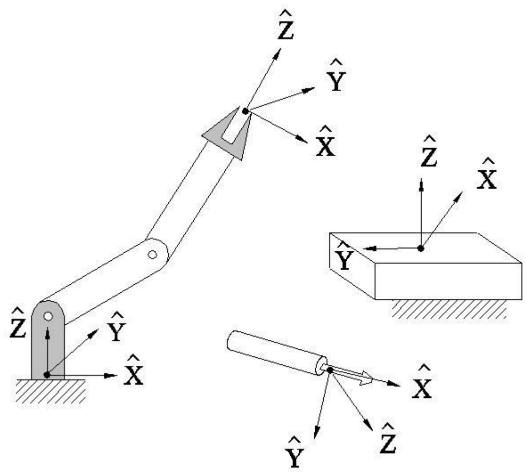
- **Mechanical System**
  - Manipulator: (hand) allows for placement of end effector in space.
  - End-Effector: (what the hand is holding) task accomplishment: welding rod, laser, tool.
  - Actuators: allow manipulator to move.
    - electric (DC brush motors, brushless motors, AC stepper motors).
    - hydraulics.
    - pneumatics.
  - Drives and Transmissions:
    - gears;
    - cables - tendon drives;
    - direct drive (no transmission, directly linked to joint).
- **Sensors:**
  - Internal State:
    - encoders (binary rotation information);
    - resolvers;
    - potentiometers;
    - gyroscopes - good, but they can drift and require re-calibration, which is expensive;
    - accelerometers - not too dependable, but cheap.
  - Interaction:
    - force sensors;
    - strain gauges;
    - load cells.
  - External:
    - visual sensors (digital cameras);
    - acoustic rangefinders;
    - lasers (triangulation).
- **Controller:** uses information from sensors as feedback to control position, velocity, acceleration (i.e. motion of the manipulator).

1.1 From Here to Dynamics

To get to the point where we have enough tools to deal with serial robot (multi-body) dynamics, we must first consider, in the order given below, the following.

1. Description of Position and Orientation
   We are constantly concerned with the position and orientation of objects in 3-D space. The objects are the links of the robot, parts and tools with which it deals, and other objects in the robot's environment.

   To describe the position and orientation (pose) of an object, we usually attach a reference coordinate frame rigidly to it. We then describe the pose of this frame with respect to some other reference frame.

   Since we may be interested in representing the pose of one reference frame with respect to more than one other frame, we have to consider *transforming* the coordinates of points in one frame to the coordinates of the same point expressed in another reference frame. First we will consider conventions and methodologies for dealing with this description of a *pose*, and the mathematics of manipulating these quantities with respect to various coordinate frames.

2. Mappings: changing descriptions from frame to frame.

3. Operators: translations, rotations.

4. Transformation arithmetic.

5. Transformation equations.

1.2 Manipulator Kinematics

Kinematics is the study of motion without regard to the forces which cause the motion. We must examine the following.

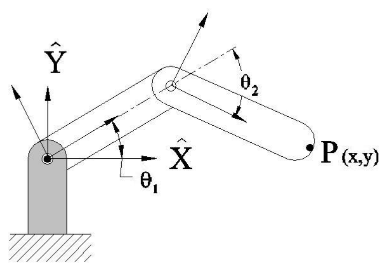
1. Link description.

2. Joint description.

3. DH conventions: methods for affixing frames to links.

4. Forward Kinematics
   This is the problem whereby given a set of particular variable joint inputs (1 for each DOF of the manipulator), determine the pose of the robot. Since we can control the joint inputs and know their values, this is known as the *forward* problem.

   - Position Kinematics
     e.g. given $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ determine $\begin{bmatrix} x \\ y \end{bmatrix}$.



   - Differential Kinematics
     - Velocity Level: $\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}. \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$
     - Acceleration Level: $\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}$.

5. Inverse Kinematics
   This is the problem whereby given a desired (feasible) pose of the manipulator, determine all joint input values necessary to attain the pose, if they exist.

   - Position Level: $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \Rightarrow \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$.
   - Velocity Level: $\dot{\mathbf{x}} \Rightarrow \dot{\boldsymbol{\theta}}$.
   - Acceleration Level: $\ddot{\mathbf{x}} \Rightarrow \ddot{\boldsymbol{\theta}}$.

1.3 Jacobians: Velocities and Static Forces

It turns out that velocities as well as static forces and moments lead to a matrix quantity called the *Jacobian of the Manipulator*. In this regard, manipulator velocities and static forces are considered to be dual quantities. Thus both velocities and static forces can be studied by considering the same Jacobian.

1.4 Dynamics

The study of motion including the forces which cause the motion.

1.4.1 Newton's Equation for Linear Acceleration

$$\sum \mathbf{F} \;=\; \dot{\mathbf{G}} \;=\; m\frac{d\mathbf{v}}{dt} \;=\; m\mathbf{a}, \;\; \mathbf{G} \;=\; m\mathbf{v}.$$

The resultant of all forces acting on a system is equivalent to the systems time rate of change of linear momentum ($\mathbf{G}$). Also, the resultant of moments acting on a system equals the time rate of change of angular momentum ($\mathbf{H}$).

1.4.2 Euler's Equation for angular acceleration

$$\begin{aligned} \sum \mathbf{M} &= \dot{\mathbf{H}} = \frac{d(\mathbf{I}\boldsymbol{\omega})}{dt}, \;\; \mathbf{H} = \mathbf{I}_c\boldsymbol{\omega}, \\ &= \mathbf{I}_c\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_c\boldsymbol{\omega}, \\ &= \mathbf{I}_c\boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{I}_c\boldsymbol{\omega}, \end{aligned}$$

where $\boldsymbol{\omega} \times \mathbf{I}_c\boldsymbol{\omega}$ vanishes for planar systems.

- Iterative Newton-Euler Dynamic Formulation
  - *Outward* iterations, from the base, to compute velocities and accelerations.
  - *Inward* iterations, towards the base, to compute forces and torques.
- Closed Form Dynamic Equations. Obtained by applying iterative Newton-Euler equations symbolically to $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\ddot{\boldsymbol{\theta}}$ to investigate the structure of the equations.
- The *State-Space* Equation.

  $$\boldsymbol{\tau} = \mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{g}(\boldsymbol{\theta}),$$

  where $\mathbf{M}(\boldsymbol{\theta})$ is the $m \times n$ mass matrix, $\mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is the $n \times 1$ vector of centrifugal and Coriolis terms, $\mathbf{g}(\boldsymbol{\theta})$ is the $n \times 1$ vector of gravity terms. The term *state-space* is used because $\mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is both position and velocity dependent.

- The *Configuration Space* Equation. The velocity dependent terms may be written in a different way, allowing us to write the dynamics equations as:

$$\boldsymbol{\tau} = \mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{B}(\boldsymbol{\theta})[\dot{\boldsymbol{\theta}}\,\dot{\boldsymbol{\theta}}] + \mathbf{c}(\boldsymbol{\theta})[\dot{\boldsymbol{\theta}}^{\,2}] + \mathbf{g}(\boldsymbol{\theta}),$$

  where $\mathbf{B}(\boldsymbol{\theta})$ is an $n \times \dfrac{n(n-1)}{2}$ matrix of Coriolis terms, $[\dot{\boldsymbol{\theta}}\;\;\dot{\boldsymbol{\theta}}]$ is an $\dfrac{n(n-1)}{2} \times 1$ vector of joint velocity products given by:

$$\begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \\ \dot{\theta}_1 & \dot{\theta}_3 \\ \dot{\theta}_1 & \dot{\theta}_4 \\ \vdots & \vdots \\ \dot{\theta}_2 & \dot{\theta}_3 \\ \dot{\theta}_2 & \dot{\theta}_4 \\ \vdots & \vdots \\ \dot{\theta}_{n-1} & \dot{\theta}_n \end{bmatrix},$$

  where $\mathbf{c}(\boldsymbol{\theta})$ is an $n \times n$ matrix of centrifugal coefficients, and $[\dot{\boldsymbol{\theta}}^{\mathbf{2}}]$ is an $n \times 1$ vector:

$$\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \\ \vdots \\ \dot{\theta}_n^2 \end{bmatrix}.$$

  This formulation is called *configuration space* because the coefficient matrices are functions of one position only.

  It is often convenient to express the dynamic equations in one of the above ways. It hides some details, but shows the structure of the equations.

1.5 Lagrangian Formulation of Manipulator Dynamics

The Newton-Euler formulation is a *force balance* approach to dynamics, whereas the Lagrangian formulation is an *energy balance* approach. Of course, for the same manipulator, both will give identical equations of motion. For some situations one formulation may be easier to use than the other.

Chapter 2

Descriptions of Position and Orientation: Pose

Robotic manipulation, by definition, implies that parts and tools will be moved around in space somehow, by some sort of mechanism. This naturally leads to the need of representing positions and orientations of the parts, tools, and of the mechanism itself. To define and manipulate mathematical quantities which represent position and orientation of an object, also called the *pose* of the object, we must define coordinate reference frames and develop conventions their for representation.

Many of the ideas developed here in the context of *the pose of an object* form the basis for our approach to analysis of linear and angular velocities as well as for forces and torques.

2.1 Position of a Point

Suppose we want to describe the position of a point in space. A good way to proceed is to first define a coordinate reference frame. To do that, we must decide which type of coordinates to use. We have many choices.

2.1.1 Rectangular Cartesian Coordinates

These coordinates are expressed as distances along three mutually perpendicular coordinate axes. For instance, the position of point P in frame $\{0\}$ can be expressed as a position vector whose components are the distances along the corresponding axes, described by the unit vectors $\hat{\mathbf{X}}_0$, $\hat{\mathbf{Y}}_0$, $\hat{\mathbf{Z}}_0$.

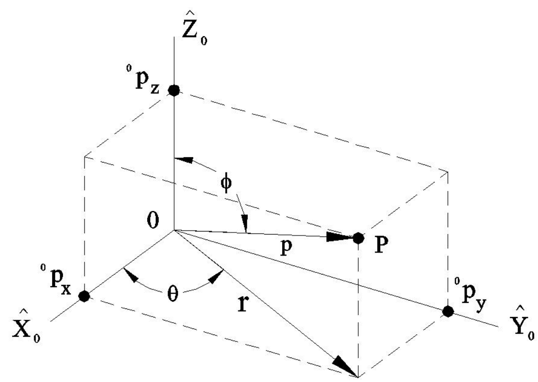


Figure 2.1: Rectangular Cartesian coordinates.

Notation:

Unit Vector

$$\begin{bmatrix} \hat{X}_0 \\ \hat{Y}_0 \\ \hat{Z}_0 \end{bmatrix}$$

Unit vectors have a magnitude of 1. The lower right subscript indicates the reference frame.

Position Vector

$$^0\mathbf{p} = \begin{bmatrix} ^0p_x \\ ^0p_y \\ ^0p_z \end{bmatrix}$$

The upper left superscript indicates the coordinate system to which the point is referenced. Individual elements are given subscripts x, y, z.

Position vectors always originate at the origin of the reference frame in which they are described. When defining a reference frame an origin *must* be specified along with the unit vectors defining its axes directions. The elements of a position vector can be regarded as the projections of the position vector onto the corresponding axis.i.e.

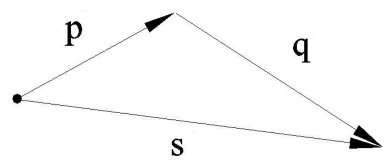
$$\begin{aligned} ^0p_x &= \mathbf{p} \cdot \hat{\mathbf{X}}_0 &= \begin{bmatrix} ^0p_x & ^0p_y & ^0p_z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= |\mathbf{p}| \cos(\theta_x), \\ ^0p_y &= \mathbf{p} \cdot \hat{\mathbf{Y}}_0 &= \begin{bmatrix} ^0p_x & ^0p_y & ^0p_z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= |\mathbf{p}| \cos(\theta_y), \\ ^0p_z &= \mathbf{p} \cdot \hat{\mathbf{Z}}_0 &= \begin{bmatrix} ^0p_x & ^0p_y & ^0p_z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= |\mathbf{p}| \cos(\theta_z). \end{aligned}$$

In summary, we will describe the position of a point in space with a position vector, whose elements are rectangular Cartesian coordinates. There are other possibilities: cylindrical coordinates $P(r, \theta, z)$; or spherical coordinates $P(\rho, \phi, \theta)$. However, it is commonplace to use rectangular Cartesian coordinates.

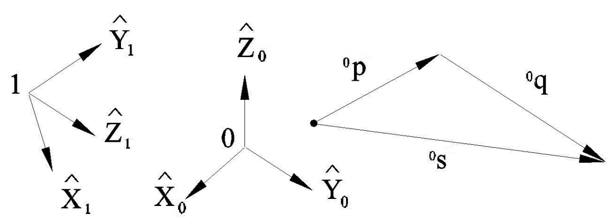
2.1.2 Important Fact of Life: Vector Addition

We will often be representing the same point in more than one coordinate system, so we have to be careful. Consider vectors $\mathbf{p}$ and $\mathbf{q}$. Their sum $\mathbf{s}$ is:

Graphical Addition: $\mathbf{p} + \mathbf{q} = \mathbf{s}$.



Algebraic Addition: First we must define a frame of reference in which the vectors are represented.



$$\begin{aligned} {}^0\mathbf{p} + {}^0\mathbf{q} &= {}^0\mathbf{s}, \\ \Rightarrow \begin{bmatrix} {}^0\mathbf{p}_x + {}^0\mathbf{q}_x \\ {}^0\mathbf{p}_y + {}^0\mathbf{q}_y \\ {}^0\mathbf{p}_z + {}^0\mathbf{q}_z \end{bmatrix} &= \begin{bmatrix} {}^0\mathbf{s}_x \\ {}^0\mathbf{s}_y \\ {}^0\mathbf{s}_z \end{bmatrix}, \\ \textit{and } {}^1\mathbf{p} + {}^1\mathbf{q} &= {}^1\mathbf{s}. \end{aligned}$$

But in general ${}^1\mathbf{p} + {}^0\mathbf{q} = \mathit{Junk}!$

2.2 Orientation of a (Rigid) Body

A point is just a point: it has position, but its orientation is ambiguous because it has no dimensions, only location. When we have an additional point in a different location, and the distance between the points is frozen in time for the instant considered, then we can consider the notion of orientation.

A rigid body may be considered to be a collection of points. To describe the orientation of a rigid body in space, we may rigidly attach (paint on) a coordinate system to the body and then give a description of this coordinate system with respect to some other reference coordinate system.

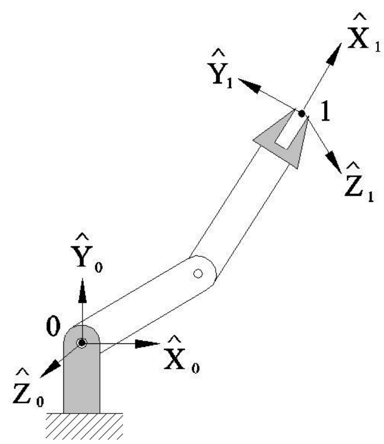


Figure 2.2: Reference coordinate systems.

In Figure 2.2 coordinate system $\{1\}$ has been attached to the gripper in a known way. We also know everything about the stationary (non moving) system $\{0\}$, rigidly attached to the ground. A description of $\{1\}$ relative to $\{0\}$ now suffices to specify the orientation of the gripper, at least relative to $\{0\}$.

So far, we see that positions of points can be described with vectors, while orientations of bodies can be described with body-fixed coordinate systems. One way to describe coordinate system $\{1\}$ is to write the unit vectors corresponding to its three principal axes (we will always use orthogonal principal axes) in terms of those of coordinate system $\{0\}$. We write the unit vectors of $\{1\}$ as: $\hat{\mathbf{X}}_1$, $\hat{\mathbf{Y}}_1$, $\hat{\mathbf{Z}}_1$. When written in terms of $\{0\}$ they are ${}^0\hat{\mathbf{X}}_1$, ${}^0\hat{\mathbf{Y}}_1$, ${}^0\hat{\mathbf{Z}}_1$. For example, recall the components of ${}^0\hat{\mathbf{X}}_1$ are the projections of $\hat{\mathbf{X}}_1$ on to the axes of $\{0\}$ (i.e. the dot products $\hat{\mathbf{X}}_1 \cdot \hat{\mathbf{X}}_o$, etc).

$$
\begin{aligned}
{}^0\hat{\mathbf{X}}_1 &= \begin{bmatrix} \hat{\mathbf{X}}_1{}^T & \hat{\mathbf{X}}_0 \\ \hat{\mathbf{X}}_1{}^T & \hat{\mathbf{Y}}_0 \\ \hat{\mathbf{X}}_1{}^T & \hat{\mathbf{Z}}_0 \end{bmatrix} = \begin{bmatrix} \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_0) \\ \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{Y}}_0) \\ \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{Z}}_0) \end{bmatrix}, \\
\text{while } {}^0\hat{\mathbf{Y}}_1 &= \begin{bmatrix} \hat{\mathbf{Y}}_1{}^T & \hat{\mathbf{X}}_0 \\ \hat{\mathbf{Y}}_1{}^T & \hat{\mathbf{Y}}_0 \\ \hat{\mathbf{Y}}_1{}^T & \hat{\mathbf{Z}}_0 \end{bmatrix} = \begin{bmatrix} \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{X}}_0) \\ \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{Y}}_0) \\ \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{Z}}_0) \end{bmatrix}, \\
\text{and } {}^0\hat{\mathbf{Z}}_1 &= \begin{bmatrix} \hat{\mathbf{Z}}_1{}^T & \hat{\mathbf{X}}_0 \\ \hat{\mathbf{Z}}_1{}^T & \hat{\mathbf{Y}}_0 \\ \hat{\mathbf{Z}}_1{}^T & \hat{\mathbf{Z}}_0 \end{bmatrix} = \begin{bmatrix} \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{X}}_0) \\ \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{Y}}_0) \\ \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_0) \end{bmatrix}.
\end{aligned}
$$

Taken together, the set of column vectors $\begin{bmatrix} {}^0\hat{\mathbf{X}}_1 & {}^0\hat{\mathbf{Y}}_1 & {}^0\hat{\mathbf{Z}}_1 \end{bmatrix}$ describe the orientation of $\{1\}$ relative to $\{0\}$, and hence the orientation of the body to which it is attached. The result is a 3×3 matrix which describing the change in orientation. Because the change in orientation requires rotations, it is called a *rotation matrix*, and is indicated by ${}^0\mathbf{R}_1$. It has the form

$$
{}^0\mathbf{R}_1 = \begin{bmatrix} {}^0\hat{\mathbf{X}}_1 & {}^0\hat{\mathbf{Y}}_1 & {}^0\hat{\mathbf{Z}}_1 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{X}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{X}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{X}}_0 \\ \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Y}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Y}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{Y}}_0 \\ \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Z}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Z}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{Z}}_0 \end{bmatrix}. \tag{2.1}
$$

It is interesting to note that since the dot product of two unit vectors yields the cosine of the angle between them, *orientation* or *rotation* matrix components are sometimes referred to as *direction cosines.*

Closer inspection of Equation (2.1) reveals that the columns are the unit vectors of $\{1\}$ in $\{0\}$, while the rows are the unit vectors of $\{0\}$ expressed in $\{1\}$. Hence, the transpose of ${}^1\mathbf{R}_0$ is equivalent to ${}^0\mathbf{R}_1$. This suggests that the transpose of a rotation matrix is equal to its inverse:

$$
{}^0\mathbf{R}_1 = {}^1\mathbf{R}_0^T = {}^1\mathbf{R}_0^{-1}.
$$

Remember that ${}^1\hat{\mathbf{X}}_0$, ${}^1\hat{\mathbf{Y}}_0$, ${}^1\hat{\mathbf{Z}}_0$ are all unit vectors, as are ${}^0\hat{\mathbf{X}}_1$, ${}^0\hat{\mathbf{Y}}_1$, ${}^0\hat{\mathbf{Z}}_1$.

Now,

$$
\begin{aligned}
{}^0\mathbf{R}_1 &= \begin{bmatrix} \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_0) & \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{X}}_0) & \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{X}}_0) \\ \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{Y}}_0) & \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{Y}}_0) & \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{Y}}_0) \\ \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{Z}}_0) & \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{Z}}_0) & \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_0) \end{bmatrix}, \\
\text{and } {}^0\mathbf{R}_1^T = {}^1\mathbf{R}_0 &= \begin{bmatrix} \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_0) & \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{Y}}_0) & \cos(\hat{\mathbf{X}}_1, \hat{\mathbf{Z}}_0) \\ \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{X}}_0) & \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{Y}}_0) & \cos(\hat{\mathbf{Y}}_1, \hat{\mathbf{Z}}_0) \\ \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{X}}_0) & \cos(\hat{\mathbf{Z}}_1, \hat{\mathbf{Y}}_0) & \cos \hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_0 \end{bmatrix}, \\
\text{finally, } {}^0\mathbf{R}_1\, {}^0\mathbf{R}_1^T = {}^0\mathbf{R}_1\, {}^1\mathbf{R}_0 &= \begin{bmatrix} {}^1\hat{\mathbf{X}}_0 \cdot {}^1\hat{\mathbf{X}}_0 & {}^1\hat{\mathbf{X}}_0 \cdot {}^1\hat{\mathbf{Y}}_0 & {}^1\hat{\mathbf{X}}_0 \cdot {}^1\hat{\mathbf{Z}}_0 \\ {}^1\hat{\mathbf{Y}}_0 \cdot {}^1\hat{\mathbf{X}}_0 & {}^1\hat{\mathbf{Y}}_0 \cdot {}^1\hat{\mathbf{Y}}_0 & {}^1\hat{\mathbf{Y}}_0 \cdot {}^1\hat{\mathbf{Z}}_0 \\ {}^1\hat{\mathbf{Z}}_0 \cdot {}^1\hat{\mathbf{X}}_0 & {}^1\hat{\mathbf{Z}}_0 \cdot {}^1\hat{\mathbf{Y}}_0 & {}^1\hat{\mathbf{Z}}_0 \cdot {}^1\hat{\mathbf{Z}}_0 \end{bmatrix}, \\
&= \begin{bmatrix} {}^1\hat{\mathbf{X}}_0 & {}^1\hat{\mathbf{Y}}_0 & {}^1\hat{\mathbf{Z}}_0 \end{bmatrix} \begin{bmatrix} {}^1\hat{\mathbf{X}}_0{}^T \\ {}^1\hat{\mathbf{Y}}_0{}^T \\ {}^1\hat{\mathbf{Z}}_0{}^T \end{bmatrix}, \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}.
\end{aligned}
$$

2.2.1 Properties of ${}^0\mathbf{R}_1$

From linear algebra we know that the inverse of a matrix with orthonormal columns *is* equal to its transpose. We have just shown this with geometric arguments.

1. ${}^0\mathbf{R}_1$ is orthogonal and orthonormal because

   $$|{}^0\hat{\mathbf{X}}_1| = |{}^0\hat{\mathbf{Y}}_1| = |{}^0\hat{\mathbf{Z}}_1| = 1 \text{ (3 conditions for orthogonality)},$$

   and ${}^0\hat{\mathbf{X}}_1 \cdot {}^0\hat{\mathbf{Y}}_1 = {}^0\hat{\mathbf{X}}_1 \cdot {}^0\hat{\mathbf{Z}}_1 = {}^0\hat{\mathbf{Y}}_1 \cdot {}^0\hat{\mathbf{Z}}_1 = 0$ (3 conditions for orthonormality).

   (6 conditions, 3 independent numbers).

2. $\mathrm{Det}({}^0\mathbf{R}_1) = 1 \Rightarrow$ Proper.

3. ${}^0\mathbf{R}_1^{-1} = {}^0\mathbf{R}_1^T$.

4. ${}^0\mathbf{R}_1^T\, {}^0\mathbf{R}_1 = \mathbf{I}$.

Example: Consider a rotation of θ degrees about the $\hat{\mathbf{Z}}_0$ axis. We have,

$$
\begin{aligned}
\hat{\mathbf{X}}_1 \cdot \hat{\mathbf{X}}_0 &= \cos(\theta), \\
\hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Y}}_0 &= \cos(\theta), \\
\hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{X}}_0 &= \cos(\theta + 90^o) = -\sin(\theta), \\
\hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Y}}_0 &= \cos(90^o - \theta) = \sin(\theta), \\
\hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Z}}_0 = \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Z}}_0 &= \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{X}}_0 = \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{Y}}_0 = 0, \\
\hat{\mathbf{Z}}_0 \cdot \hat{\mathbf{Z}}_1 &= 1.
\end{aligned}
$$

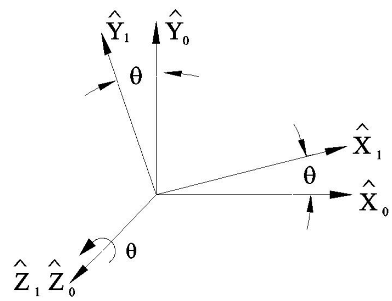


Figure 2.3: Rotation of θ degrees about the $\hat{\mathbf{Z}}_0$ axis.

$$\mathbf{R}_{Z_0}(\theta) \Rightarrow \begin{bmatrix} \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{X}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{X}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{X}}_0 \\ \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Y}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Y}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{Y}}_0 \\ \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Z}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Z}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{Z}}_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.3 Description of a Reference Frame

The information needed to completely specify the whereabouts of a rigid body is a position and an orientation. For position, we can choose an arbitrary point of the rigid body. For convenience, the point whose position we will describe shall be the origin of the body-fixed frame. The situation of a position and an orientation set arises so often in robot kinematic/dynamic analysis that we will define an entity called a *frame*, which is a set of four vectors: one specifying position, the other three specifying orientation. Equivalently, a frame can be thought of as a position vector and a rotation matrix.

For example, Frame 1 is described by ${}^0\mathbf{R}_1$ and ${}^0\mathbf{p}_{1_{\mathrm{ORG}}}$, where ${}^0\mathbf{p}_{1_{\mathrm{ORG}}}$ is the vector which locates the origin of Frame 1. So we have

$$\{1\} = \{{}^0\mathbf{R}_1,\ {}^0\mathbf{p}_{1_{\mathrm{ORG}}}\}.$$

For example, in Figure 2.4 we have three frames and a known reference system. Frames $\{1\}$ and $\{3\}$ are known relative to $\{0\}$, and Frame $\{2\}$ is known relative to Frame $\{1\}$.

In summary, a frame can be used as a description of one coordinate system relative to another. It may be thought of as generalizing the notions of position and orientation, since it encompasses both. Positions could be represented by a frame whose rotation is the identity matrix. Likewise, an orientation could be

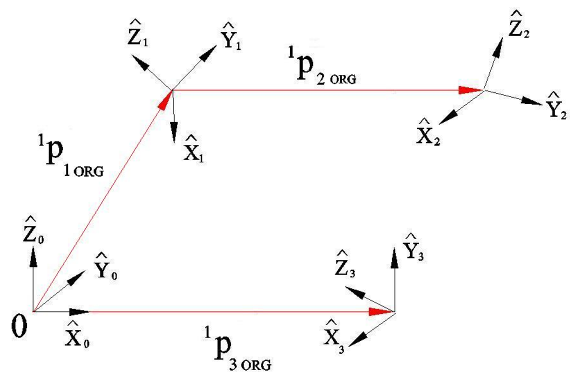


Figure 2.4: Reference frames.

represented by a frame whose position vector part was the zero vector.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

2.4 Mappings: Changing Descriptions from One Frame to Another

In a great many of the problems in robotics, we are concerned with expressing the same quantity in terms of various reference coordinate systems. We have discussed descriptions of positions, orientations, and frames, now we consider the mathematics of mapping so we can change descriptions from frame to frame.

2.4.1 Translated Frames

We wish to change the description of point P from frame $\{1\}$ to $\{0\}$. Here, $\{1\}$ differs from $\{0\}$ by a translation described by ${}^0\mathbf{p}_{1_{\mathrm{ORG}}}$, a vector locating the origin of $\{1\}$ relative to $\{0\}$. Because ${}^0\mathbf{p}_{1_{\mathrm{ORG}}}$ and ${}^1\mathbf{p}$ are defined in frames with the same orientation, we can use simple vector addition to obtain

$${}^0\mathbf{p} = {}^1\mathbf{p} + {}^0\mathbf{p}_{1_{\mathrm{ORG}}}.$$

(Only in this special situation can we add vectors defined in different frames).

Note: The concept of mapping, changing the description from one frame to another, is extremely important for robotics. The quantity itself (here, a point in space) is not changed; only its description is changed.

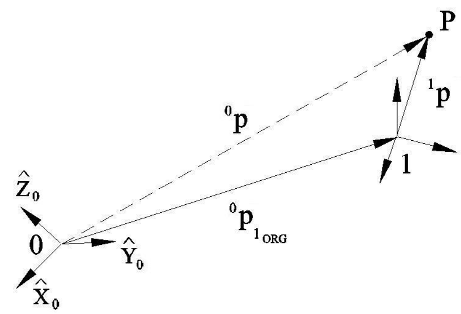


Figure 2.5: Translated frames.

The vector ${}^0\mathbf{p}_{1_{\mathrm{ORG}}}$ defines this mapping: this, along with the knowledge that orientation is constant, is the only information needed to perform the change in description.

2.4.2 Rotated Frames

To change a description from one frame to another sharing the same origin, but with a different orientation, we can use the rotation matrix $\mathbf{R}$. We wish to change the description of P from ${}^1\mathbf{p}$ to ${}^0\mathbf{p}$. To compute the components of ${}^0\mathbf{p}$, we project ${}^1\mathbf{p}$ onto the unit vectors of $\{0\}$. As discussed earlier, this can be accomplished with

$${}^0\mathbf{p} \;=\; {}^0\mathbf{R}_1\, {}^1\mathbf{p}.$$

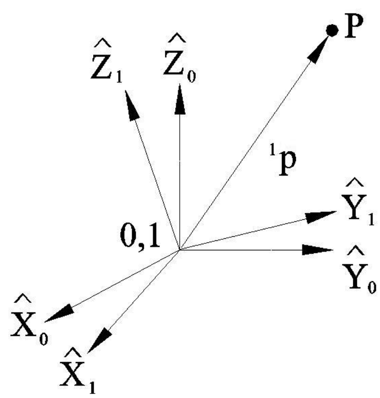
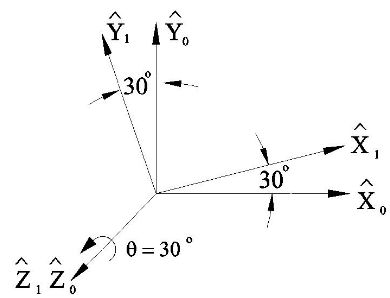


Figure 2.6: Rotated frames.

Note: A useful way to view the notation we have adopted is to imagine the right subscripts “cancel” the left superscripts. For instance, in the previous example the 1’s may be thought of as “canceling out”.

Example: Frame $\{1\}$ is rotated by 30^o relative to frame $\{0\}$ about $\hat{\mathbf{Z}}_0$.



$$
{}^0\mathbf{R}_1 = \begin{bmatrix} \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{X}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{X}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{X}}_0 \\ \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Y}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Y}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{Y}}_0 \\ \hat{\mathbf{X}}_1 \cdot \hat{\mathbf{Z}}_0 & \hat{\mathbf{Y}}_1 \cdot \hat{\mathbf{Z}}_0 & \hat{\mathbf{Z}}_1 \cdot \hat{\mathbf{Z}}_0 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},
$$

$$
\begin{aligned}
r_{11} &= \cos(30), \\
r_{12} &= \cos(30+90) = -\sin(30), \\
r_{21} &= \cos(90-30) = \sin(30), \\
r_{22} &= \cos(30), \\
r_{33} &= \cos(0), \\
r_{31} = r_{32} &= r_{13} = r_{23} = 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{this gives } {}^0\mathbf{R}_1 &= \begin{bmatrix} \cos(30) & -\sin(30) & 0 \\ \sin(30) & \cos(30) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \\
\text{Given } {}^1\mathbf{p} &= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \\
\text{we calculate } {}^0\mathbf{p} &= {}^0\mathbf{R}_1\, {}^1\mathbf{p} = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix}.
\end{aligned}
$$

2.5 General Displacements: Homogeneous Transformations

For the *general* case of mapping, the new frame (called the *image-space*) has undergone both a translation and rotation, at least as far as practical robot joints are concerned. To map ${}^1\mathbf{p}$ to $\{0\}$ we can proceed by first changing ${}^1\mathbf{p}$

to its description relative to an intermediate frame with the same orientation as $\{0\}$, but whose origin is coincident with that of $\{1\}$. This is done by pre-multiplying ${}^1\mathbf{p}$ with ${}^0\mathbf{R}_1$. We can then account for the translation component with simple vector addition. This gives

$$ {}^0\mathbf{p} \; = \; {}^0\mathbf{R}_1 \, {}^1\mathbf{p} + {}^0\mathbf{p}_{1_{\mathrm{ORG}}}. $$

Note, the 1's "cancel", giving all vectors in $\{0\}$, which may then be added. However, the above displacement representation is *not* a linear transformation, for the simple reason that the translation of the sum of two vectors $\mathbf{x}$ and $\mathbf{y}$, by the amount $\mathbf{d}$ is $T(\mathbf{x}+\mathbf{y}) = \mathbf{x}+\mathbf{y}+\mathbf{d}$, and not the sum of the translation of each vector separately, which is $T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{x}+\mathbf{y}+2\mathbf{d}$. Thus, general displacements in 3-D space cannot be represented by 3×3 transformations.

The inconvenience is removed by embedding 3-D Euclidean space, E^3, in E^4, as the 3-dimensional hyperplane H. Identifying E^3 with H changes each 3-D coordinate vector into one that is 4-D. We usually specify the hyperplane H by giving a value to the fourth coordinate. This value is arbitrary. For convenience, we choose 1. A displacement of ${}^0\mathbf{p} \; = \; {}^0\mathbf{R}_1 \, {}^1\mathbf{p} + {}^0\mathbf{p}_{1_{\mathrm{ORG}}}$ of E^3 becomes a linear transformation ${}^0\mathbf{T}_1$ of E^4, given by

$$ \begin{aligned} \left[\begin{array}{c} {}^0\mathbf{p} \\ \hline 1 \end{array}\right] &= \left[\begin{array}{c|c} {}^0\mathbf{R}_1 & {}^0\mathbf{p}_{1_{\mathrm{ORG}}} \\ \hline 0\;0\;0 & 1 \end{array}\right] \left[\begin{array}{c} {}^1\mathbf{p} \\ \hline 1 \end{array}\right], \\ &= {}^0\mathbf{T}_1 \left[\begin{array}{c} {}^1\mathbf{p} \\ \hline 1 \end{array}\right], \\ &= {}^0\mathbf{T}_1 \, {}^1\mathbf{p}. \end{aligned} \tag{2.2} $$

When ${}^1\mathbf{p}$ is pre-multiplied by the 4×4 matrix ${}^0\mathbf{T}_1$, it is obvious from context that ${}^1\mathbf{p}$ has four coordinates, and that the fourth coordinate of $({}^1\mathbf{p})_4 = 1$. The 4×4 matrix $\mathbf{T}$ is called the *homogeneous transform* representing the displacement. The term *homogeneous* refers to the fact that the coordinate vector $\left[\begin{array}{c} {}^1\mathbf{p} \\ \hline 1 \end{array}\right]$ may be interpreted as homogeneous coordinates of a 3-D projective space (4-D homogeneous coordinate space).

Now is the perfect time for a brief review of homogeneous coordinates.

2.5.1 Homogeneous Coordinates

Let O be the origin of the Cartesian coordinate system, shown in Figure 2.7. Let S be a distinct point in the plane. The ray passing through O and S is described by the coordinate pair (x, y). Another distinct point $Q \neq O$, on ray OS is described by the pair (μ_x, μ_y), where $\mu \in \mathcal{R}$ (ie. a real number). As $\mu \to \pm\infty$ the seemingly meaningless pair (∞, ∞) is obtained.

To remedy this representational problem, the point pairs may be represented by two ratios, given by ordered triples (x_0, x_1, x_2). If $x_0 \neq 0$, then the point S can be uniquely described as:

$$ x = \frac{x_1}{x_2} \; , \; y = \frac{x_2}{x_0}. $$

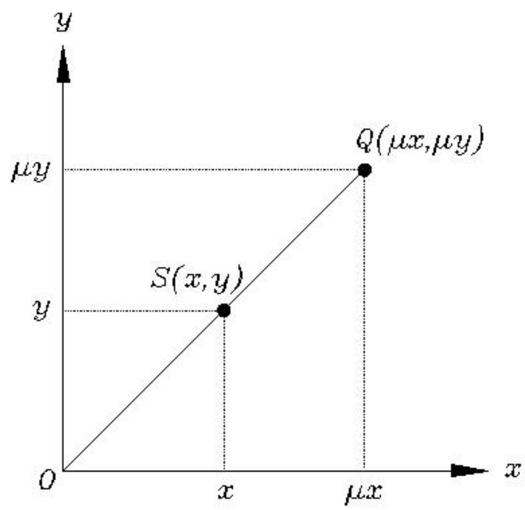


Figure 2.7: Cartesian coordinates in E^2.

Then any triple of the form $(\lambda x_0, \lambda x_1, \lambda x_2)$ (for $\lambda \neq 0$) describes exactly the same point S. In other words, two real points are equal if the triples representing them are proportional. This is because

$$\frac{\lambda x_1}{\lambda x_0} = \frac{x_1}{x_0} = x, \quad \text{and} \quad \frac{\lambda x_2}{\lambda x_0} = y.$$

The coordinates $(x_0 : x_1 : x_2)$ are called *homogeneous coordinates.* When $x_0 = 1$ the Cartesian coordinate pair (x, y) is recovered.

The Cartesian coordinates $(\mu x, \mu y)$, $\mu \neq 0$, of the family of points on the ray through Q in Figure 2.7 can be expressed in homogeneous coordinates as ratios:

$$(\mu x, \mu y) = (x_0 : \mu x_1 : \mu x_2) = (\frac{x_0}{\mu} : x_1 : x_2).$$

In E^2, as $\mu \to \pm\infty$, the homogeneous coordinates $(0 : x_1 : x_2)$ are obtained. There is no point on the line OS to which this triple can correspond because E^2 is unbounded. However, in the projective extension of the Euclidean plane[1] the triple $(0 : x_1 : x_2)$ describes the *point at infinity (ideal point)* on the line OS. Since the same triple is obtained regardless if $\mu \to +\infty$ or $\mu \to -\infty$, a unique point at infinity is associated with the line OS in E^2. Hence, an ordinary line adjoined by its point at infinity is a closed curve.

The triple $(0 : 0 : 0)$ describes neither an ideal point nor a real point on OS. $(x : y : 0) = (0 : 0 : 0)$ seems to imply that $S = O$, which is a contradiction in

[1]The *projective plane*, P_2, can be thought of as the unbounded Euclidean plane, E_2, to which the *line at infinity* has been added, thereby imposing a bound on E_2.

the construction of ray OS. The trivial triple $(0:0:0)$ is therefore not included in the point set comprising the projective extension of E^2.

All lines in E^2 which are extended to their points at infinity have the homogenizing coordinate $x_0 = 0$. The totality of all the existing points at infinity (with the exception of $(0:0:0)$) are described by $x_0 = 0$. The extended Euclidean plane which includes all the points at infinity is called the *projective plane* P_2. Since $x_0 = 0$ is a linear equation, it represents the *line at infinity*.

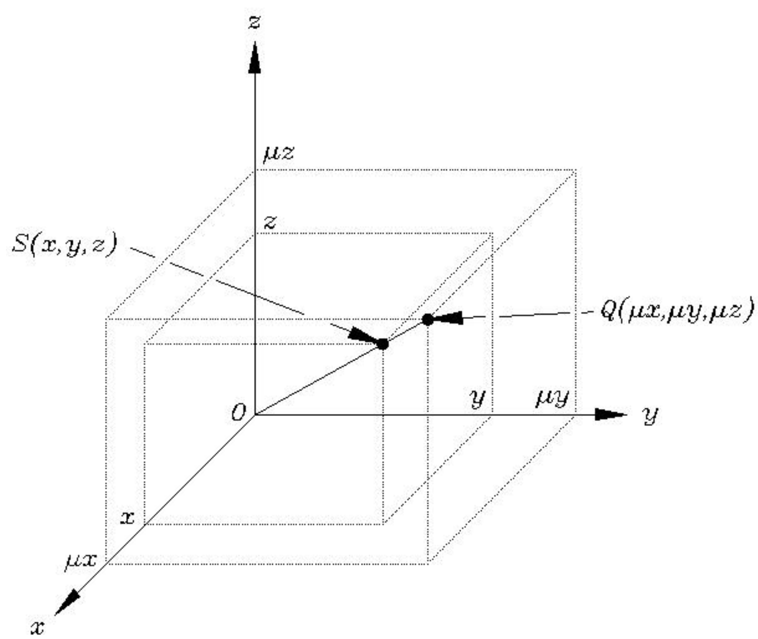


Figure 2.8: Cartesian coordinates in E^3.

Entirely analogous statements can be made for 3-D Euclidean space, E^3. This space is covered by a Cartesian coordinate system with origin O and axes x, y, z. The axes are usually defined as orthogonal. Such an orthogonal Cartesian system is illustrated in Figure 2.8. The homogeneous coordinates (x_0, x_1, x_2, x_3) of the point $S \in E^3$ are defined as:

$$x = \frac{x_1}{x_0},\ y = \frac{x_2}{x_0},\ z = \frac{x_3}{x_0},\ \ x_0 \neq 0.$$

As in two dimensional projective space, when $x_0 = 1$ the Cartesian coordinate triple (x, y, z) is recovered.

It should be noted that in general the choice of homogenizing coordinate is arbitrary. Over the course of time the following conventions have been developed:

1. In North America and the British Commonwealth the homogenizing coordinate is taken to be the last one. The coordinate indices begin with 1. In the plane $(x_1 : x_2 : x_3)$ represent the coordinates of a point, with x_3 the homogenizing coordinate. In space, a point is described with $(x_1 : x_2 : x_3 : x_4)$, with x_4 being the homogenizing coordinate. In general, the homogenizing coordinate in an n-D space has the index $n + 1$.

2. In most other places the first coordinate, given the index 0, is taken to be the homogenizing one. Thus, x_0 represents the homogenizing coordinate regardless of the dimension of the coordinate space.

Both conventions shall be employed henceforth. This is to underscore the idea that such a restriction is arbitrary and unnecessary in the context of projective geometry. However, where required the homogenizing coordinate shall be explicitly identified.

2.5.2 A Note on Free and Position Vectors

Free vectors represent a magnitude and direction, but represent a vector field (i.e. couples, moments, angular velocities and accelerations, angular and linear momentum, etc.) and hence pass through every point in space. Thus, they may be conveniently represented by setting the homogenizing coordinate equal to zero:

$$\mathbf{v}_f = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}.$$

Position vectors locate a point in space, and thus must pass through a specific point. They my be conveniently represented by setting the homogenizing coordinate equal to one:

$$\mathbf{v}_p = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix}.$$

<u>Example: Homogeneous Transform Mapping</u>

Consider frame $\{1\}$ rotated with respect to frame $\{0\}$ about $\hat{\mathbf{Z}}$ by 30^o and translated 10 units along $\hat{\mathbf{X}}_0$ and 5 units along $\hat{\mathbf{Y}}_0$. Find ${}^0\mathbf{p}$ when ${}^1\mathbf{p} = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$.

<u>Solution:</u> The definition of Frame $\{1\}$ in Frame $\{0\}$ is:

$${}^0\mathbf{T}_1 = \begin{bmatrix} \cos(30) & -\sin(30) & 0 & 10 \\ \sin(30) & \cos(30) & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We add a fourth coordinate of 1 to ${}^1\mathbf{p}$ and pre-multiply with ${}^0\mathbf{T}_1$:

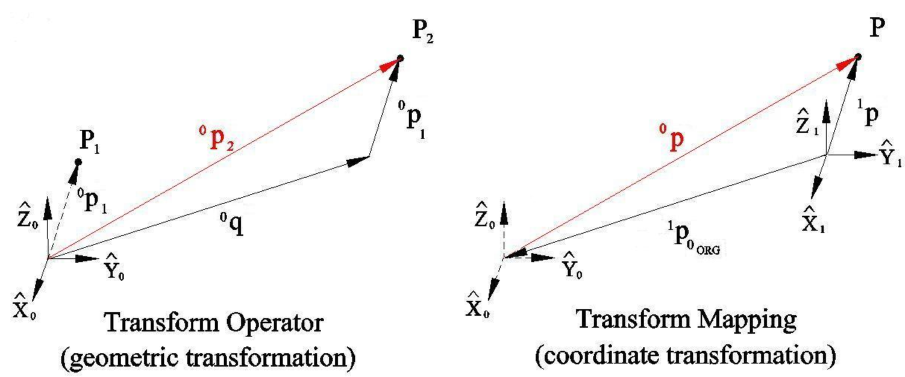
$$ {}^0\mathbf{p} \;=\; {}^0\mathbf{T}_1 \, {}^1\mathbf{p} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}. $$

2.6 Transform Operators

The same mathematical forms we have used to map points from one frame to another can also be interpreted as operators: matrix operators, which translate points and rotate vectors. We will now examine this interpretation of the transforms we have discussed.

2.6.1 Translation Operators

Let's first consider a simple illustration: translation of a point. Using the operator interpretation of the homogeneous transform, to translate the actual point, we only need one coordinate system. The distinction between the two interpretations is: we may consider that either an object has moved relative to a frame (operator) or, that the frame has moved relative to the object (transform). For instance, we may consider that a vector has moved *forward* relative to a frame, or that that frame has moved *backward* relative to the vector.



Here, vector ${}^0\mathbf{p}_1$ is translated by vector ${}^0\mathbf{q}$. This moves the point P_1 to P_2 by ${}^0\mathbf{q}$. Because all vectors are described in $\{0\}$, no subscripts or superscripts are needed for $\mathbf{T}$. The new vector, ${}^0\mathbf{p}_2$ is calculated as

$$ {}^0\mathbf{p}_2 \;=\; {}^0\mathbf{p}_1 + {}^0\mathbf{q} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} {}^0\mathbf{p}_1 \\ \hline 1 \end{array}\right] = \mathbf{T}\, {}^0\mathbf{p}_1. $$

The mapping interpretation gives:

$$ {}^{0}\mathbf{p} \; = \; {}^{1}\mathbf{p} - {}^{1}\mathbf{p}_{0_{\mathrm{ORG}}} \; = \; {}^{0}\mathbf{T}_{1}\, {}^{1}\mathbf{p}. $$

The sign change caused by moving $\{0\}$ backwards $(-\,{}^{1}\mathbf{p}_{0_{\mathrm{ORG}}})$ is all the difference. If we use ${}^{0}\mathbf{p}_{1_{\mathrm{ORG}}}$ instead, the two interpretations are isomorphic!

2.6.2 Rotation Operators

A rotation matrix operates on a vector, to change it to a new vector by rotation about some axis. When used as an operator, no subscripts or superscripts appear since it is not seen as transforming reference frames. Hence, we write

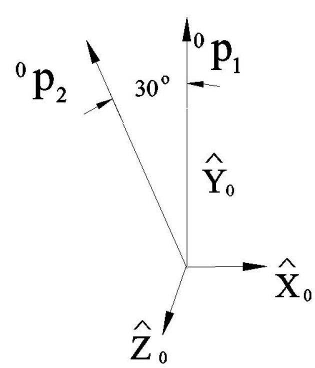
$$ {}^{0}\mathbf{p}_{2} = \mathbf{R}\, {}^{0}\mathbf{p}_{1}. $$

Note: for a pure rotation:

$$ \mathbf{T} = \left[\begin{array}{c|c} & 0 \\ \mathbf{R} & 0 \\ & 0 \\ \hline 0\;0\;0 & 1 \end{array} \right]. $$

As for translations, the mathematics describing a mapping involving rotation and a rotation operator is the same, only the interpretation is different. We may consider that a vector has been rotated about an axis in one sense (positive or negative) or that the frame in which the vector is described has been rotated about the same axis, but in the *opposite* sense.

<u>Example:</u> We wish to compute vector ${}^{0}\mathbf{p}_{2}$ obtained by rotating ${}^{0}\mathbf{p}_{1}$ about $\hat{\mathbf{Z}}_{0}$ by $30°$, where ${}^{0}\mathbf{p}_{1} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

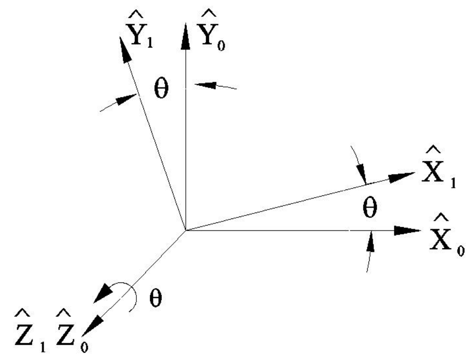


Solution: From our earlier rotation example we know the rotation matrix about the $\hat{\mathbf{Z}}$-axis is:

$$\mathbf{R}_{\hat{\mathbf{Z}}_0} = \begin{bmatrix} \cos(30) & -\sin(30) & 0 \\ \sin(30) & \cos(30) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Rightarrow \ {}^0\mathbf{p}_2 = \mathbf{R}_{\hat{\mathbf{Z}}_0}(30^o)\,{}^0\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix}.$$

Note that $\mathbf{p} = \mathbf{R}_0\ {}^0\mathbf{p}$ and ${}^0\mathbf{p}_2 = \mathbf{R}\ {}^0\mathbf{p}_1$ implement the same mathematics. Only



the meaning of $\mathbf{R}$ is different. i.e. $\mathbf{R}_{\hat{\mathbf{Z}}_0}(\theta)$ rotates frame $\{0\}$ *forwards* by θ^o, while $\mathbf{R}_{\hat{\mathbf{Z}}_1}(\theta)$ rotates frame $\{1\}$ *backwards* by θ^o. In this case ${}^1\mathbf{R}_0 = \mathbf{R}$, but ${}^0\mathbf{R}_1 = \mathbf{R}^{-1}$.

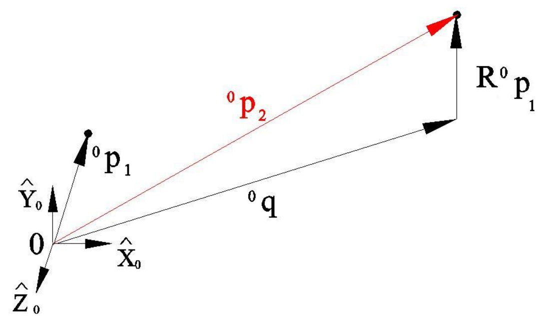
2.6.3 Transform Operator

The matrix operator $\mathbf{T}$ rotates and translates a vector ${}^0\mathbf{p}_1$ to give a new one, ${}^0\mathbf{p}_2$. Subscripts and superscripts are not required for $\mathbf{T}$ because only one coordinate frame is used:

$${}^0\mathbf{p}_2 = \mathbf{T}\,{}^0\mathbf{p}_1.$$

The transform which rotates the vector by $\mathbf{R}$ and translates it by $\mathbf{q}$ is the same as the one which describes a frame rotated by $\mathbf{R}$ and translated by $\mathbf{q}$, relative to the same frame.

Example: Given ${}^0\mathbf{p}_1 = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$, we wish to rotate it about $\hat{\mathbf{Z}}_0$ by 30^o and translate it 10 units along $\hat{\mathbf{X}}_0$ and 5 units along $\hat{\mathbf{Y}}_0$. Find ${}^0\mathbf{p}_2$.



Solution:

$$\mathbf{T} = \begin{bmatrix} \cos(30) & -\sin(30) & 0 & 10 \\ \sin(30) & \cos(30) & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Rightarrow {}^0\mathbf{p}_2 = \mathbf{T}\,{}^0\mathbf{p}_1 = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \end{bmatrix}.$$

Note that $\mathbf{T}$ is the same as in the example from Section 2.5.1. That solution is numerically equivalent to the one above, but the interpretations are vastly different.

2.6.4 Summary

We have introduced the 4×4 homogeneous transform matrix, containing orientation and position data, as a general tool to describe a *frame*. We have further discussed three interpretations of the homogeneous transform:

1. Description of a Frame: ${}^0\mathbf{T}_1$ describes Frame $\{1\}$ relative to Frame $\{0\}$. The columns of ${}^0\mathbf{R}_1$ are orthogonal unit vectors defining the directions of the principal axes of $\{1\}$ in $\{0\}$, and ${}^0\mathbf{p}_{1_{\mathrm{ORG}}}$ locates the origin of $\{1\}$ in $\{0\}$.

2. Transform Mapping: ${}^0\mathbf{T}_1$ maps ${}^1\mathbf{p} \rightarrow {}^0\mathbf{p}$.

3. Transform Operator: $\mathbf{T}$ operates on ${}^0\mathbf{p}_1$ to give ${}^0\mathbf{p}_2$.

The terms *frame* and *transform* are used to refer to both a position and orientation. *Frame* is usually used for a description, while *transform* is usually used when a mapping or operator is implied.

2.7 Transformation Arithmetic

General displacements, represented by 4×4 homogeneous transforms, are a mathematical group under multiplication, when considered as the set of all possible displacements in 3-D space. This means that every homogeneous transform is invertible, and any two may be multiplied, yielding another invertible transform. Additionally every set of three transforms are associative. Under addition they are a commutative group.

2.7.1 Compound Transformations

If Frame $\{3\}$ is known relative to $\{2\}$, and $\{2\}$ relative to $\{1\}$, and $\{1\}$ relative to $\{0\}$, we can transform ${}^3\mathbf{p}$ into ${}^2\mathbf{p}$ into ${}^1\mathbf{p}$ into ${}^0\mathbf{p}$.
i.e.

$$\begin{aligned} {}^2\mathbf{p} &= {}^2\mathbf{T}_3\, {}^3\mathbf{p}, \\ {}^1\mathbf{p} &= {}^1\mathbf{T}_2\, {}^2\mathbf{p}, \\ {}^0\mathbf{p} &= {}^0\mathbf{T}_1\, {}^1\mathbf{p}. \end{aligned}$$

Combining these three equations, we get:

$${}^0\mathbf{p} = {}^0\mathbf{T}_1\, {}^1\mathbf{T}_2\, {}^2\mathbf{T}_3\, {}^3\mathbf{p},$$

from which we get:

$${}^0\mathbf{T}_3 = {}^0\mathbf{T}_1\, {}^1\mathbf{T}_2\, {}^2\mathbf{T}_3.$$

Again, the subscript and superscript notation makes these manipulations easy to follow. In general:

$${}^0\mathbf{T}_n = {}^0\mathbf{T}_1\, {}^1\mathbf{T}_2 \ldots {}^{n-1}\mathbf{T}_n.$$

2.7.2 Special Transforms

Pure translation by $\mathbf{q}$:

$$\mathbf{T} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

Pure rotation about axis k by θ:

$$\mathbf{T} = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathbf{R}_k(\theta) & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

2.7.3 Computational Considerations

While homogeneous transforms are useful as a conceptual tool, typical transformation software used in industrial manipulation systems does not make direct use of them since the time multiplying by 1's and 0's is wasted. Moreover, the order in which transformations are applied can make a large difference in the amount of computation required to compute the same quantity. For example, if $\mathbf{T}$ is 4×4 and the $\mathbf{p}$ are 4×1 and we wish to multiply

$$^0\mathbf{p} \;=\; {}^0\mathbf{T}_1 \, {}^1\mathbf{T}_2 \, {}^2\mathbf{p}.$$

We have two possibilities. We can transform the vectors with the matrices one-at-a-time, or first multiply the matrices together:

1. $^0\mathbf{p} \;=\; {}^0\mathbf{T}_1({}^1\mathbf{T}_2 \, {}^2\mathbf{p})$.
2. $^0\mathbf{p} \;=\; ({}^0\mathbf{T}_1 \, {}^1\mathbf{T}_2) \, {}^2\mathbf{p}$.

The first option requires 32 multiplications and 24 additions, but the second requires 80 multiplications and 60 additions. Less than half of the second's operations are required by the first. Of course, in some cases, some of the $\mathbf{T}$'s may be constant, and there may be many $^2\mathbf{p}_i$'s which need to be transformed. Additionally, we may be looking for a *symbolic* solution. Here it is more efficient to compute $^0\mathbf{T}_2$ once, and use it for all required transformations.

2.7.4 Inverse of a Homogeneous Transform

It is often necessary to compute the inverse of a known transform, sometimes numerical and sometimes symbolic transforms. We want to find a computationally simple method that takes advantage of the transform's inherent structure. Recall, the transform is a homogeneous representation of

$$^0\mathbf{p} \;=\; {}^0\mathbf{R}_1 \, {}^1\mathbf{p} + {}^0\mathbf{b},$$

$$\Rightarrow \;\; {}^0\mathbf{p} \;=\; {}^0\mathbf{T}_1 \, {}^1\mathbf{p}.$$

Here, $^0\mathbf{b} \;=\; {}^0\mathbf{p}_{1_{\mathrm{ORG}}}$ for short. We wish to compute $^1\mathbf{p}$, so we require $^0\mathbf{T}_1^{-1}$. Go back to the first equation and isolate $^1\mathbf{p}$:

$$\begin{aligned}
{}^0\mathbf{p} - {}^0\mathbf{b} \;&=\; {}^0\mathbf{R}_1 \, {}^1\mathbf{p}, \\
{}^0\mathbf{R}_1^{-1}({}^0\mathbf{p} - {}^0\mathbf{b}) \;&=\; {}^0\mathbf{R}_1^{-1} \, {}^0\mathbf{R}_1 \, {}^1\mathbf{p} \;=\; {}^1\mathbf{p}, \\
\Rightarrow \;\; {}^1\mathbf{p} \;&=\; {}^0\mathbf{R}_1^T \, {}^0\mathbf{p} - {}^0\mathbf{R}_1^T \, {}^0\mathbf{b}, \\
\Rightarrow \;\; {}^0\mathbf{T}_1^{-1} \;&=\; \left[\begin{array}{c|c} {}^0\mathbf{R}_1^T & -\,{}^0\mathbf{R}_1^T \, {}^0\mathbf{b} \\ \hline 0\;0\;0 & 1 \end{array}\right], \\
&=\; \left[\begin{array}{c|c} {}^0\mathbf{R}_1^T & -\,{}^0\mathbf{R}_1^T \, {}^0\mathbf{p}_{1_{\mathrm{ORG}}} \\ \hline 0\;0\;0 & 1 \end{array}\right].
\end{aligned}$$

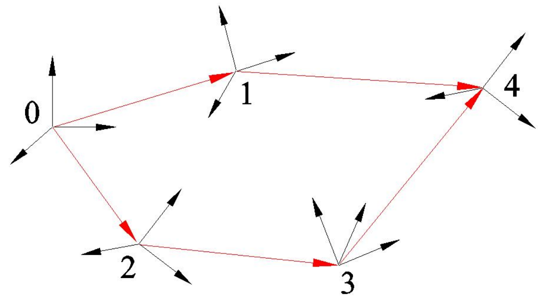


Figure 2.9: Frame transformations.

2.8 Transform Equations

Figure 2.9 shows that frame {4} can be expressed as the product of transformations in many ways. In particular as:

$$^0\mathbf{T}_4 = {}^0\mathbf{T}_1\,{}^1\mathbf{T}_4, \tag{2.3}$$

$$^0\mathbf{T}_4 = {}^0\mathbf{T}_2\,{}^2\mathbf{T}_3\,{}^3\mathbf{T}_4. \tag{2.4}$$

We can combine Equations (2.3) and (2.4) to yield a single transform equation

$$^0\mathbf{T}_1\,{}^0\mathbf{T}_4 = {}^0\mathbf{T}_2\,{}^2\mathbf{T}_3\,{}^3\mathbf{T}_4. \tag{2.5}$$

n transform equations can be used to solve for n unknown transforms. Suppose in (2.5) we had one unknown transform, $^1\mathbf{T}_4$. We easily determine its solution as:

$$^1\mathbf{T}_4 = {}^0\mathbf{T}_1{}^{-1}\,{}^0\mathbf{T}_2\,{}^2\mathbf{T}_3\,{}^3\mathbf{T}_4 = {}^1\mathbf{T}_0\,{}^0\mathbf{T}_2\,{}^2\mathbf{T}_3\,{}^3\mathbf{T}_4,$$

where we have just pre-multiplied both sides of (2.5) by $^0\mathbf{T}_1{}^{-1}$.

The graphical frame representation used the arrow pointing from one origin to another. The *inverse* transform simply involves changing the direction of the arrow.

2.9 Other Representations of Orientation

So far we have specified orientations with a 3×3 rotation matrix. Is it possible to describe an orientation with fewer than nine numbers? To answer this, we note that a rotation matrix is *proper orthogonal*, ie. it's determinant is always $+1$. Cayley's Theorem for proper orthogonal matrices states that for any proper orthogonal matrix $\mathbf{R}$, there exists a skew-symmetric matrix, $\mathbf{S}$, such that

$$\mathbf{R} = (\mathbf{I} - \mathbf{S})^{-1}\,(\mathbf{I} + \mathbf{S}),$$

where $\mathbf{I}$ is the 3×3 identity matrix. A *skew-symmetric* matrix has the following properties:

1. $\mathbf{S} = -\mathbf{S}^T$.

2. A 3×3 skew-symmetric matrix is specified by three parameters (s_x, s_y, s_z).

$$\mathbf{S} = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}.$$

Hence, any 3×3 rotation matrix can be determined by just three independent parameters. But we already alluded to this. There are six constraints on the nine elements of $\mathbf{R}$. Each column is a unit vector orthogonal to the other two. Hence

$$\mathbf{R} = [\hat{\mathbf{X}}\ \hat{\mathbf{Y}}\ \hat{\mathbf{Z}}],$$

with:

$$\begin{aligned} |\hat{\mathbf{X}}| &= 1, \\ |\hat{\mathbf{Y}}| &= 1, \\ |\hat{\mathbf{Z}}| &= 1, \\ \hat{\mathbf{X}} \cdot \hat{\mathbf{Y}} &= 0, \\ \hat{\mathbf{X}} \cdot \hat{\mathbf{Z}} &= 0, \\ \hat{\mathbf{Y}} \cdot \hat{\mathbf{Z}} &= 0. \end{aligned}$$

So, the next question is how do you conversely represent an orientation in 3-D space with three parameters?

It's important to point out that one problem representing orientations is that rotation matrices don't, in general, commute.

2.9.1 Fixed Angles, Euler Angles, Unit Quaternions

A human operator at a computer terminal whose job it is to type in desired robot hand orientations doesn't want to input nine-element proper orthogonal matrices. The following representations require three angles, or four number that obey one constraint.

X-Y-Z Fixed Angles

Here, the orientations of frame $\{1\}$ with respect to $\{0\}$ is described in the following way: Start with the two frames coincident, then

1. Rotate $\{1\}$ about $\hat{\mathbf{X}}_0$ by an angle γ.

2. Rotate $\{1\}$ about $\hat{\mathbf{Y}}_0$ by an angle β.

3. Rotate $\{1\}$ about $\hat{\mathbf{Z}}_0$ by an angle α.

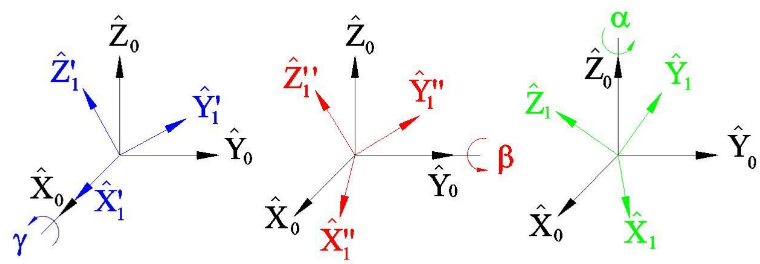


Figure 2.10: X-Y-Z fixed angles.

The word *fixed* refers to the fact that the rotations are specified about the fixed (ie. non-moving) reference frame. Some call this convention *roll, pitch, yaw* angles. However, this name is frequently given to other related but different conventions.

The equivalent rotation matrix is easily derived:

$$\begin{aligned}
{}^0\mathbf{R}_{1_{XYZ}}(\gamma,\beta,\alpha) &= {}^0\mathbf{R}_{1_Z}(\alpha)\,{}^0\mathbf{R}_{1_Y}(\beta)\,{}^0\mathbf{R}_{1_X}(\gamma), \\
&= \begin{bmatrix} \mathrm{c}\,\alpha & -\mathrm{s}\,\alpha & 0 \\ \mathrm{s}\,\alpha & \mathrm{c}\,\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\,\beta & 0 & \mathrm{s}\,\beta \\ 0 & 1 & 0 \\ -\mathrm{s}\,\beta & 0 & \mathrm{c}\,\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\,\gamma & -\mathrm{s}\,\gamma \\ 0 & \mathrm{s}\,\gamma & \mathrm{c}\,\gamma \end{bmatrix}, \\
&= \begin{bmatrix} \mathrm{c}\,\alpha\,\mathrm{c}\,\beta & \mathrm{c}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{s}\,\gamma - \mathrm{s}\,\alpha\,\mathrm{c}\,\gamma & \mathrm{c}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{c}\,\gamma + \mathrm{s}\,\alpha\,\mathrm{s}\,\gamma \\ \mathrm{s}\,\alpha\,\mathrm{c}\,\beta & \mathrm{s}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{s}\,\gamma + \mathrm{c}\,\alpha\,\mathrm{c}\,\gamma & \mathrm{s}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{c}\,\gamma - \mathrm{c}\,\alpha\,\mathrm{s}\,\gamma \\ -\mathrm{s}\,\beta & \mathrm{c}\,\beta\,\mathrm{s}\,\gamma & \mathrm{c}\,\beta\,\mathrm{c}\,\gamma \end{bmatrix}.
\end{aligned} \tag{2.6}$$

Where c = cos and s = sin in (2.6). The order of matrix concatenation results from the order of rotation: ${}^0\mathbf{R}_{1_X}$ is *operated* on by ${}^0\mathbf{R}_{1_Y}$, and the product of ${}^0\mathbf{R}_{1_Y}\,{}^0\mathbf{R}_{1_X}$ is *operated* on by ${}^0\mathbf{R}_{1_Z}$. Equation (2.6) is correct *only* for rotations performed in the specified order.

The inverse problem, extracting X-Y-Z fixed angles from a rotation matrix is also of interest. We have nine equations (with six dependencies) and three unknowns:

$${}^0\mathbf{R}_{1_{XYZ}}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

It can be shown that

$$\begin{aligned}
\beta &= \arctan 2\left(-r_{31},\ \pm\sqrt{r_{11}^2 + r_{21}^2}\right), \\
\alpha &= \arctan 2\left(\frac{r_{21}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\right), \\
\gamma &= \arctan 2\left(\frac{r_{32}}{\cos\beta}, \frac{r_{33}}{\cos\beta}\right).
\end{aligned}$$

Where $\arctan 2(y,x)$ is the two-argument arc-tangent function. Most programming language libraries have it pre-defined. It is defined by:

$$\begin{aligned}
\arctan_2(y,x) \equiv \tan^{-1}(\frac{y}{x}) &= \theta \text{ if } x > 0, \\
\tan^{-1}(\frac{y}{x}) + \pi \,\mathrm{sgn}(y) &= \theta + \pi \,\mathrm{sgn}(y) \text{ if } x < 0, \\
\tan^{-1}(\infty) \,\mathrm{sgn}(y) &= \frac{\pi}{2} \mathrm{sgn}(y) \text{ if } x = 0.
\end{aligned}$$

$$\text{where} \quad \mathrm{sgn}(y) = \begin{cases} 1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0 \end{cases}$$

The troubling thing is that there are two distinct solutions. We must live with this. Looking at α and γ, we can get into trouble if $\beta = \pm 90^o$, since $\cos(\pm 90^o) = 0$. The solution degenerates. In these cases only the sum, or difference, of α and γ may be computed.

Z-Y-X Euler Angles

Another possible description of orientation of $\{1\}$ with respect to $\{0\}$. Start with frames $\{1\}$ and $\{0\}$ coincident.

1. Rotate $\{1\}$ about $\hat{\mathbf{Z}}_1$ by an angle α.
2. Rotate $\{1\}$ about $\hat{\mathbf{Y}}_1$ by an angle β.
3. Rotate $\{1\}$ about $\hat{\mathbf{X}}_1$ by an angle γ.

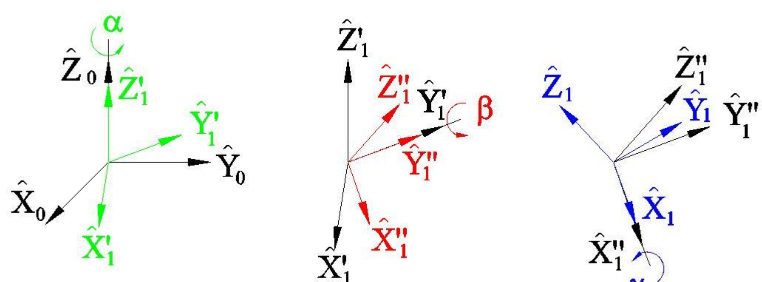


Figure 2.11: Z-Y-X Euler angles.

Here, each rotation is performed about an axis of the *moving* frame $\{1\}$, rather than the fixed frame $\{0\}$. These rotations are called *Euler Angles*. Each rotation takes place about an axis whose location depends upon the previous rotations. Rotation matrices parameterized by Z-Y-X Euler angles are indicated by

$${}^0\mathbf{R}_{Z'Y'X'}(\alpha,\beta,\gamma).$$

The *primes* added to the subscripts distinguishes *Euler angles* from *fixed angles*.

To determine the equivalent rotation matrix, we use the intermediate frames $\{1'\}$ and $\{1''\}$. Thinking of the rotations as descriptions of these frames with $\{1'''\} \equiv \{1\}$ as the final pose, we have immediately:

$$
\begin{aligned}
{}^{0}\mathbf{R}_{1} &= {}^{0}\mathbf{R}_{1'}\,{}^{1'}\mathbf{R}_{1''}\,{}^{1''}\mathbf{R}_{1} \\
&= \begin{bmatrix} \mathrm{c}\,\alpha & -\mathrm{s}\,\alpha & 0 \\ \mathrm{s}\,\alpha & \mathrm{c}\,\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\,\beta & 0 & \mathrm{s}\,\beta \\ 0 & 1 & 0 \\ -\mathrm{s}\,\beta & 0 & \mathrm{c}\,\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\,\gamma & -\mathrm{s}\,\gamma \\ 0 & \mathrm{s}\,\gamma & \mathrm{c}\,\gamma \end{bmatrix}, \\
&= \begin{bmatrix} \mathrm{c}\,\alpha\,\mathrm{c}\,\beta & \mathrm{c}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{s}\,\gamma - \mathrm{s}\,\alpha\,\mathrm{c}\,\gamma & \mathrm{c}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{c}\,\gamma + \mathrm{s}\,\alpha\,\mathrm{s}\,\gamma \\ \mathrm{s}\,\alpha\,\mathrm{c}\,\beta & \mathrm{s}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{s}\,\gamma + \mathrm{c}\,\alpha\,\mathrm{c}\,\gamma & \mathrm{s}\,\alpha\,\mathrm{s}\,\beta\,\mathrm{c}\,\gamma - \mathrm{c}\,\alpha\,\mathrm{s}\,\gamma \\ -\mathrm{s}\,\beta & \mathrm{c}\,\beta\,\mathrm{s}\,\gamma & \mathrm{c}\,\beta\,\mathrm{c}\,\gamma \end{bmatrix}.
\end{aligned}
\tag{2.7}
$$

We see, by comparing Equations (2.6) and (2.7) that the Z-Y-X Euler angle matrix is identical to the X-Y-Z fixed angle matrix!. This non-intuitive result holds in general: Three rotations about fixed orthogonal axes yields the same final orientation as the same three rotations taken in the opposite order about the axis of the moving frame when both start out coincident.

There are 11 more distinct fixed angle and 11 more Euler angle representations obtained by performing three rotations about coordinate axes in a specific order. Of the 24 sets, only 12 are distinct because of the *duality* between the fixed and Euler angle sets. Thus there are 12 unique parameterizations for orientation using three successive principal axis rotations. Usually, there is no great reason to use one over the other, indeed various robot manufacturers and investigators adopt different ones, so it is useful to list them all:

Euler angle sets:

$$
\begin{aligned}
\mathbf{R}_{X'Y'Z'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\gamma \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix}, \\
\mathbf{R}_{X'Z'Y'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma \end{bmatrix}, \\
\mathbf{R}_{Y'X'Z'}(\alpha,\beta,\gamma) &= \begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\alpha & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix}, \\
\mathbf{R}_{Y'Z'X'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta s\gamma + s\alpha s\gamma & c\alpha s\beta c\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
\mathbf{R}_{Z'X'Y'}(\alpha,\beta,\gamma) &= \begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix}, \\
\mathbf{R}_{Z'Y'X'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}, \\
\mathbf{R}_{X'Y'X'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
\mathbf{R}_{X'Z'X'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
\mathbf{R}_{Y'X'Y'}(\alpha,\beta,\gamma) &= \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix}, \\
\mathbf{R}_{Y'Z'Y'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -c\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}, \\
\mathbf{R}_{Z'X'Z'}(\alpha,\beta,\gamma) &= \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\gamma \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix}, \\
\mathbf{R}_{Z'Y'Z'}(\alpha,\beta,\gamma) &= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\gamma \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.
\end{aligned}
$$

Fixed angle sets:

$$\mathbf{R}_{XYZ}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix},$$

$$\mathbf{R}_{XZY}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta s\gamma + s\alpha s\gamma & c\alpha s\beta c\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$\mathbf{R}_{YXZ}(\alpha,\beta,\gamma) = \begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix},$$

$$\mathbf{R}_{YZX}(\alpha,\beta,\gamma) = \begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma \end{bmatrix},$$

$$\mathbf{R}_{ZXY}(\alpha,\beta,\gamma) = \begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\alpha & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix},$$

$$\mathbf{R}_{ZYX}(\alpha,\beta,\gamma) = \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\gamma \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix},$$

$$\mathbf{R}_{XYX}(\alpha,\beta,\gamma) = \begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$\mathbf{R}_{XZX}(\alpha,\beta,\gamma) = \begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix}$$

$$\mathbf{R}_{YXY}(\alpha,\beta,\gamma) = \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$\mathbf{R}_{YZY}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -c\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$\mathbf{R}_{ZXZ}(\alpha,\beta,\gamma) = \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\gamma \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix},$$

$$\mathbf{R}_{ZYZ}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\gamma \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.$$

Equivalent Angle-Axis

A well known result from classical kinematics (Euler's theorem on rotation) is that any change in orientation of a rigid body can be represented by a rotation through a certain angle about a certain axis. It can (and will) be shown that the associated rotation matrix has only one real eigenvalue. The associated eigenvector gives the direction of the rotation axis. (Note: the axis always passes through the origin).

Any orientation may be expressed with appropriate axis and angle selection. Consider the following description of {1}: Start with {1} and {0} coincident. Then, rotate {1} about the vector ${}^0\mathbf{k}$ by an angle θ according to the right-hand rule. It can be shown that the equivalent rotation matrix is:

$$\mathbf{R}_k(\theta) = \begin{bmatrix} k_x k_x \nu\theta + \mathrm{c}\,\theta & k_x k_y \nu\theta - k_z\,\mathrm{s}\,\theta & k_x k_z \nu\theta + k_y\,\mathrm{s}\,\theta \\ k_x k_y \nu\theta + k_z\,\mathrm{s}\,\theta & k_y k_y \nu\theta + \mathrm{c}\,\theta & k_y k_z \nu\theta - k_x\,\mathrm{s}\,\theta \\ k_x k_z \nu\theta - k_y\,\mathrm{s}\,\theta & k_y k_z \nu\theta + k_x\,\mathrm{s}\,\theta & k_z k_z \nu\theta + \mathrm{c}\,\theta \end{bmatrix}.$$

Where $\nu\theta = 1 - \cos(\theta)$ and ${}^0\mathbf{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$ being a *unit* vector. The sign of θ is determined by the right-hand rule with the thumb pointing in the positive direction of ${}^0\mathbf{k}$.

So, given any rotation axis and angle, we can construct an equivalent rotation matrix. When the axis is one of the principal axes of {0} we get the familiar

$$\begin{aligned} \mathbf{R}_X(\theta) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\,\theta & -\mathrm{s}\,\theta \\ 0 & \mathrm{s}\,\theta & \mathrm{c}\,\theta \end{bmatrix}, \\ \mathbf{R}_Y(\theta) &= \begin{bmatrix} \mathrm{c}\,\theta & 0 & \mathrm{s}\,\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\,\theta & 0 & \mathrm{c}\,\theta \end{bmatrix}, \\ \mathbf{R}_Z(\theta) &= \begin{bmatrix} \mathrm{c}\,\theta & -\mathrm{s}\,\theta & 0 \\ \mathrm{s}\,\theta & \mathrm{c}\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The inverse problem of determining $\mathbf{k}$ and θ from a given rotation matrix is obtained with

$$\begin{aligned} {}^0\mathbf{R}_{1k} &= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \\ \Rightarrow \quad \theta &= \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right), \\ \text{and } {}^0\mathbf{k} &= \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}. \end{aligned}$$

This gives θ as $0 < \theta < 180^o$. For every ${}^0\mathbf{k}$ and θ there is $-{}^0\mathbf{k}$ and $-\theta$ which yields the same orientation with respect to $\{0\}$ described with identical rotation matrices. We are always faced with having to choose one. More seriously, for small angles $\mathbf{k}$ becomes ill-defined. When $\theta = 0$, $\mathbf{k}$ represents a line at infinity. We are also in computational peril when $\theta = 180^o$.

Euler-Rodriguez Parameters

Another way to represent an orientation is with four numbers. In terms of the equivalent axis $\mathbf{k}$ and angle θ, the Euler-Rodriguez parameters can be defined as:

$$\begin{aligned} c_1 &= k_x \sin\left(\frac{\theta}{2}\right), \\ c_2 &= k_y \sin\left(\frac{\theta}{2}\right), \\ c_3 &= k_z \sin\left(\frac{\theta}{2}\right), \\ c_4 &= \cos\left(\frac{\theta}{2}\right). \end{aligned}$$

Recall that $\mathbf{k}$ is a unit vector, hence the four Euler-Rodriguez parameters are not independent, but are related by:

$$c_1{}^2 + c_2{}^2 + c_3{}^2 + c_4{}^2 = 1.$$

Because of the parametrization, we also observe that a non-zero condition must be satisfied

$$c_1 : c_2 : c_3 : c_4 \neq 0 : 0 : 0 : 0.$$

So, we can visualize an orientation as a point on a unit hypersphere in a 3-D projective (4-D homogeneous coordinate) space. Viewing the Euler-Rodriguez parameters as a 4×1 vector, they are known as a *unit quaternion.* The rotation matrix corresponding to a set of Euler-Rodriguez parameters is:

$$\mathbf{R}_{ER} = \begin{bmatrix} c_1^2-c_2^2-c_3^2-c_4^2 & 2(c_1c_2 - c_3c_4) & 2(c_1c_3 + c_2c_4) \\ 2(c_1c_2 + c_3c_4) & -c_1^2+c_2^2-c_3^2+c_4^2 & 2(c_2c_3 - c_1c_4) \\ 2(c_1c_3 - c_2c_4) & 2(c_2c_3 + c_1c_4) & -c_1^2-c_2^2+c_3^2+c_4^2 \end{bmatrix}. \tag{2.8}$$

The diagonal elements can be simplified to

$$\left.\begin{aligned} \mathbf{R}_{ER_{11}} &= 1 - 2c_2{}^2 - 2c_3{}^2, \\ \mathbf{R}_{ER_{22}} &= 1 - 2c_1{}^2 - 2c_3{}^2, \\ \mathbf{R}_{ER_{33}} &= 1 - 2c_1{}^2 - 2c_2{}^2. \end{aligned}\right\} \tag{2.9}$$

The inverse problem is, given a rotation matrix the equivalent Euler-Rodriguez parameters are

$$\begin{aligned} c_1 &= \frac{r_{32} - r_{23}}{4c_4}, \\ c_2 &= \frac{r_{13} - r_{31}}{4c_4}, \\ c_3 &= \frac{r_{21} - r_{12}}{4c_4}, \\ c_4 &= \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}. \end{aligned}$$

If $c_4 = 0$, then $\theta = \pm\, 180^o$, and we can only determine the ratios of k_x, k_y, k_z. Note that the Euler-Rodriguez parameters are the *invariants* of a rotation. That is, when expressed as a rotation matrix, we can determine what remains invariant under the rotation by determining the eigenvectors of $\mathbf{R}$. It turns out that there is only one real eigenvalue for any rotation matrix. The corresponding single real eigenvector *is* the direction of the axes of rotation, $\mathbf{k}$.

$$\text{The real eigenvalue is } \underline{\textit{always}} + 1.$$
$$\text{The real eigenvector is } \underline{\textit{always}} \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}.$$

A Note on Line Bound and Free Vectors:

When represented by homogeneous coordinates, a line bound vector, like a force, has a non-zero homogenizing coordinate. Free vectors, like the linear velocity of a rigid body, are represented by setting the homogenizing coordinate to zero. This states that only the direction (and magnitude) of the vector are important.

The Euler-Rodriguez parameters can be obtained directly from Cayley's Theorem for proper orthonormal matrices:

$$\mathbf{R} = (\mathbf{I} - \mathbf{S})^{-1}. \tag{2.10}$$

Expanding, we get matrix $\mathbf{R}$ in terms of $S_1,\ S_2,\ S_3$. These are called the *Rodriguez parameters.*

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \tan\left(\frac{\theta}{2}\right) \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix},$$

with $\mathbf{k}$ as the rotation axis. We can *homogenize* $\mathbf{R}$ by setting

$$S_1 = \frac{c_1}{c_4},\ \ S_2 = \frac{c_2}{c_4},\ \ S_3 = \frac{c_3}{c_4},$$

and we obtain the rotation matrix in terms of the Euler-Rodriguez parameters.

Chapter 3

Kinematics

Kinematics is the study of motion without regard to the forces causing the motion. This involves working with position, velocity, acceleration, and even higher order derivatives of the position variables with respect to time, or any other variable(s). These are the geometric, and time-based, properties of motion. The relations between these motions and the forces and torques causing them is *dynamics*. More on dynamics later. For now, we will only consider static situations.

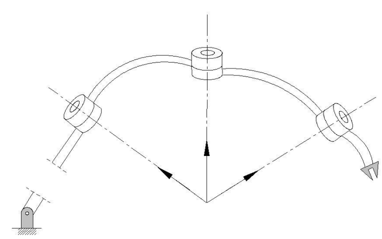


Figure 3.1: Spherical joint constructed from 3 R-pairs.

3.1 Link Description

A robot may be thought of as a *kinematic chain*, i.e., a set of rigid links connected by joints which allow relative motion between the links. Joints are also called *kinematic pairs*. The term *lower pair* is used when the relative motion involves sliding surface contact. *Higher pairs* involve point, or line contact (like a cam and follower, or mating spur gears).

We will only consider 1 DOF joints, namely prismatic (P) pairs and revolute (R) pairs. All of the remaining lower-pairs can be constructed as combinations

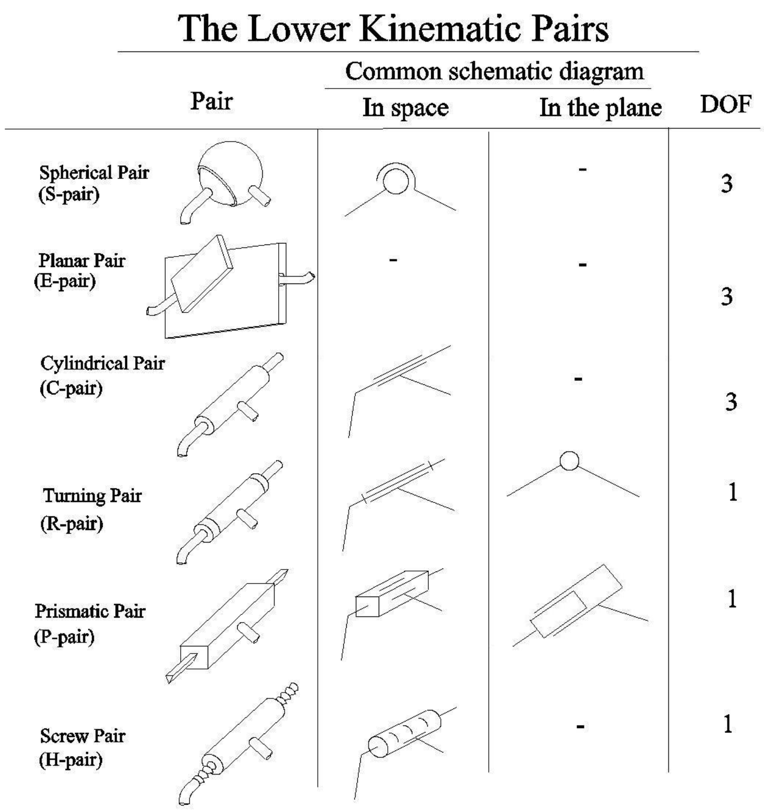


Figure 3.2: Lower kinematic pairs.

of P- and R-pairs. For example, a spherical joint is constructed with 3 R-pairs with intersecting axes, as shown in Figure 3.1.

Any lower pair joint with n DOF can be modeled as n joints with 1 DOF connecting $n-1$ links of zero length!

3.1.1 Convention

The links are numbered starting from the fixed base, called link 0. The first moving body is link 1, and so on, out to the free end of the "arm", which is link n. Attributes of a single link include material, its strength and stiffness, the location of any type of bearings in the joint, the shape, mass and inertia, etc. But for kinematics, a link is a simple rigid body defining the relationship between neighboring axes. Joint axes are defined by lines.

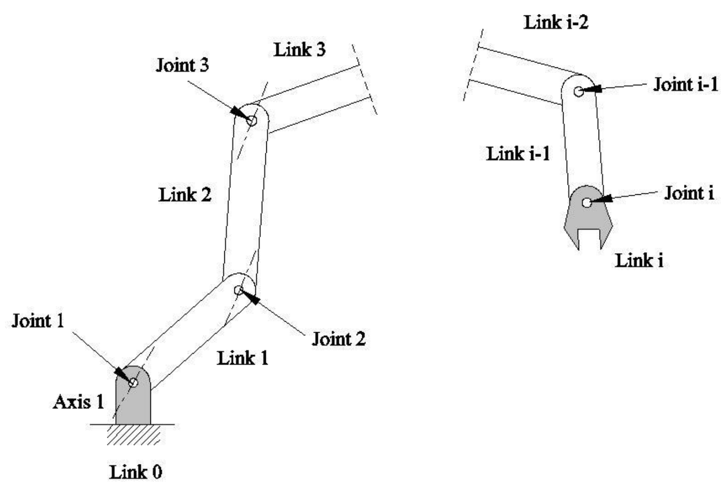


Figure 3.3: Convention for numbering links (joint axes are normal to the page).

3.2 Modified Denavit-Hartenberg (MDH) Link Parameters

One of the most fundamental problems in describing a working environment in which one or more robots operate, together with supporting equipment and devices, is how to explain the relative positions of the various components. This is crucial since many robot operations are pose driven. For example, a robot has to pick up a part from a certain location at a certain time, put it down in a clamping device, then, after collecting a tool, perform a machining operation on the part. From a kinematics point of view, the description of this problem reduces to determining the relationships between the reference frames attached to each part.

The Denavit-Hartenberg (DH) parametrisation involves the allocation of coordinate frames to each robot link using a set of rules to locate the origin of the frame and the orientation of the axes. The position of consecutive links is then defined by a homogeneous transformation matrix, which transforms coordinates in the frame attached to link n into those of the frame attached to link $n-1$. Concatenating all n transformations maps the tool point from the tool frame to the base frame, i.e., the forward kinematics. This transformation is obtained from simpler ones representing translations along and rotations about the principal axes. There are many variations on the DH method, we will examine two.

IMPORTANT NOTE: In the textbook, *Introduction to Robotics* by John J. Craig used in this course, the DH parametrisation used is modified from the original one introduced by Denavit and Hartenberg in 1955. Hence, in these notes we will call the ones used by Craig the Modified Denavit-Hartenberg (MDH) parameters to differentiate them from the original DH parameters. We will stop making the distinction once we get to Subsection 3.3.1, and simply refer to the MDH parameters as DH parameters, because the goal here is simply to show that different conventions exist. Reference to the book by Craig will be much simpler if we also call them DH parameters. But, for now we will keep the distinction between MDH and DH parameters.

3.2.1 Link Length a_{i-1}

The length of link $i-1$, indicated by a_{i-1}, is the perpendicular distance between axis $i-1$ and axis i. If the two axes intersect, $a_{i-1} = 0$.

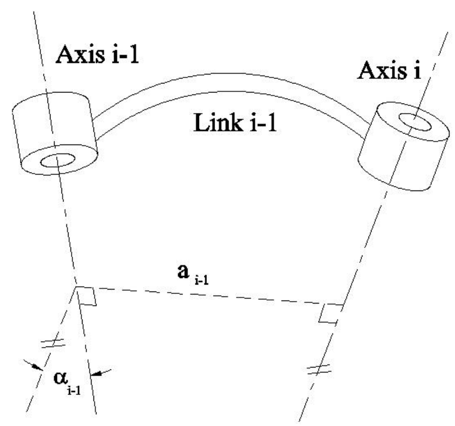


Figure 3.4: Link length a_{i-1}.

3.2.2 Link Twist α_{i-1}

This is the angle between axis $i-1$ and axis i. Imagine that axis $i-1$ and i, if they don't intersect, are in parallel planes. a_{i-1} is the normal distance between the two planes. The angle between the two axes can be measured by projecting the axes into one of those parallel planes. This angle is measured from axis $i-1$ to axis i in the right hand sense about a_{i-1} (when a_{i-1} is taken as the directed line segment from axis $i-1$ to axis i). If the axes intersect, twist is measured in the plane containing the axes, but the sense is arbitrary.

Length a_{i-1} and twist α_{i-1} define the relationship between any two lines in space, and hence, between any two joint axes.

3.2.3 Link Offset and Joint Angle: d_i and θ_i

An additional two quantities are sufficient to completely specify how two links are connected: the link offset d_i and the joint angle θ_i. Neighboring links have an axis in common, and these two parameters concern this axis.

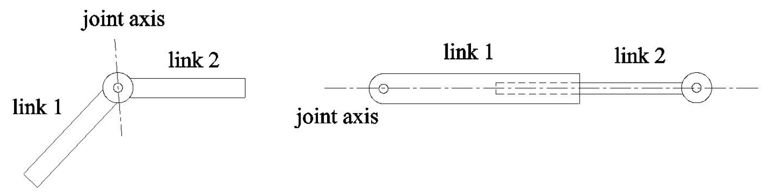


Figure 3.5: Joint axes.

Link Offset d_i

This is the distance, measured along the common axis from link $i-1$ to link i. Each link has a link-fixed reference frame associated with it. The pose of this frame is the pose of the link. Depending on how the frames are attached, there may be an offset between the origins of two neighboring link frames in a specific coordinate direction (discussed in greater detail later). The link offset

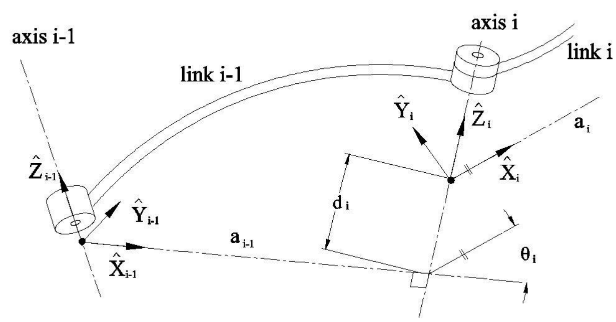


Figure 3.6: Link offset d_i and joint angle θ_i.

distance d_i is the signed distance measured along axis i from the point where a_{i-1} intersects axis i to the point where a_i intersects axis i, see Figure 3.6.

Joint Angle θ_i

This parameter describes the amount of rotation about the common axis between link $i-1$ and link i, which is the angle between a_{i-1} and a_i.

Summary

The following summarises the parameters:

d_i is fixed, i.e. constant, for an R-pair but variable for a P-pair;

θ_i is constant for a P-pair but variable for an R-pair;

a_{i-1} is always constant;

α_{i-1} is always constant.

First and Last Links in the Chain:

$$a_0 = a_n = 0,$$
$$\alpha_0 = \alpha_n = 0.$$

If joint 1 is an R-pair: $d_1 = 0$ and the zero for θ_1 is arbitrary.

If joint 1 is a P-pair: $\theta_1 = 0$ and the zero position for d_1 is arbitrary.

Analogous statements apply for joint n. The idea is to assign zero to arbitrary quantities to simplify computations later.

3.2.4 Denavit-Hartenberg (DH) Parameters

The kinematics of any (serial) robot can be described by specifying the values of the four numbers $(a_i,\ \alpha_i,\ d_i,\ \theta_i)$ for each link. These are called *DH parameters*. The MDH parameters are $(a_{i-1},\ \alpha_{i-1},\ d_i,\ \theta_i)$. For a robot with six joints, 18 numbers completely describe the constant portions of its kinematics.

To visualize the four DH parameters, consider two arbitrary neighboring links $i-1$ and i shown in Figure 3.7.

θ_i = joint angle: the angle from $\hat{\mathbf{X}}_{i-1}$ to $\hat{\mathbf{X}}_i$ measured about $\hat{\mathbf{Z}}_{i-1}$.
α_i = link twist: the angle from $\hat{\mathbf{Z}}_{i-1}$ to $\hat{\mathbf{Z}}_i$ measured about $\hat{\mathbf{X}}_i$.
a_i = link length: the distance from $\hat{\mathbf{Z}}_{i-1}$ to $\hat{\mathbf{Z}}_i$ measured along $\hat{\mathbf{X}}_i$.
d_i = link offset: the distance from $\hat{\mathbf{X}}_{i-1}$ to $\hat{\mathbf{X}}_i$ measured along $\hat{\mathbf{Z}}_{i-1}$.

Convention for Affixing DH Frames to Links

The procedure for assigning the origin and axes for link i are:

1. Identify all joint axes. Consider neighbours $i-1$, i and $i+1$.
2. Identify the common perpendicular between the two axes i and $i+1$, or their point of intersection. At the point of intersection, or where the common perpendicular meets the $(i+1)^{\text{st}}$ joint axis, assign the link frame origin, O_i.

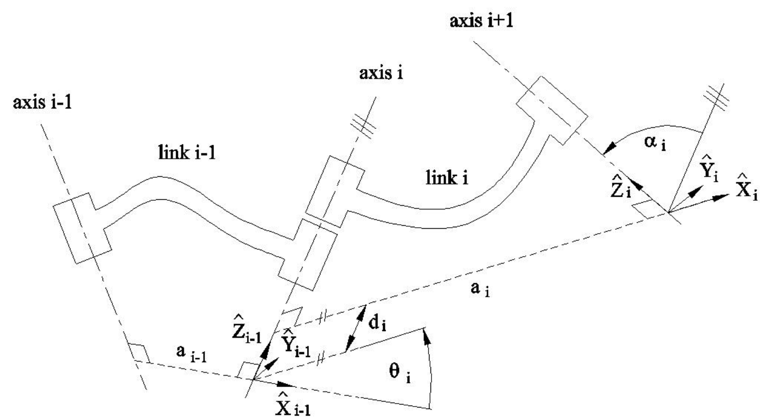


Figure 3.7: DH Parameters.

3. For $\{0\}$ and $\{1\}$, ensure the axes are aligned when $\theta_1 = 0$. An n-joint manipulator has $n-1$ frames. Assign the tool frame $\{T\}$ to align with $\{n-1\}$.

4. Assign the z_i axis to point along joint axis $i+1$.

5. Assign the x_i axis to point along the common normal between joint axes i and $i+1$. If the axes are parallel, any convenient normal can be selected. If the axes intersect assign x_i to be perpendicular to the plane containing z_{i-1} and z_i.

6. Assign the y_i axis to complete a right-handed coordinate system.

NOTE: the frame assignments are not unique. For instance, when the z_i axis is aligned with the $(i+1)^{\text{st}}$ joint axis there is a choice of direction for z_i.

3.2.5 Modified Denavit-Hartenberg (MDH) Parameters

The four MDH parameters are the following.

$$\begin{aligned}
\theta_i &= \text{joint angle: the angle from } \hat{\mathbf{X}}_{i-1} \text{ to } \hat{\mathbf{X}}_i \text{ measured about } \hat{\mathbf{Z}}_i. \\
\alpha_i &= \text{link twist: the angle from } \hat{\mathbf{Z}}_i \text{ to } \hat{\mathbf{Z}}_{i+1} \text{ measured about } \hat{\mathbf{X}}_i. \\
a_i &= \text{link length: the distance from } \hat{\mathbf{Z}}_i \text{ to } \hat{\mathbf{Z}}_{i+1} \text{ measured along } \hat{\mathbf{Z}}_i. \\
d_i &= \text{link offset: the distance from } \hat{\mathbf{X}}_{i-1} \text{ to } \hat{\mathbf{X}}_i \text{ measured along } \hat{\mathbf{Z}}_i.
\end{aligned}$$

The procedure for assigning the origin and axes for link i are:

1. Identify all joint axes. Consider neighbors i and $i+1$.

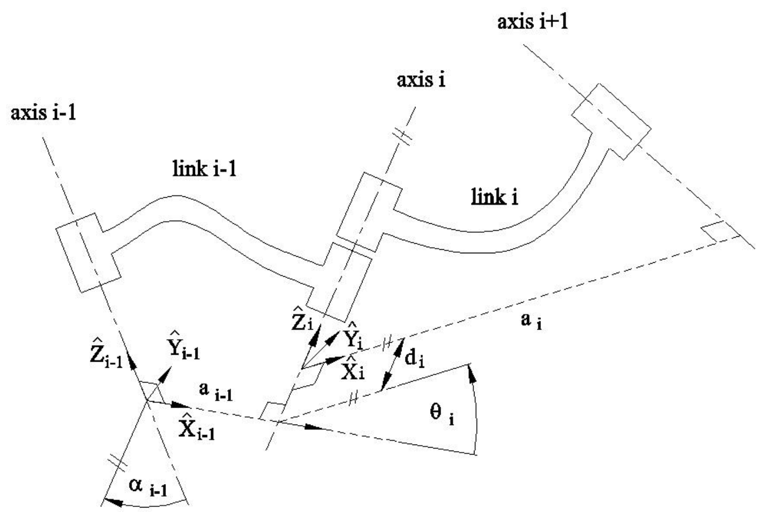


Figure 3.8: MDH Parameters.

2. Identify the common perpendicular between the two axes, or their point of intersection. At the point of intersection, or where the common perpendicular meets the i^{th} joint axis, assign the link frame origin, O_i. *Important:* assign $\{0\}$ to match $\{1\}$ when $\theta_1 = 0$. Assign the axes of $\{T\}$ to align with those of $\{n\}$.

3. Assign the $\hat{\mathbf{Z}}_i$ axis to point along joint axis i.

4. Assign the $\hat{\mathbf{X}}_i$ axis to point along the common normal between joint axes i and $i+1$. If axes are parallel, any convenient normal can be selected. If they intersect, assign $\hat{\mathbf{X}}_i$ to be perpendicular to the plane containing $\hat{\mathbf{Z}}_i$ and $\hat{\mathbf{Z}}_{i+1}$.

5. Assign the $\hat{\mathbf{Y}}_i$ axis to complete a right-handed coordinate system.

MDH Example 1

Assign the link frames and MDH parameters to the planar arm (RRR) in Figure 3.9:

Frames:

{0} Fixed to the base and aligns with $\{1\}$ when $\theta_1 = 0$. $\hat{\mathbf{Z}}_0$ is aligned with joint axes 1, pointing out of the page.

{1} Fixed to link $\{1\}$, $\hat{\mathbf{X}}_1$ in direction of common perpendicular from joint axis 1 to joint axis 2. $\hat{\mathbf{X}}_1$ points out of the page.

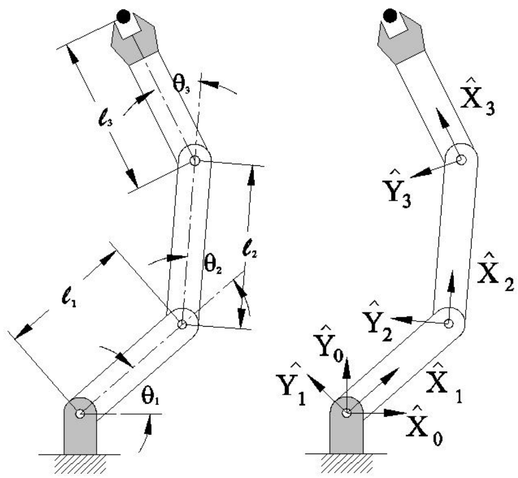


Figure 3.9: MDH Example 1.

{2} Since the arm is planar, all $\hat{\mathbf{Z}}_1$ are parallel. Thus $\hat{\mathbf{X}}_2$ points towards $\hat{\mathbf{Z}}_3$

{3} All joints are R-pairs, when at 0 degrees all $\hat{\mathbf{X}}$ axes must align. Assign $\hat{\mathbf{X}}_3$ along length of last link.

Table 3.1: MDH Parameters for Example 1

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	l_1	0	θ_2
3	0	l_2	0	θ_3

All $\hat{\mathbf{Z}}$ axes are parallel and the arm is planar, therefore $\alpha_i = d_i = 0$, no link twist, no offset. Note l_3 is not a parameter. Final offsets of the end effector reference points are discussed later.

MDH Example 2

Assign link frames and MDH parameters to the spatial RRP arm in Figure 3.10.

Frames:

{0} As assigned in MDH Example 1.

{1} As assigned in MDH Example 1.

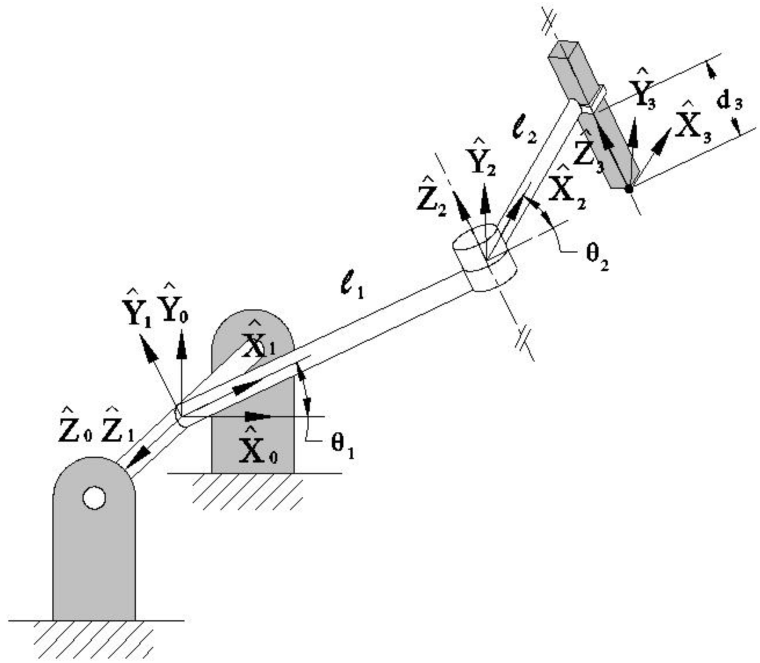


Figure 3.10: MDH Example 2.

{2} Joint axis 2 is not parallel to 1, but perpendicular. $\hat{\mathbf{Z}}_2$ may point *up* or *down*. Let's choose *up*. $\hat{\mathbf{X}}_2$ points in direction of common perpendicular with $\hat{\mathbf{Z}}_3$.

{3} Joint 3 is a P-pair whose translation direction is parallel to $\hat{\mathbf{Z}}_2$. For simplicity, choose $\hat{\mathbf{X}}_3$ in the same direction as $\hat{\mathbf{X}}_2$.

Table 3.2: MDH Parameters for Example 2

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90^o	l_1	0	θ_2
3	0	l_2	d_3	0

Here there are link offsets (d) and twists (α). This is due to the nature of the joints and their orientations. Because we selected $\hat{\mathbf{X}}_3$ parallel to $\hat{\mathbf{X}}_2$, then $\theta_3 = 0$.

Comparison of DH and MDH Forms

1. **DH Form:** The origin of frame i, O_i, is located on the axis of joint $i+1$.
   MDH Form: The origin of frame i, O_i, is located on the axis of joint i.

2. a_i is always the length of link i, but it is the distance from $\hat{\mathbf{Z}}_{i-1}$ to $\hat{\mathbf{Z}}_i$ in the DH form, and the distance from $\hat{\mathbf{Z}}_i$ to $\hat{\mathbf{Z}}_{i+1}$ in the MDH form. Similar for α_i.

Example 3 - Planar RPR

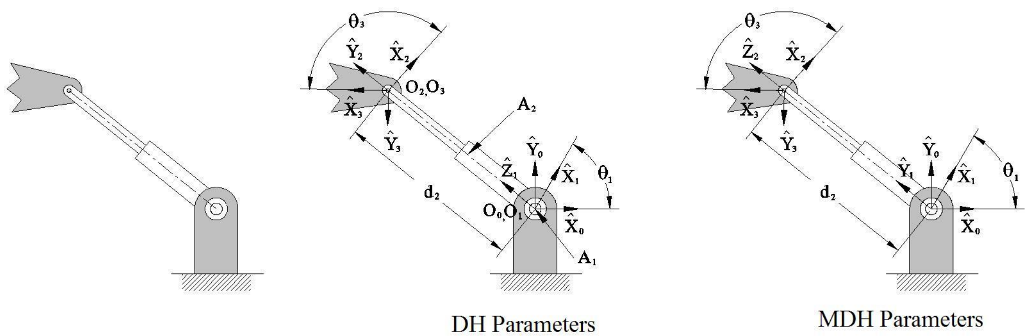


Figure 3.11: DH and MDH Example 3.

Table 3.3: DH Parameters for Example 3

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	0	θ_1
2	$\pi/2$	0	d_2	0
3	0	0	0	θ_3

Table 3.4: MDH Parameters for Example 3

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	d_2	0
3	$\pi/2$	0	0	θ_3

Example 4 - Planar RRR:

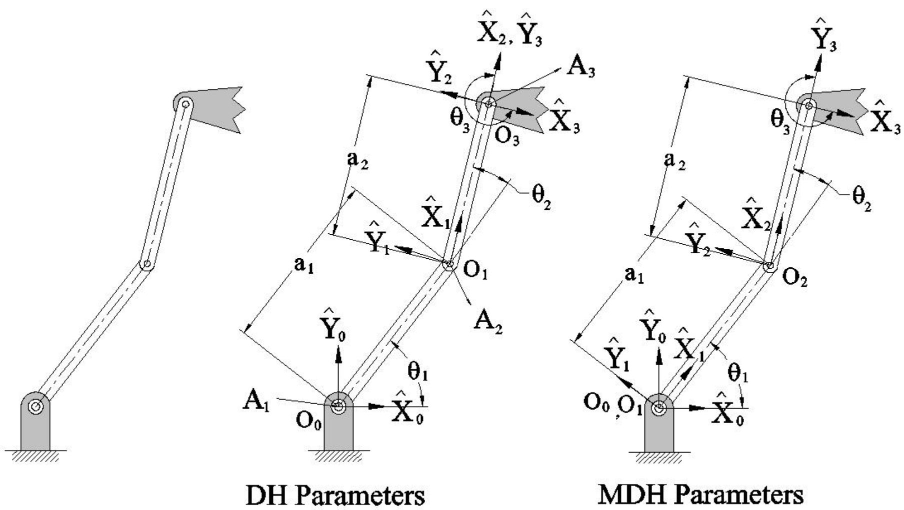


Figure 3.12: DH and MDH Example 4.

Table 3.5: DH Parameters for Example 4

i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2
3	0	0	0	θ_3

Table 3.6: MDH Parameters for Example 4

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	a_1	0	θ_2
3	0	a_2	0	θ_3

3.3 MDH Parameters in Transformations

The goal is to determine the individual transforms which describe $\{i\}$ with respect to $\{i-1\}$. In general, ${}^{i-1}\mathbf{T}_i$ is a function of the four MDH parameters $(a_{i-1},\ \alpha_{i-1},\ d_i,\ \theta_i)$. However, if each joint permits only one DOF, then ${}^{i-1}\mathbf{T}_i$ depends on the single joint variable (either θ_i or d_i), the others being fixed design parameters.

The forward kinematics problem involves determining the pose of the end-effector given a set of values for the joint variables. Having defined n frames for each of the robot's n links, we have broken the forward kinematics problem into n sub-problems. To solve the sub-problems, i.e. find each ${}^{i-1}\mathbf{T}_i$, we will further sub-divide each problem into four sub-problems - one for each MDH parameter (only one being variable). Each sub-transform is a function of one MDH parameter and is simple enough that it can be determined by inspection! We begin by introducing three intermediate frames $\{P\}$, $\{Q\}$, and $\{R\}$.

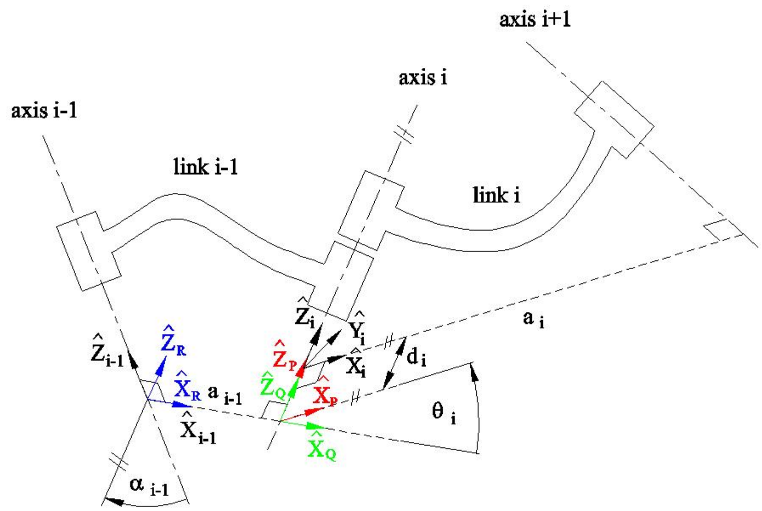


Figure 3.13: Derivation of MDH form transformations.

Frame $\{R\}$ differs from $\{i-1\}$ by only rotation α_{i-1}. Frame $\{Q\}$ differs from $\{R\}$ by a transformation a_{i-1}. Frame $\{P\}$ differs from $\{Q\}$ by a rotation θ_i. Frame $\{i\}$ differs from $\{P\}$ by a translation d_i. To write the transformation which transforms vectors in $\{i\}$ to $\{i-1\}$, we can write

$${}^{i-1}\mathbf{p} \;=\; {}^{i-1}\mathbf{T}_R\,{}^{R}\mathbf{T}_Q\,{}^{Q}\mathbf{T}_P\,{}^{P}\mathbf{T}_i\,{}^{i}\mathbf{p},$$

or

$${}^{i-1}\mathbf{p} \;=\; {}^{i-1}\mathbf{T}_i\,{}^{i}\mathbf{p},$$

where:

$$\begin{aligned} {}^{i-1}\mathbf{T}_i &= {}^{i-1}\mathbf{T}_R\ {}^{R}\mathbf{T}_Q\ {}^{Q}\mathbf{T}_P\ {}^{P}\mathbf{T}_i, \\ &= \mathbf{R}_{\hat{\mathbf{X}}_{i-1}}(\alpha_{i-1})\ \boldsymbol{\tau}_{\hat{\mathbf{X}}_{i-1}}(a_{i-1})\ \mathbf{R}_{\hat{\mathbf{Z}}_i}(\theta_i)\ \boldsymbol{\tau}_{\hat{\mathbf{Z}}_i}(d_i), \\ &= S_{\hat{\mathbf{X}}_{i-1}}(a_{i-1},\alpha_{i-1})\ S_{\hat{\mathbf{Z}}_i}(d_i,\theta_i). \end{aligned}$$

Where $\mathbf{S}_{\hat{\mathbf{Q}}}(r,\theta)$ is *screw* notation, meaning a translation along axis $\hat{\mathbf{Q}}$ by distance r, and a rotation about the same axis by angle θ. Transforming coordinates from i to $i-1$ may be thought of as

${}^{i-1}\mathbf{T}_R$: Rotate about $\hat{\mathbf{X}}_R(x_{i-1})$ by α_{i-1}.

${}^{R}\mathbf{T}_Q$: Translate along $\hat{\mathbf{X}}_Q(x_{i-1})$ by θ_i.

${}^{P}\mathbf{T}_i$: Translate along $\hat{\mathbf{Z}}_i$ a distance d_i.

We can write out the transformations by inspection, easily.

$${}^{i-1}\mathbf{T}_i =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\,\alpha_{i-1} & -\mathrm{s}\,\alpha_{i-1} & 0 \\ 0 & \mathrm{s}\,\alpha_{i-1} & \mathrm{c}\,\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\,\theta_i & -\mathrm{s}\,\theta_i & 0 & 0 \\ \mathrm{s}\,\theta_i & \mathrm{c}\,\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} \mathrm{c}\,\theta_i & -\mathrm{s}\,\theta_i & 0 & a_{i-1} \\ \mathrm{s}\,\theta_i\,\mathrm{c}\,\alpha_{i-1} & \mathrm{c}\,\theta_i\,\mathrm{c}\,\alpha_{i-1} & -\mathrm{s}\,\alpha_{i-1} & -\mathrm{s}\,\alpha_{i-1}d_i \\ \mathrm{s}\,\theta_i\,\mathrm{s}\,\alpha_{i-1} & \mathrm{c}\,\theta_i\,\mathrm{s}\,\alpha_{i-1} & \mathrm{c}\,\alpha_{i-1} & \mathrm{c}\,\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because of the recursive relationship among the parameter indices, this formulation is particularly well suited to the derivation of the manipulator dynamics equations.

3.3.1 The Forward Kinematics Problem

IMPORTANT NOTE: For the remainder of this course we will use the MDH parameters as they are found in Craig's book. Henceforth we shall refer to them only as DH parameters, bearing in mind we *really* mean the modified DH parameters.

The forward kinematics problem for a serial manipulator reduces to, at worst, n matrix multiplications, one for each DOF. Using DH notation, we can summarize the procedure in 7 steps:

1. Identify and draw joint axes.
    - R-pair: axis of rotation.

- P-pair: direction of translation

2. Determine link lengths
(common normals between adjacent axes), a_{i-1}, $a_0 = a_n = 0$.

3. Affix the DH frames to the links.

   - If $a_{i-1} = 0$ (intersecting $i-1$ and i axes), then $\hat{\mathbf{X}}_{i-1} \perp (\hat{\mathbf{Z}}_{i-1}, \hat{\mathbf{Z}}_i)$. Place O_{i-1} at the intersection of $\hat{\mathbf{Z}}_{i-1}$ and $\hat{\mathbf{Z}}_i$.
   - Measure angles according to right-hand rule.

4. Determine link twists, α_{i-1}, $\alpha_0 = \alpha_n = 0$
(angle between $\hat{\mathbf{Z}}_{i-1}$ and $\hat{\mathbf{Z}}_i$ measured about $\hat{\mathbf{X}}_{i-1}$).

5. Determine link offsets, d_i.
(Distance from $\hat{\mathbf{X}}_{i-1}$ to $\hat{\mathbf{Z}}_i$ measured along $\hat{\mathbf{Z}}_i$).

6. Determine joint angles θ_i.
(Angle between $\hat{\mathbf{X}}_{i-1}$ and $\hat{\mathbf{X}}_i$ measured about $\hat{\mathbf{Z}}_i$).

   - if joint 1 = R-pair, then $d_i = 0$ and θ_i has arbitrary zero position.
   - if joint 1 = P-pair, then $\theta_1 = 0$ and d_i has arbitrary zero.

7. Determine transforms ${}^{i-1}\mathbf{T}_i$ and multiply to obtain ${}^{0}\mathbf{T}_n$.

Joint Limits

Real joints have limits: angular for R-pairs; length for P-pairs. The are also not perfect, but we will assume the errors to be negligibly small.

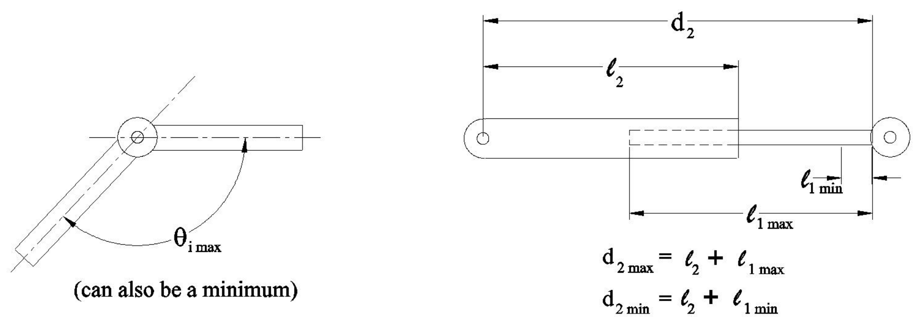


Figure 3.14: Joint limits.

3.3.2 Actuator, Joint, Cartesian Space

Joint Space

A serial manipulator with n DOF has a position and orientation that can be specified by n joint variables. Joint space is the coordinate space whose components are the joint variables, e.g. $[\theta_1, \theta_2, d_3, \theta_4]$.

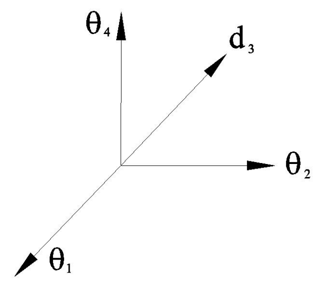


Figure 3.15: $4D$ joint space.

Cartesian Space

The space where position of the end-effector is measured along orthogonal axes and orientation is measured according to the connections discussed earlier.

Actuator Space

Sometimes two differential actuators are used to move one joint, or a linear actuator is used to move an R-pair through a 4-bar. This is the space where pose has been specified with actuator values. The components are actuator positions/angles recorded by sensors.

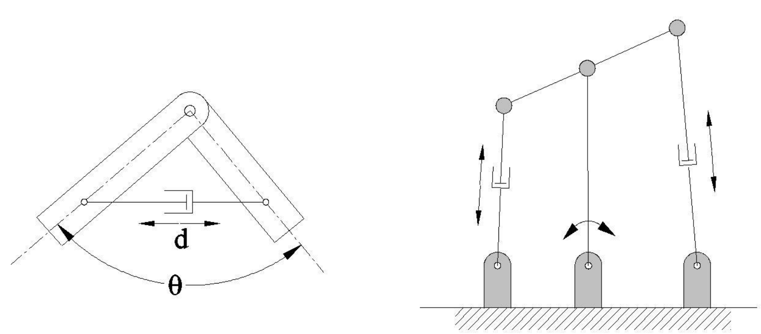


Figure 3.16: Actuator space.

3.3.3 Standard Frames

The base frame $\{B\}$

$\{B\}$ is located at the base of the manipulator. It is merely another name for frame $\{O\}$. It is affixed to a non-moving part of the robot, sometimes called link 0.

The station frame $\{S\}$

$\{S\}$ is located in a task-relevant location. In Figure 3.17 it is at the corner of a table upon which the robot is to work. As far as the user of this robot system is concerned, $\{S\}$ is the universe frame and all actions of the robot are made relative to it. It is sometimes called the *task frame*, the *world frame*, or the *universe frame*. The station frame is always specified with respect to the base frame, that is, ${}^{B}\mathbf{T}_{S}$.

The wrist frame $\{W\}$

$\{W\}$ is affixed to the last link of the manipulator. It is another name for frame $\{N\}$, the link frame attached to the last link of the robot. Very often $\{W\}$ has its origin fixed at a point called the *wrist* of the manipulator, and $\{W\}$ moves with the last link of the manipulator. It is defined relative to the base frame, that is, $\{W\} = {}^{B}T_{W} = {}^{0}T_{N}$.

The tool frame $\{T\}$

$\{T\}$ is affixed to the end of any tool the robot happens to be holding. When the hand is empty, $\{T\}$ is usually located with its origin between the fingertips of the robot. The tool frame is always specified with respect to the wrist frame. In Figure 3.17 the tool frame is defined with its origin at the tip of a pin that the robot it holding.

The goal frame $\{G\}$

$\{G\}$ is a description of the location to which the robot is to move the tool. Specifically this means that at the end of the motion, the tool frame should be brought to coincidence with the goal frame. $\{G\}$ is always specified relative to the station frame. In Figure 3.17 the goal is located at a hole into which we want the pin to be inserted. All robot motions may be described in terms of these frames without loss of generality. Their use helps to give us a standard language for talking about robot tasks.

3.3.4 The Inverse Kinematics Problem

As we have seen, the forward kinematics problem for serial manipulators reduces to matrix multiplication. That is, given a set of feasible joint variables, we

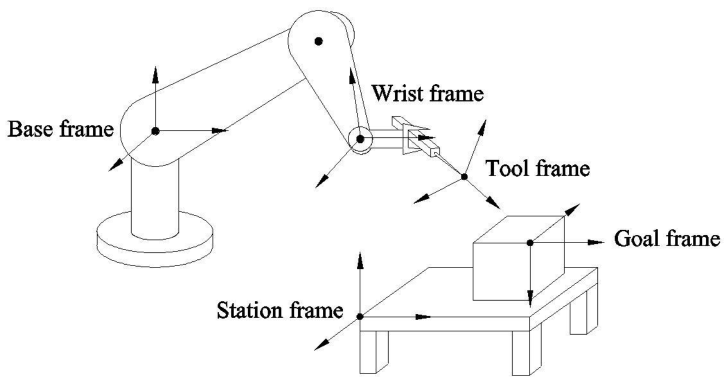


Figure 3.17: The standard frames.

can find the position and orientation of the end-effector by concatenating the appropriate matrices.

Now we have to consider a somewhat nastier problem: Given the position and orientation of the tool relative to the work station, determine the joint variables which achieve the desired result. Depending on the kinematic architecture of the manipulator, this can be an extremely difficult problem from a computational point of view.

Existence of Solutions

Solutions to the inverse kinematics problem don't always exist. This raises the issue of the workspace. There are two main definitions, one more inclusive than the other:

1. Reachable Workspace: The volume of space the end-effector can reach in *at least one* orientation.

2. Dextrous Workspace: The volume of space the end-effector can reach with *any* orientation.

The dextrous workspace, if it exists, is clearly a subset of the reachable workspace.

1. If $l_1 \neq l_2$, dextrous workspace $= \{0\}$.

2. If $l_1 = l_2$, dextrous workspace $=$ 1 point, $(0, 0)$.

3. If a zero-length 3^{rd} link is connected to the free end of l_2 with an R-pair, then dextrous workspace $=$ reachable workspace. (i.e. can move end-effector to any point in workspace and give it any orientation).

4. As l_3 increases in length, the dextrous workspace decreases in size.

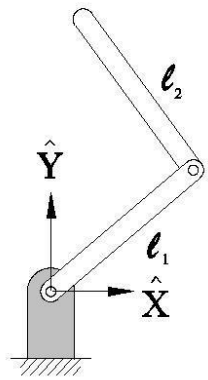


Figure 3.18: Planar $2R$ linkage.

Joint Limits

Due to design limitations, it may be that some joints in a manipulator will have a restricted range of motion. Sometimes, joint limits are imposed, designed-in, to avoid singularities (more on that later). Joint limits reduce both dextrous and reachable workspace. When a manipulator has less than 6 DOF it cannot

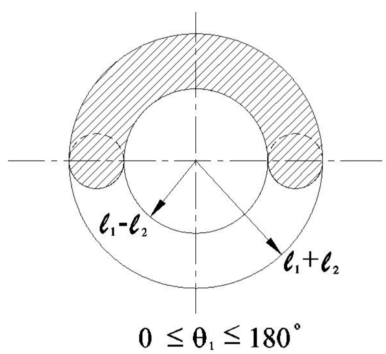


Figure 3.19: Joint limits.

attain general goal poses in 3-D space. In many real industrial applications, manipulators with 4 or 5 DOF are used. They cannot attain general poses. The workspaces of such manipulators must be carefully studied so that the controller can be programmed to avoid inaccessible poses. This raises an interesting question: What is the nearest attainable pose?

Another point of interest is that the workspace depends on the tool frame, since the end-effector reference point used to specify the workspace is usually on the tool. Usually, the tool transformation is computed independently of the forward kinematics and the inverse kinematics, so, often we must consider the workspace of the wrist frame. Hence, the workspace computed by the designer is different from the one imagined by the user.

Multiple Solutions

A problem encountered in solving the inverse kinematics problem is the existence of multiple solutions. This fact may cause problems when the controller has to choose one. How should the decision be made? A very reasonable choice is the *closest* solution.

If the manipulator is to move from A to B, a good choice might be the solution that minimizes the amount each joint must move. This suggests we should use the *present* position as an input parameter to our inverse kinematics algorithm.

But how do you define *close*? Most serial arms have three bit *rough positioning* links, then three small *orienting* links near the end-effector. Here, weights would be used so that the selection favors moving smaller joints rather than larger ones, when a choice exists. Of course, the *closer* solution may cause a collision (see Figure 3.20). Sadly, this means we need to calculate all possible real solutions.

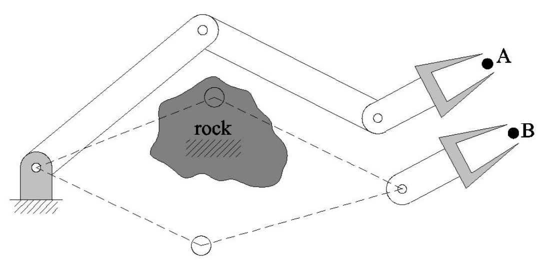


Figure 3.20: Here, the closer solution causes a collision with the rock.

Method of Solution

Solvability: A manipulator is *solveable* if all inverse kinematics solutions can be determined for a given pose.

An important condition in this definition is that we require all solutions. Some iterative numerical procedures may not be able to find all solutions. Those that can, like Morgan's Polynomial Continuation Method, are mathematically well beyond the scope of this course. So let's split all solution strategies into two classes:

1. Closed form solutions.
2. Numerical solutions.

For some applications, like path planning, the inverse kinematics must be computed at rates on the order of 30Hz, often faster. So, iterative numerical solutions, are in general, too slow compared to a corresponding closed form solution. Hence, we will only consider closed form solutions.

Here, that means solutions based on analytic equations of degree 4 or less. Otherwise we're back to numerical methods anyway. A major recent result in kinematics is that all 6 DOF serial manipulators (most general) are solvable (R and P pairs only).

There are two main philosophies for developing closed form solutions:

1. Algebraic.
2. Geometric.

Of course, the algebra is based on geometry and the geometry requires some algebra, but they present two distinct approaches to the problem.

Comparing Algebra and Geometry: Forward Kinematics Example

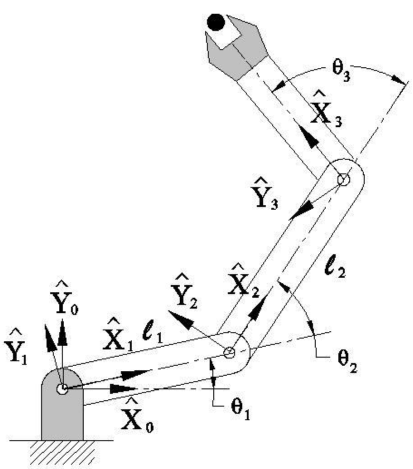


Figure 3.21: Planar $3R$ linkage.

1. <u>By inspection and straight matrix multiplication:</u>

$$
\begin{aligned}
{}^0\mathbf{T}_1 &= \begin{bmatrix} \mathrm{c}\,\theta_i & -\mathrm{s}\,\theta_i & 0 & 0 \\ \mathrm{s}\,\theta_i & \mathrm{c}\,\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^1\mathbf{T}_2 &= \begin{bmatrix} \mathrm{c}\,\theta_i & -\mathrm{s}\,\theta_i & 0 & l_1 \\ \mathrm{s}\,\theta_i & \mathrm{c}\,\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^2\mathbf{T}_3 &= \begin{bmatrix} \mathrm{c}\,\theta_i & -\mathrm{s}\,\theta_i & 0 & l_2 \\ \mathrm{s}\,\theta_i & \mathrm{c}\,\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Useful Identities:

$$
\begin{aligned}
\mathrm{c}(\theta_1 + \theta_2) &= c_{12} = c_1 c_2 - s_1 s_2, \\
\mathrm{s}(\theta_1 + \theta_2) &= s_{12} = c_1 s_2 + c_2 s_1, \\
\mathrm{c}(\theta_1 - \theta_2) &= c_1 c_2 + s_1 s_2, \\
\mathrm{s}(\theta_1 - \theta_2) &= s_1 c_2 - s_2 c_1.
\end{aligned}
$$

Combine:

$$
\begin{aligned}
{}^0\mathbf{T}_3 &= {}^0\mathbf{T}_1\, {}^1\mathbf{T}_2\, {}^2\mathbf{T}_3, \\
&= \begin{bmatrix} c_1 c_2 - s_1 s_2 & -(c_1 s_2 + c_2 s_1) & 0 & l_1 c_1 \\ c_2 s_1 + c_1 s_2 & -s_1 s_2 + c_1 c_2 & 0 & l_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^2\mathbf{T}_3, \\
&= \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 \\ s_{12} & c_{12} & 0 & l_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^2\mathbf{T}_3, \\
&= \begin{bmatrix} c_{12} c_3 - s_{12} s_3 & -(c_{12} s_3 + s_{12} c_3) & 0 & l_2 c_{12} + l_1 c_1 \\ s_{12} c_3 + c_{12} s_3 & -s_{12} s_3 + c_{12} c_3 & 0 & l_2 s_{12} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
&= \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

2. <u>DH Parameters:</u>

Table 3.7: DH Parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	l_1	0	θ_2
3	0	l_2	0	θ_3

$$
\begin{aligned}
{}^{i-1}\mathbf{T}_i &= \begin{bmatrix} \mathrm{c}\,\theta_i & -\mathrm{s}\,\theta_i & 0 & a_{i-1} \\ \mathrm{s}\,\theta_i\,\mathrm{c}\,\alpha_{i-1} & \mathrm{c}\,\theta_i\,\mathrm{c}\,\alpha_{i-1} & -\mathrm{s}\,\alpha_{i-1} & -\mathrm{s}\,\alpha_{i-1}d_i \\ \mathrm{s}\,\theta_i\,\mathrm{s}\,\alpha_{i-1} & \mathrm{c}\,\theta_i\,\mathrm{s}\,\alpha_{i-1} & \mathrm{c}\,\alpha i-1 & \mathrm{c}\,\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^{0}\mathbf{T}_1 &= \begin{bmatrix} \mathrm{c}\,\theta_1 & -\mathrm{s}\,\theta_1 & 0 & 0 \\ \mathrm{s}\,\theta_1 & \mathrm{c}\,\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^{1}\mathbf{T}_2 &= \begin{bmatrix} \mathrm{c}\,\theta_2 & -\mathrm{s}\,\theta_2 & 0 & l_1 \\ \mathrm{s}\,\theta_2 & \mathrm{c}\,\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^{2}\mathbf{T}_3 &= \begin{bmatrix} \mathrm{c}\,\theta_3 & -\mathrm{s}\,\theta_3 & 0 & l_2 \\ \mathrm{s}\,\theta_3 & \mathrm{c}\,\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Let's consider the end-effector reference point to be the origin of the wrist frame, in this case, $\{3\}$ (O_3). The *tool point* is known with respect to $\{3\}$ and is a constant job-specific transformation. If the tool can reach its point, the wrist will be in a necessary corresponding pose. Without loss of generality, the inverse kinematics problem then involves determining all sets of joint angles given a pose of $\{3\}$. That is, given the (x, y) coordinates of O_3 in $\{0\}$, and the orientation of $\{3\}$ in $\{0\}$, indicated by ϕ. So, all attainable poses must have the form implied by the transformation

$$
{}^{B}\mathbf{T}_W = \begin{bmatrix} \mathrm{c}\,\phi & -\mathrm{s}\,\phi & 0 & x \\ \mathrm{s}\,\phi & \mathrm{c}\,\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Equating ${}^0\mathbf{T}_3$ and ${}^B\mathbf{T}_W$ we get four non-linear equations:

$$\begin{aligned} \mathrm{c}\,\phi &= c_{123}, && (3.1)\\ \mathrm{s}\,\phi &= s_{123}, && (3.2)\\ x &= l_1c_1 + c_2c_{12}, && (3.3)\\ y &= l_1s_1 + l_2s_{12}. && (3.4) \end{aligned}$$

$x,\ y,\ \phi$ are given, determine $\theta_1,\ \theta_2,\ \theta_3$.

Algebraic Solution:

1. Clearly, from (3.1) and (3.2),
$$\phi = \theta_1 + \theta_2 + \theta_3 \qquad (3.5)$$

2. Square and add (3.3) and (3.4):
Right Hand Side:
$$l_1{}^2\, c_1{}^2 + l_2{}^2\, c_{12}{}^2 + 2l_1l_2c_1c_{12} + l_1{}^2\ s_1{}^2 + l_2{}^2\ s_{12}{}^2 + 2l_1l_2s_1s_{12}$$
$$\begin{aligned} &= l_1{}^2 + l_2{}^2 + 2l_1l_2(c_1c_{12} + s_1s_{12}),\\ &= l_1{}^2 + l_2{}^2 + 2l_1l_2(c_1(c_1c_2 - s_1s_2) + s_1(c_1s_2 + c_2s_1)),\\ &= l_1{}^2 + l_2{}^2 + 2l_1l_2(c1^2\, c_2 - c_1s_1s_2 + c_1s_1s_2 + c_2s_1{}^2),\\ &= l_1{}^2 + l_2{}^2 + 2l_1l_2c_2(c_1{}^2 + s_1{}^2),\\ \Rightarrow\ & l_1{}^2 + l_2{}^2 + 2l - 1l_2c_2 \ = \ x^2y^2. \end{aligned}$$
Solve for c_2:
$$\begin{aligned} c_2 &= \frac{x^2 + y^2 - l_1{}^2 - l_2{}^2}{2l_1l_2},\\ \Rightarrow\ \theta_2 &= c^{-1}\left(\frac{x^2 + y^2 - l_1{}^2 - l_2{}^2}{2l_1l_2}\right). \qquad (3.6) \end{aligned}$$
Equation(3.6) must have a value $-1 \ \leq \ c_2 \ \leq \ 1$. In the solution algorithm this physical constraint would be checked to determine if any real solutions exist. If (3.6) is not viable, this means that the desired pose is too far away for the manipulator to reach.

3. Assuming (3.6) is viable we can always write
$$s_2 = \pm\sqrt{1 - c_2}.$$
Then
$$\theta_2 = \arctan_2(s_2, c_2),$$
by using the sine and cosine of the joint angle and then using $\arctan_2$ we ensure we have all solutions, and the solved angle is in the correct quadrant.
Note: the (+) means *elbow-up* and (-) means *elbow-down*.

4. Now that we have θ_2, we can solve Equations (3.3) and (3.4) for θ_2. Rewriting, we have:

$$\begin{aligned} x &= l_1 c_1 + l_2(c_1 c_2 - s_1 s_2) \;=\; (l_1 + c_2 c_2)c_1 - (l_2 x_2)s_1, \\ y &= l_1 s_1 + l_2(c_1 s_2 + c_2 s_1) \;=\; (l_1 + l_2 c_2)s_1 + (l_2 s_2)c_1. \end{aligned}$$

   These equations are linear in c_1 and s_1:

$$\begin{aligned} x &= k_1 c_1 - k_2 s_1, \\ y &= k_2 c_1 + k_1 s_1, \\ \text{where } k_1 &= l_1 + l_2 c_2, \\ k_2 &= -l_2 s_2. \end{aligned}$$

   Using Cramer's Rule:

$$c_1 = \frac{\begin{bmatrix} x & -k_2 \\ y & k_1 \end{bmatrix}}{\begin{bmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{bmatrix}}, \quad s_1 = \frac{\begin{bmatrix} k_1 & x \\ k_2 & y \end{bmatrix}}{\begin{bmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{bmatrix}}.$$

$$\theta_1 = \arctan_2(s_1, c_1).$$

5. Finally,

$$\theta_3 = \phi - \theta_1 - \theta_2.$$

   Note that in this equation and the one above it, there are two values, one for each θ_2, and hence, two solutions.

An algebraic approach to solving the inverse kinematics problem involves manipulating the system of equations into a form for which a solution is known.

Geometric Solution:

In the general 3-D case we try to decompose the inverse kinematics problem into several planar problems. But for our example, we are already in the plane. We are given the pose of link 3. We must determine all solutions for θ_1, θ_2, θ_3.

1. Use law of cosines to solve for θ_2:

$$\begin{aligned} A^2 &= B^2 + C^2 - 2BC\cos\alpha, \\ \cos(\theta_2 + 180^o) &= \cos(\theta_2 - 180^o) \;=\; -\cos\theta_2, \\ \Rightarrow\; l_2 &= l_1 + l_2 - l_1 l_2 \cos(\theta \pm 180^o), \\ x^2 + y^2 &= {l_1}^2 + {l_2}^2 + s l_1 l_2 c_2, \\ \Rightarrow\; c_2 &= \frac{x^2 + y^2 - {l_1}^2 - {l_2}^2}{2 l_1 l_2} \;\Rightarrow\; \theta_2. \end{aligned} \tag{3.7}$$

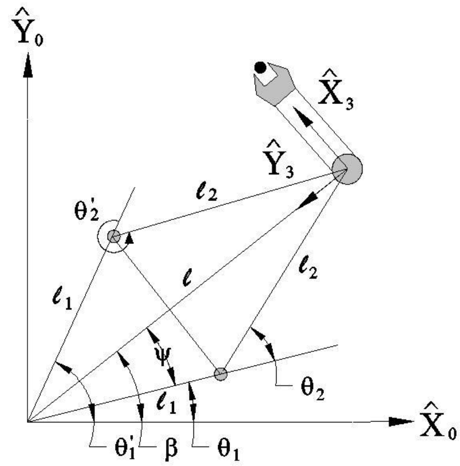


Figure 3.22: IK example.

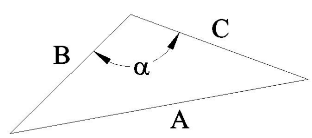


Figure 3.23: Law of cosines.

For this triangle to be real, the distance $\sqrt{x^2+y^2}$ must be less than or equal to $l_1 + l_2$. This condition fails when the goal point is out of the workspace.

The second solution is found by symmetry:

$$\theta_2 = -\theta_2{}'.$$

2. Now we must determine θ_1. This can be done by first finding β, then ψ. Then we have:

$$\theta_1 = \beta \pm \ \psi \ \Leftrightarrow \ \begin{cases} +\text{ if } \theta_2 \ < \ 0 \\ -\text{ if } \theta_2 \ > \ 0 \end{cases}$$

Since we are given the (x, y) coordinates of O_3,

$$\beta = \arctan_2(y, x).$$

Again we use the law of cosines to find ψ:

$$\psi = \cos^{-1}\left(\frac{x^2+y^2-{l_1}^2-{l_2}^2}{2l_1\sqrt{x^2+y^2}}\right).$$

3. Angles in the plane add linearly. So the sum of the three joint angles must be the orientation of link 3. This gives:

$$\theta_3 \ = \ \phi - \theta_1 - \theta_2.$$

Polynomial Method:

Transcendental equations can, sometimes, be more trouble than they are worth. We can use the half-angle substitutions to transform the system into polynomials, which may, or may not be "easier" to solve. But, this is beyond the scope of this course.

$$\mu = \tan\frac{\theta}{2}, \quad \frac{1-\mu^2}{1+\mu^2} = \cos\theta, \quad \frac{2\mu}{1+\mu^2} = \sin\theta.$$

Pieper's 3-Intersecting-Axes Solution

Generally, a 6 DOF serial robot does not have closed form inverse kinematics solutions, but some important special cases do. We will look at a method, developed by D.Pieper in his Ph.D. thesis from 1968 (his Ph.D. supervisor was Bernie Roth), that can be used in the case where all six joints are R-pairs with the last three having mutually intersecting axes.

Pieper's solution works in the following way: When axes of $\{4\}$, $\{5\}$, $\{6\}$ intersect, O_4, O_5, O_6 are all located at the point of intersection.

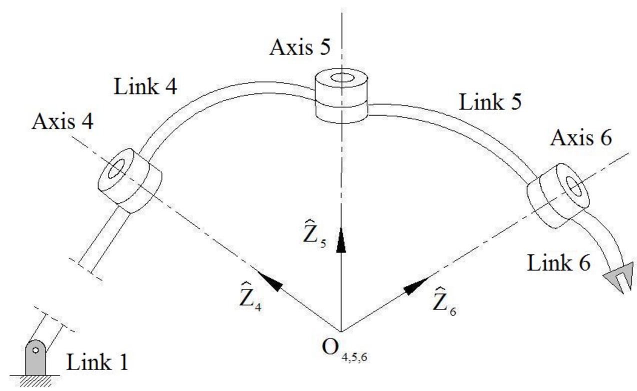


Figure 3.24: Pieper's 3 intersecting axes solution.

The orientation of $\{4\}$ depends on the previous joints only, θ_1, θ_2, θ_3. ${}^0\mathbf{T}_6$ is given:

$${}^0\mathbf{T}_6 = \left[\begin{array}{c|c} {}^0\mathbf{R}_6 & {}^0\mathbf{p}_{6_{\mathrm{ORG}}} \\ \hline 0\;0\;0 & 1 \end{array}\right],$$

but,

$$\begin{aligned}
{}^{0}\mathbf{p}_{6_{\mathrm{ORG}}} &= {}^{0}\mathbf{p}_{5_{\mathrm{ORG}}} = {}^{0}\mathbf{p}_{4_{\mathrm{ORG}}}, \\
\Rightarrow {}^{0}\mathbf{p}_{4_{\mathrm{ORG}}} &= f(\theta_1, \theta_2, \theta_3), \\
\Rightarrow {}^{0}\mathbf{p}_{4_{\mathrm{ORG}}} &= {}^{0}\mathbf{T}_3\, {}^{3}\mathbf{p}_{4_{\mathrm{ORG}}}.
\end{aligned}$$

Which gives three equations and 3 unknowns, so we can find θ_1, θ_2, θ_3. So

$${}^{0}\mathbf{R}_3 = g(\theta_1, \theta_2, \theta_3)$$

is now known, and thus ${}^{0}\mathbf{R}_6 = {}^{0}\mathbf{R}_3\, {}^{3}\mathbf{R}_6$ and ${}^{3}\mathbf{R}_6 = {}^{3}\mathbf{R}_0\, {}^{0}\mathbf{R}_6$. Thus we can find θ_4, θ_5, θ_6 since

$${}^{3}\mathbf{R}_6 = h(\theta_4, \theta_5, \theta_6).$$

3.4 Differential Kinematics

So far, we have examined the issues surrounding static positioning problems: position level kinematics. If we want to consider motions, ie. how the manipulator gets from A to B and examine some details of how it gets there we have to consider linear and angular velocities of the links. We will assume our transformations to be smooth functions of time. Hence, the position vector of a point P will be mapped smoothly into a new position. For the position level, we had the:

1. Forward Kinematics Problem: Given the joint variables vector $\mathbf{q} = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_6 \end{bmatrix}$, determine the end-effector pose vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$, $\mathbf{x} = f(\mathbf{q})$.

   At the velocity level, this translates to: Given the joint rate vector $\dot{\mathbf{q}}$ determine the end-effector velocity vector $\mathbf{v} = \dot{\mathbf{x}}$.

2. Inverse Kinematics Problem: Given the end-effector velocity vector $\mathbf{v}$, determine the joint rate vector $\dot{\mathbf{q}}$.

We will also look at the effects of static forces applied to the end-effector on the joint torques required to maintain static equilibrium. A nice result is that $\mathbf{v}$ and $\dot{\mathbf{q}}$ are related by the same matrix operator, a *Jacobian*, as are $\mathbf{f}$ and $\boldsymbol{\tau}$.

$$\mathbf{v} = \mathbf{J}\dot{\mathbf{q}}, \;\; \boldsymbol{\tau} = \mathbf{J}^T \mathbf{f}.$$

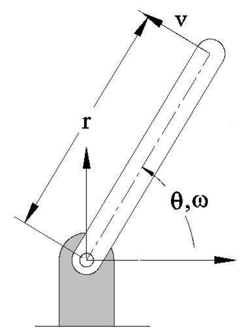


Figure 3.25: Linear velocity $\mathbf{v}$ caused by angular velocity $\boldsymbol{\omega}$: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

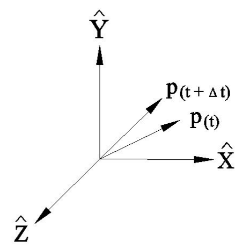


Figure 3.26: Displacement of point p after time interval Δt.

3.4.1 Differentiation of Vectors, Matrices, Representation of Angular Velocity, Notation

Linear Velocity of One Point

As with any vector, a velocity may be described in any frame. The velocity of a position vector of a point is the (here, linear) velocity of the point represented by the position vector. Suppose point p is moving relative to frame $\{1\}$. The velocity of ${}^1\mathbf{p}$ with respect to $\{1\}$ is obtained by differentiation:

$$\mathbf{v}_p \;=\; \frac{d}{dt}\,{}^1\mathbf{p} \;=\; \lim_{\Delta t \to 0} \frac{{}^1\mathbf{p}(t+\Delta t) - {}^1\mathbf{p}(t)}{\Delta t}.$$

If $\mathbf{p}$ is not changing relative to $\{1\}$, but $\{1\}$ is changing relative to $\{0\}$, then:

$$\frac{d}{dt}\,{}^1\mathbf{p} \;=\; 0 \;\neq\; \frac{d}{dt}\,{}^0\mathbf{p}$$

in general.

Thus it is important to indicate the frame in which the vector is differentiated. But we may want to express this velocity in some other frame. So, we also have to indicate the frame of reference. Let's adopt the following convention:

$${}^a\mathbf{v}_p^b.$$

It is the velocity of point p relative to frame $\{b\}$, but expressed in frame $\{a\}$. Where a is the frame in which the velocity vector is expressed, b is the frame of differentiation, and p indicates it is the velocity of point p. If $\{a\}$ and $\{b\}$ are the same, the upper-right superscript is not always needed. This is also true if $\{b\} = \{0\}$, this should be clear from context. Also:

$$^{a}\mathbf{p}^{\,b},$$

where a denotes "expressed in frame $\{a\}$", and b indicates the position vector is relative to $\{b\}$.

In order to add vectors, they MUST be expressed in the same frame.

Angular Velocity of One Body

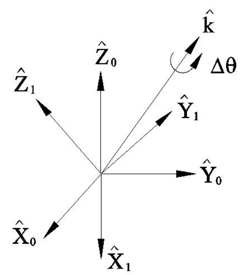


Figure 3.27: Angular velocity.

Linear Velocity describes a property of a *point.*

Angular Velocity describes a property of a *body.*

Note that all lines in a rigid body have the same ω.

$$^{0}\boldsymbol{\omega}_1^2 = \hat{\mathbf{k}} \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \hat{\mathbf{k}}\dot{\theta}.$$

In words, we can say $^{0}\boldsymbol{\omega}_1^2$ = *angular velocity of* $\{1\}$ *relative to* $\{2\}$*, expressed in* $\{0\}$.

The physical meaning of the angular velocity vector $\boldsymbol{\omega}$ is that the change in orientation about a particular axis $\hat{\mathbf{k}}$. Vector $\hat{\mathbf{k}}$ is the *instantaneous axis of rotation.* Taking $\hat{\mathbf{k}}$ as a unit vector scaled by the speed of rotation, $\dot{\theta}$, yields $\boldsymbol{\omega}$.

Derivative of a Rotation Matrix

When two frames, with coincident origins, rotate with respect to each other, the rotation matrices change with respect to time. The question is: What is the time derivative of an orthogonal matrix?

$$\frac{d}{dt}{}^0\mathbf{R}_1(t) \;=\; {}^0\dot{\mathbf{R}}_1.$$

For an $n \times n$ proper orthogonal matrix $\mathbf{R}$, we know that:

$$\mathbf{R}\mathbf{R}^T \;=\; \mathbf{I}_n. \tag{3.8}$$

Differentiating (3.8) with respect to time gives:

$$\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}} \;=\; \mathbf{0}_n. \tag{3.9}$$

We can rearrange (3.9) as:

$$\dot{\mathbf{R}}\mathbf{R}^T + (\dot{\mathbf{R}}\mathbf{R}^T)^T \;=\; \mathbf{0}. \tag{3.10}$$

Since:

$$\begin{aligned}
\mathbf{R} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \\
\text{then } \dot{\mathbf{R}} &= \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}, \\
\mathbf{R}\dot{\mathbf{R}}^T &= \begin{bmatrix} 0 & \cos(\theta)^2 + \sin(\theta)^2 \\ -\sin(\theta)^2 - \cos(\theta)^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
\dot{\mathbf{R}}\mathbf{R}^T &= \begin{bmatrix} 0 & \cos(\theta)^2 + \sin(\theta)^2 \\ -\sin(\theta)^2 - \cos(\theta)^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
(\mathbf{R}\dot{\mathbf{R}}^T) + (\dot{\mathbf{R}}\mathbf{R}^T)^T &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0.
\end{aligned}$$

Let's call $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$, and then from (3.10):

$$\mathbf{\Omega} + \mathbf{\Omega}^T \;=\; \mathbf{0}. \tag{3.11}$$

Equation (3.11) is the definition of a skew-symmetric matrix. Also we see that, by the definition of $\boldsymbol{\omega}$:

$$\dot{\mathbf{R}} \;=\; \boldsymbol{\omega}\mathbf{R}. \tag{3.12}$$

This is very nice, but now, what are the elements of $\boldsymbol{\omega}$? Let's directly differentiate $\mathbf{R}$:

$$\dot{\mathbf{R}} \;=\; \lim_{\Delta t \to 0} \frac{\mathbf{R}(t + \Delta t) - \mathbf{R}(t)}{\Delta t}, \tag{3.13}$$

but,

$$\mathbf{R}(t + \Delta t) \;=\; \mathbf{R}_k(\Delta\theta)\mathbf{R}(t), \tag{3.14}$$

where a rotation of $\delta\theta$ has occurred about axis $\hat{\mathbf{k}}$ over the interval Δt. We can use (3.14) to rewrite (3.13) as:

$$\dot{\mathbf{R}} \;=\; \left(\lim_{\Delta t \to 0} \frac{\mathbf{R}_x(\Delta\theta) - \mathbf{I}}{\Delta_t} \right) \mathbf{R}(t). \tag{3.15}$$

Then, what does a differential rotation matrix about an arbitrary axis $\hat{\mathbf{k}}$ look like? Consider the following rotation matrices:

1. Small rotation about $\hat{\mathbf{X}}$ by δx:

$$\mathbf{R}_{\hat{\mathbf{X}}}(\delta x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\,\delta x & -\mathrm{s}\,\delta x \\ 0 & \mathrm{s}\,\delta x & \mathrm{c}\,\delta x \end{bmatrix}.$$

2. Small rotation about $\hat{\mathbf{Y}}$ by δy:

$$\mathbf{R}_{\hat{\mathbf{Y}}}(\delta y) = \begin{bmatrix} \mathrm{c}\,\delta y & 0 & \mathrm{s}\,\delta y \\ 0 & 1 & 0 \\ -\mathrm{s}\,\delta y & 0 & \mathrm{c}\,\delta y \end{bmatrix}.$$

3. Small rotation about $\hat{\mathbf{Z}}$ by δz:

$$\mathbf{R}_{\hat{\mathbf{Z}}}(\delta z) = \begin{bmatrix} \mathrm{c}\,\delta z & -\mathrm{s}\,\delta z & 0 \\ \mathrm{s}\,\delta z & \mathrm{c}\,\delta z & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For differential rotations, $\delta x \simeq \delta y \simeq \delta z \simeq 0$.

$$\begin{aligned} \Rightarrow\ \cos\delta x &\simeq \cos\delta y \simeq \cos\delta z \simeq 1, \\ \sin\delta x &\simeq \delta x;\ \sin\delta y \simeq \delta y\ \ \sin\delta z \simeq \delta z, \end{aligned}$$

$$\mathbf{R}_{\hat{\mathbf{X}}}(\delta x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\delta x \\ 0 & \delta x & 1 \end{bmatrix},\ \mathbf{R}_{\hat{\mathbf{Y}}}(\delta y) = \begin{bmatrix} 1 & 0 & \delta y \\ 0 & 1 & 0 \\ -\delta y & 0 & 1 \end{bmatrix},\ \mathbf{R}_{\hat{\mathbf{Z}}}(\delta z) = \begin{bmatrix} 1 & -\delta z & 0 \\ \delta z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When higher order products of differentials are ignored, differential rotation matrices are commutative. i.e.:

$$\begin{aligned}
\mathbf{R}_{\hat{\mathbf{X}}}(\delta x) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -dx \\ 0 & dx & 1 \end{bmatrix}, \\
\mathbf{R}_{\hat{\mathbf{Y}}}(\delta y) &= \begin{bmatrix} 1 & 0 & dy \\ 0 & 1 & 0 \\ -dy & 0 & 1 \end{bmatrix}, \\
\mathbf{R}_{\hat{\mathbf{Z}}}(\delta z) &= \begin{bmatrix} 1 & -dz & 0 \\ dz & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\text{Let } \mathbf{P1} \;=\; \mathbf{R}_{\hat{\mathbf{X}}}(\delta x)\mathbf{R}_{\hat{\mathbf{Y}}}(\delta y)\mathbf{R}_{\hat{\mathbf{Z}}}(\delta z) &= \begin{bmatrix} 1 & -dz & dy \\ dxdy + dz & -dxdydz + 1 & -dx \\ -dy + dxdz & dydz + dx & 1 \end{bmatrix}.
\end{aligned}$$

After setting higher order differentials to zero
(i.e. products of differential angles):

$$\mathbf{P1} \;=\; \begin{bmatrix} 1 & -dz & dy \\ dz & 1 & -dx \\ -dy & dx & 1 \end{bmatrix}.$$

Ignoring products, any rearrangement of the three differential rotation matrices yields the same result:

$$\mathbf{P2} \;=\; \begin{bmatrix} 1 & -dz + dxdy & dxdz + dy \\ dz & 1 & -dx \\ -dy & dydz + dx & -dxdydz + 1 \end{bmatrix} = \begin{bmatrix} 1 & -dz & dy \\ dz & 1 & -dx \\ -dy & dx & 1 \end{bmatrix}.$$

Thus, we can represent the differential rotation matrix about an arbitrary axis $\hat{\mathbf{k}}$ as the product of the components of the differential rotation matrices about the $\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}$ axes:

$$\mathbf{R}_{\hat{\mathbf{k}}}(\Delta\theta) \;=\; \mathbf{R}_{\hat{\mathbf{X}}}(\delta x)\mathbf{R}_{\hat{\mathbf{Y}}}(\delta y)\mathbf{R}_{\hat{\mathbf{Z}}}(\delta z). \tag{3.16}$$

When higher order products are ignored, multiplication in any order in (3.16) yields:

$$\mathbf{R}_{\hat{\mathbf{k}}}(\Delta\theta) \;=\; \begin{bmatrix} 1 & -\delta z & \delta y \\ \delta z & 1 & -\delta x \\ -\delta y & \delta x & 1 \end{bmatrix}. \tag{3.17}$$

Now substitute (3.17) into (3.15) and set $\delta x = k_{\hat{\mathbf{X}}}\Delta\theta,\;\; \delta y = k_{\hat{\mathbf{Y}}}\Delta\theta,\;\; \delta z = k_{\hat{\mathbf{Z}}}\Delta\theta$ and take the limit. We get:

$$\dot{\mathbf{R}} \;=\; \begin{bmatrix} 0 & -k_{\hat{\mathbf{Z}}}\dot{\theta} & k_{\hat{\mathbf{Y}}}\dot{\theta} \\ k_{\hat{\mathbf{Z}}}\dot{\theta} & 0 & -k_{\hat{\mathbf{X}}}\dot{\theta} \\ -k_{\hat{\mathbf{Y}}}\dot{\theta} & k_{\hat{\mathbf{X}}}\dot{\theta} & 0 \end{bmatrix} \mathbf{R}(t) \;=\; \boldsymbol{\theta}\mathbf{R}(t). \tag{3.18}$$

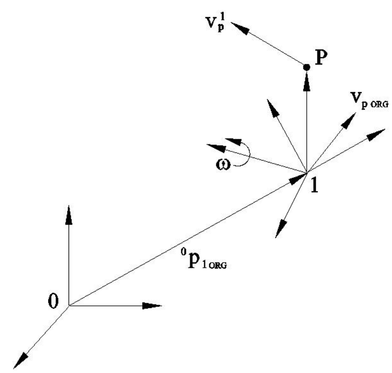
Since $\boldsymbol{\omega} = \hat{\mathbf{k}}\dot{\theta}$, we see that $\boldsymbol{\Omega}$ is the *angular velocity cross-product matrix operator.*

$$\boldsymbol{\Omega} \;=\; \dot{\mathbf{R}}\mathbf{R}^T \;=\; \begin{bmatrix} 0 & -\omega_{\hat{\mathbf{Z}}} & \omega_{\hat{\mathbf{Y}}} \\ \omega_{\hat{\mathbf{Z}}} & 0 & -\omega_{\hat{\mathbf{X}}} \\ -\omega_{\hat{\mathbf{Y}}} & \omega_{\hat{\mathbf{X}}} & 0 \end{bmatrix}, \tag{3.19}$$

and hence $\dot{\mathbf{R}} = \boldsymbol{\Omega}\mathbf{R} = \boldsymbol{\omega} \times \mathbf{R}$, where $\boldsymbol{\omega} \times \mathbf{R}$ is the angular velocity operator.

$$\begin{aligned}
{}^0\mathbf{R}_1 &= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & -\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\boldsymbol{\omega} \times {}^0\mathbf{R}_1 &= \begin{bmatrix} -\omega_z \sin\phi & -\omega_z \cos\phi & \omega_y \\ \omega_z \cos\phi & -\omega_z \sin\phi & -\omega_x \\ \omega_x \sin\phi - \omega_y \cos\phi & \omega_x \cos\phi + \omega_y \sin\phi & 0 \end{bmatrix}, \\
\boldsymbol{\Omega}{}^0\mathbf{R}_1 &= \begin{bmatrix} -\omega_z \sin\phi & -\omega_z \cos\phi & \omega_y \\ \omega_z \cos\phi & -\omega_z \sin\phi & -\omega_x \\ \omega_x \sin\phi - \omega_y \cos\phi & \omega_x \cos\phi + \omega_y \sin\phi & 0 \end{bmatrix}.
\end{aligned}$$

Relative Linear Velocity (time derivative) Transformations



The velocity of ${}^1\mathbf{p}$ with respect to $\{1\}$, expressed in $\{1\}$, ${}^1\mathbf{v}_P{}^1$ is known. We know how $\{1\}$ is translating and rotating with respect to $\{0\}$. What does the total velocity of ${}^1\mathbf{p}$ look like to a stationary observer in $\{0\}$? i.e., what is ${}^0\mathbf{v}_P$?

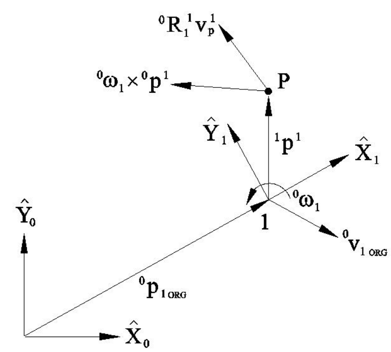
At any time t, the position of ${}^0\mathbf{p}$ is:

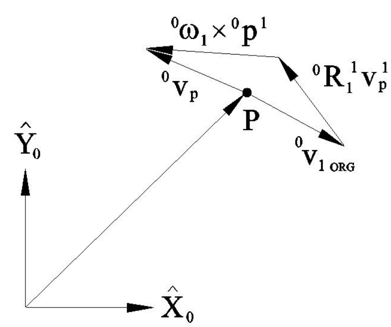
$${}^0\mathbf{p} \;=\; {}^0\mathbf{p}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\, {}^1\mathbf{p}^1,$$

where the first superscript 1 in ${}^1\mathbf{p}^1$ indicates that this is expressed in $\{1\}$, and the second superscript indicates that it is relative to frame $\{1\}$. Then:

$$
\begin{aligned}
\Rightarrow \frac{d}{dt}\,{}^0\mathbf{p} &= \frac{d}{dt}\,{}^0\mathbf{p}_{1_{\mathrm{ORG}}} + \frac{d}{dt}({}^0\mathbf{R}_1\,{}^1\mathbf{p}^1), \\
\Rightarrow {}^0\mathbf{v}_P &= {}^0\mathbf{v}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1 + {}^0\dot{\mathbf{R}}_1\,{}^1\mathbf{p}^1, \\
&= {}^0\mathbf{v}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1 + {}^0\boldsymbol{\Omega}_1\,{}^0\mathbf{R}_1\,{}^1\mathbf{p}^1, \\
&= {}^0\mathbf{v}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1 + ({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1)\,{}^1\mathbf{p}^1, \\
&= {}^0\mathbf{v}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1 + {}^0\boldsymbol{\omega}_1 \times ({}^0\mathbf{R}_1\,{}^1\mathbf{p}^1), \\
\text{where } \boldsymbol{\Omega} &= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.
\end{aligned}
\tag{3.20}
$$

Where:

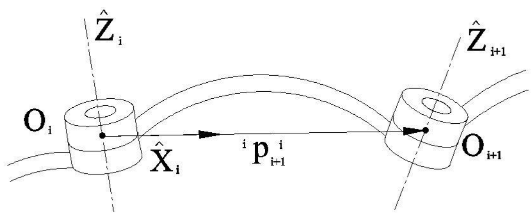




${}^0\mathbf{v}_P$ is the velocity of P expressed in $\{0\}$,
${}^0\mathbf{v}_{1_{\mathrm{ORG}}}$ is the velocity of origin of $\{1\}$ expressed in $\{0\}$,
${}^1\mathbf{v}_P{}^1$ is the velocity of P relative to $\{1\}$ expressed in $\{1\}$,
${}^0\boldsymbol{\omega}_1$ is the angular velocity of $\{1\}$ with respect to $\{0\}$ expressed in $\{0\}$,
${}^1\mathbf{p}$ is the position of P expressed in $\{1\}$, and
${}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1$ is the relative velocity of $\mathbf{p}$ with respect to $\{1\}$ expressed in $\{0\}$.

Velocity Propagation

Now we examine the problem of determining the relative velocities (linear and angular) of the links of a serial robot. We consider the propagation of velocity between adjacent links i and $i+1$ starting from the base, link 0. The velocity of link i is specified by vectors $\mathbf{v}_i$ and $\boldsymbol{\omega}_i$, which may be expressed in any frame, even $\{i\}$. The velocity of link $i+1$ will be that of link i plus new components added by joint $i+1$(i.e. the relative velocity of $\{i+1\}$ in $\{i\}$).



We can use Equation (3.20) with point $P = O_{i+1}$. Then set $\{0\} = \{i+1\}$, $\{1\} = \{i\}$:

$$ {}^{i+1}\mathbf{v}_{i+1} = {}^{i+1}\mathbf{v}_i + {}^{i+1}\mathbf{R}_i \, {}^i\mathbf{v}_{i+1}{}^i + {}^{i+1}\boldsymbol{\Omega}_i \, {}^{i+1}\mathbf{R}_i \, {}^i\mathbf{p}_{i+1}{}^i. \tag{3.21} $$

But ${}^{i+1}\boldsymbol{\Omega}_i = {}^{i+1}\boldsymbol{\omega}_i{}^x$, (i.e. CPM of ${}^{i+1}\boldsymbol{\omega}_i$). A theorem from linear algebra says if:

$$ \begin{aligned} {}^0\mathbf{a} &= {}^0\mathbf{R}_1 \, {}^1\mathbf{a}, \\ \Rightarrow \; {}^0\mathbf{a}^x &= ({}^0\mathbf{R}_1 \, {}^1\mathbf{a})^x, \\ &= {}^0\mathbf{R}_1 \, {}^1\mathbf{a}^x \, {}^0\mathbf{R}_1{}^T, \\ &= {}^0\mathbf{R}_1 \, {}^1\mathbf{a}^x \, {}^1\mathbf{R}_0. \end{aligned} \tag{3.22} $$

As shown below:

$$
\begin{aligned}
\text{Let}^1\mathbf{a} &= \begin{bmatrix} {}^1a_x \\ {}^1a_y \\ {}^1a_z \end{bmatrix}, \\
{}^0\mathbf{R}_1 &= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
{}^0\mathbf{a} \;=\; {}^0\mathbf{R}_1\,{}^1\mathbf{a} &= \begin{bmatrix} \cos\phi\,{}^1a_x - \sin\phi\,{}^1a_y \\ \sin\phi\,{}^1a_x + \cos\phi\,{}^1a_y \\ {}^1a_z \end{bmatrix}, \\
{}^1\mathbf{a}^x &= \begin{bmatrix} 0 & -{}^1a_z & {}^1a_y \\ {}^1a_z & 0 & -{}^1a_x \\ -{}^1a_y & {}^1a_x & 0 \end{bmatrix}, \\
{}^0\mathbf{a}^x &= \begin{bmatrix} 0 & -{}^1a_z & \sin\phi\,{}^1a_x + \cos\phi\,{}^1a_y \\ {}^1a_z & 0 & -\cos\phi\,{}^1a_x + \sin\phi\,{}^1a_y \\ -\sin\phi\,{}^1a_x - \cos\phi\,{}^1a_y & \cos\phi\,{}^1a_x - \sin\phi\,{}^1a_y & 0 \end{bmatrix}, \\
{}^0\mathbf{R}_1\,{}^1\mathbf{a}^x\,{}^0\mathbf{R}_1{}^T &= \begin{bmatrix} 0 & -{}^1a_z & \sin\phi\,{}^1a_x + \cos\phi\,{}^1a_y \\ {}^1a_z & 0 & -\cos\phi\,{}^1a_x + \sin\phi\,{}^1a_y \\ -\sin\phi\,{}^1a_x - \cos\phi\,{}^1a_y & \cos\phi\,{}^1a_x - \sin\phi\,{}^1a_y & 0 \end{bmatrix}. \\
&\Rightarrow\; {}^{i+1}\mathbf{\Omega}_i \;=\; {}^{i+1}\mathbf{R}_i\,{}^i\mathbf{\Omega}_i\,{}^i\mathbf{R}_{i+1}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
{}^{i+1}\mathbf{v}_i &= {}^{i+1}\mathbf{R}_i\,{}^i\mathbf{v}_i, \\
{}^{i+1}\mathbf{v}_{i+1}{}^i &= {}^{i+1}\mathbf{R}_i\,{}^i\mathbf{v}_{i+1},
\end{aligned}
$$

and sub this into (3.21):

$$
{}^{i+1}\mathbf{R}_i({}^i\boldsymbol{\omega}_i \times {}^i\mathbf{p}_{i+1}{}^i) \;=\; {}^{i+1}\mathbf{R}_i\,{}^i\mathbf{\Omega}_i\,{}^i\mathbf{p}_{i+1}{}^i,
$$

then

$$
{}^{i+1}\mathbf{v}_{i+1} \;=\; {}^{i+1}\mathbf{R}_i\,{}^i\mathbf{v}_i + {}^{i+1}\mathbf{v}_{i+1}{}^i + {}^{i+1}\mathbf{R}_i\,{}^i\mathbf{\Omega}_i\,{}^i\mathbf{p}_{i+1}{}^i. \tag{3.23}
$$

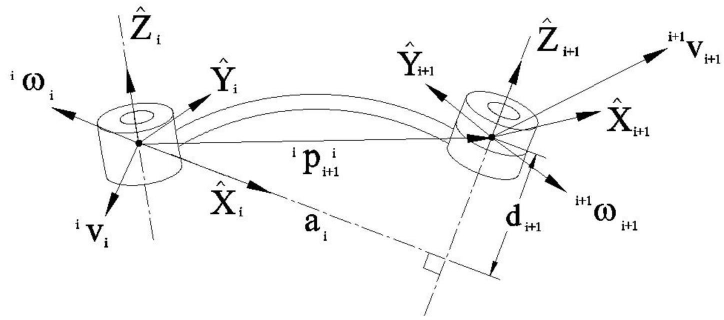
Relative Angular Velocities

These, like any vector, can only be added when expressed in the same frame. The angular velocity of link $i+1$ is the same as that of link i plus a new component caused by the angular velocity at joint $i+1$, of link $i+1$ relative to link i.

We can write this as:

$$
{}^{i+1}\boldsymbol{\omega}_{i+1} \;=\; {}^{i+1}\boldsymbol{\omega}_{i+1}{}^i + {}^{i+1}\boldsymbol{\omega}_i. \tag{3.24}
$$

Remember, we can represent a vector in any frame. The absolute angular velocity of link $i+1$ in $\{i+1\}$ equals the absolute angular velocity of link i in $\{i+1\}$ plus the relative angular velocity of link $i+1$ with respect to link i in $\{i+1\}$.



If Joint $\{i+1\}$ is an R-Pair

${}^{i+1}\mathbf{v}_{i+1}{}^{i} = 0$, i.e. no relative linear velocity between O_i and O_{i+1}. The relative angular velocity of link i+1 relative to i: ${}^{i+1}\boldsymbol{\omega}_{i+1}{}^{i} = {}^{i+1}\hat{\mathbf{Z}}_{i+1}\dot{\theta}_{i+1}$,$\Rightarrow$ link i is fixed.

$$\begin{aligned}
\Rightarrow \quad {}^{i+1}\mathbf{v}_{i+1} &= {}^{i+1}\mathbf{R}_i({}^i\mathbf{v}_i + {}^i\boldsymbol{\omega}_i \times {}^i\mathbf{p}_{i+1}), \\
&= {}^{i+1}\mathbf{R}_i\,({}^i\mathbf{v}_i + {}^i\boldsymbol{\Omega}_i\,{}^i\mathbf{p}_{i+1}), \\
{}^{i+1}\boldsymbol{\omega}_{i+1} &= {}^{i+1}\hat{\mathbf{Z}}_{i+1}\dot{\theta}_{i+1} + {}^{i+1}\mathbf{R}_i\,{}^i\dot{\omega}_i, \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + {}^{i+1}\mathbf{R}_i\,{}^i\boldsymbol{\omega}_i.
\end{aligned} \tag{3.25}$$

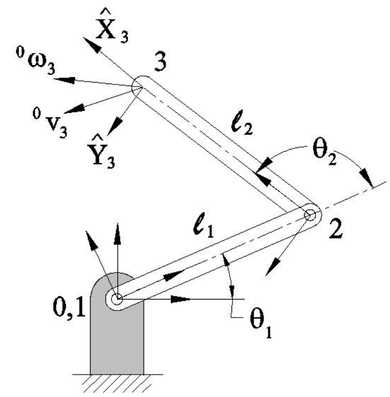
If Joint $\{i+1\}$ is a P-Pair

$${}^{i+1}\mathbf{R}_i\mathbf{v}_{i+1}{}^{i} = \dot{d}_{i+1}\,{}^{i+1}\hat{\mathbf{Z}}_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} \text{ and } {}^{i+1}\boldsymbol{\omega}_{i+1}{}^{i} = 0 \text{ because } \dot{\theta}_{i+1} = 0$$

$$\begin{aligned}
\Rightarrow \quad {}^{i+1}\mathbf{v}_{i+1} &= {}^{i+1}\mathbf{R}_i\,({}^i\mathbf{v}_i + {}^i\boldsymbol{\omega}_i \times {}^i\mathbf{p}_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{\mathbf{Z}}_{i+1}, \\
{}^{i+1}\boldsymbol{\omega}_{i+1} &= {}^{i+1}\mathbf{R}_i\,{}^i\boldsymbol{\omega}_i.
\end{aligned} \tag{3.26}$$

Applying Equations (3.25) and (3.26) iteratively from link to link, we can compute ${}^n\mathbf{v}_n$ and ${}^n\boldsymbol{\omega}_n$. The fact that they are in terms of the end-effector frame turns out to be useful. In terms of the base frame, one multiplication - that of ${}^0\mathbf{R}_n$ is required.

Example: Compute the velocity of the tip of the 2R manipulator as functions of joint rates. Supply ${}^3\mathbf{v}_3$, ${}^3\boldsymbol{\omega}_3$, ${}^0\mathbf{v}_3$, and ${}^0\boldsymbol{\omega}_3$.



Solution: We first assign frames.
Since all joints are revolute, we need Equations (3.25).

$$
\begin{aligned}
{}^0\mathbf{T}_1 &= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^1\mathbf{T}_2 &= \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^2\mathbf{T}_3 &= \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}
$$

And ${}^0\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, ${}^1\mathbf{p}_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$, ${}^2\mathbf{p}_3 = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}$.

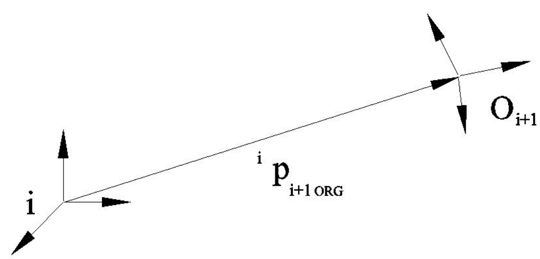
Angular Velocities:

$$i=0: \quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}^{1}\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + 0,$$

$$\Rightarrow \quad {}^{1}\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}.$$

$$\begin{aligned} i=1: \quad {}^{2}\boldsymbol{\omega}_2 &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + {}^{2}\mathbf{R}_1\, {}^{1}\boldsymbol{\omega}_1 \\ &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_2 & -s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}, \\ \Rightarrow \quad {}^{2}\boldsymbol{\omega}_2 &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \\ {}^{3}\boldsymbol{\omega}_3 &= {}^{2}\boldsymbol{\omega}_2. \end{aligned}$$

Note on homogeneous transforms:



$${}^{i}\mathbf{p}_{O_{i+1}} =^{i} \mathbf{T}_{i+1}\, {}^{i+1}\mathbf{p} = \left[\begin{array}{ccc|c} & & & x \\ & {}^{i}\mathbf{R}_{i+1,3\times 3} & & y \\ & & & z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] {}^{i+1}\mathbf{p},$$

where x, y, z are the coordinates of the origin of $\{i+1\}$ expressed in $\{i\}$.

Linear Velocities:

$$
\begin{aligned}
i=0: \quad {}^{i+1}\mathbf{v}_{i+1} \;=\; {}^{1}\mathbf{v}_1 &= {}^{1}\mathbf{R}_0({}^{0}\mathbf{v}_0 + {}^{0}\boldsymbol{\omega}_0 \times {}^{0}\mathbf{p}_1), \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\
i=1: \quad {}^{2}\mathbf{v}_2 &= {}^{2}\mathbf{R}_1\,({}^{1}\mathbf{v}_1 + {}^{1}\boldsymbol{\omega}_1 \times {}^{1}\mathbf{p}_2), \\
&= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix}, \\
&= \begin{bmatrix} l_1\dot{\theta}_1 s_2 \\ l_1\dot{\theta}_1 c_2 \\ 0 \end{bmatrix}, \\
i=2: \quad {}^{3}\mathbf{v}_3 &= {}^{3}\mathbf{R}_2({}^{2}\mathbf{v}_2 + {}^{2}\boldsymbol{\omega}_2 \times {}^{2}\mathbf{p}_3), \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} l_1\dot{\theta}_1 s_2 \\ l_1\dot{\theta}_1 c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta}_1 - \dot{\theta}_2 & 0 \\ \dot{\theta}_1\dot{\theta}_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \right), \\
&= \begin{bmatrix} l_1\dot{\theta}_1 s_2 \\ l_1\dot{\theta}_1 s_2 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Next: } {}^{0}\mathbf{R}_3 &= {}^{0}\mathbf{R}_1\,{}^{1}\mathbf{R}_2\,{}^{2}\mathbf{R}_3, \\
&= \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{I}_{3\times 3}, \\
&= \begin{bmatrix} c_1c_2 - s_1s_2 & -(c_1s_2 + c_2s_1) & 0 \\ s_1c_2 + c_1s_2 & c_1c_2 - s_1s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
&= \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\text{where } c_{12} &= c_1c_2 - s_1s_2 = \cos(\theta_1 + \theta_2), \\
\text{and } s_{12} &= c_1s_2 + s_1c_2 = \sin(\theta_1 + \theta_2).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
{}^0\boldsymbol{\omega}_3 &= {}^0\mathbf{R}_3\,{}^3\boldsymbol{\omega}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}, \\
{}^0\mathbf{v}_3 &= {}^0\mathbf{R}_3\,{}^3\boldsymbol{\theta}_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_1 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}.
\end{aligned}
$$

after some manipulation.

The main thing to notice is that we obtain the linear and angular end-effector velocities in terms of the joint rates!

$$
\left[\begin{array}{c} {}^3\mathbf{v}_3 \\ \hline {}^3\boldsymbol{\omega}_3 \end{array}\right] = \left[\begin{array}{cc} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{array}\right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.
$$

Of course, this planar manipulator has only 2 DOF. The linear velocity has only X and Y components, and is related to the joint rates by:

$$
\begin{bmatrix} {}^3\mathbf{v}_{3_X} \\ {}^3\mathbf{v}_{3_Y} \end{bmatrix} = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.
$$

The angular velocity is clearly the sum of the joint rates.

We can also write:

$$
\begin{bmatrix} {}^0\mathbf{v}_{3_X} \\ {}^0\mathbf{v}_{3_Y} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 - l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.
$$

3.4.2 Jacobians

A *Jacobian* is a time-varying transformation that relates Cartesian end-effector velocities to the joint rates. It is a multi-dimensional derivative. Suppose we had six functions, each a function of six independent variables:

$$
\left.\begin{aligned}
y_1 &= f_1(x_1, \ldots, x_6) \\
&\vdots \\
y_6 &= f_6(x_1, \ldots, x_6)
\end{aligned}\right\} \Leftrightarrow \mathbf{y} = F(\mathbf{x}).
$$

To compute the differentials of the y_i as a function of the differentials of the x_j, we use the chain rule and get:

$$
\left.\begin{aligned}
\delta y_1 &= \tfrac{\delta f_1}{\delta x_1}(\delta x_1) + \ldots + \tfrac{\delta f_6}{\delta x_6}(\delta x_6) \\
&\vdots \\
\delta y_6 &= \tfrac{\delta f_6}{\delta x_1}(\delta x_1) + \ldots + \tfrac{\delta f_6}{\delta x_6}(\delta x_6)
\end{aligned}\right\} \Leftrightarrow \delta\mathbf{y} = \frac{\delta F}{\delta \mathbf{x}}(\delta \mathbf{x})
$$

$$\frac{\delta F}{\delta \mathbf{x}} = \mathbf{J}(\mathbf{x}).$$

Dividing both sides by the differential time element, we get:

$$\dot{y} = \mathbf{J}(x)\dot{x}.$$

Because they depend on x, and x changes with time when the robot moves, Jacobians vary with time.

The Jacobian of a manipulator transforms the *vector* of joint rates, $\dot{\mathbf{q}}$, to the Cartesian velocities of the end-effector, $\mathbf{v}$.

$$^0\mathbf{v}_E = {}^0\mathbf{J}(q)\dot{\mathbf{q}},$$

$$^0\mathbf{v}_E = \begin{bmatrix} ^0v_x \\ ^0v_y \\ ^0v_z \\ ^0\omega_x \\ ^0\omega_y \\ ^0\omega_z \end{bmatrix}, \quad \dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} \Rightarrow \text{for example: } \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \\ \dot{d}_4 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix}.$$

The Jacobian is determined by applying Equations (3.25) and (3.26). In general, they are $m \times n$ matrices.

$$\begin{aligned} n &= \text{the number of } \textit{columns} = \text{the number of joints.} \\ m &= \text{the number of } \textit{rows} = \text{DOF in Cartesian space} \\ & \text{(in the largest square sub-matrix, if not square).} \end{aligned}$$

Partition

$$\mathbf{J}_{6\times n} = \left[\begin{array}{c} \mathbf{J}_{L_{3\times n}} \\ \hline \mathbf{J}_{A_{3\times n}} \end{array}\right],$$

where $\mathbf{J}_{L_{3\times n}}$ is the effect of $\dot{\mathbf{q}}$ on linear end-effector velocity $^0\mathbf{v}_E$,and $\mathbf{J}_{A_{3\times n}}$ is the effect of $\dot{\mathbf{q}}$ on the angular velocity of the end-effector, $^0\boldsymbol{\omega}_E$.

Using the *Velocity Propagation* Equations (3.25) and (3.26), we can obtain:

$$\left.\begin{aligned} ^E\mathbf{v}_E &= {}^E\mathbf{J}_L\dot{\mathbf{q}} \\ ^E\boldsymbol{\omega}_E &= {}^E\mathbf{J}_A\dot{\mathbf{q}} \end{aligned}\right\} \Leftrightarrow {}^E\mathbf{v}_E = {}^E\mathbf{J}\dot{\mathbf{q}}.$$

The first issue is how to transform a Jacobian (because we get $^E\mathbf{J}$, but need $^0\mathbf{J}$). We have seen that we change reference frames for $\mathbf{v}$ and $\boldsymbol{\omega}$ by rotations:

$$\left.\begin{aligned} ^0\mathbf{v}_E &= {}^0\mathbf{R}_E\,{}^E\mathbf{v}_E \\ ^0\boldsymbol{\omega}_E &= {}^0\mathbf{R}_E\,{}^E\boldsymbol{\omega}_E \end{aligned}\right\} \Leftrightarrow \left[\begin{array}{c} ^0\mathbf{v}_E \\ \hline ^0\boldsymbol{\omega}_E \end{array}\right] = \left[\begin{array}{c|c} ^0\mathbf{R}_E & \mathbf{0} \\ \hline \mathbf{0} & ^0\mathbf{R}_E \end{array}\right] \left[\begin{array}{c} ^E\mathbf{v}_E \\ \hline ^E\boldsymbol{\omega}_E \end{array}\right],$$

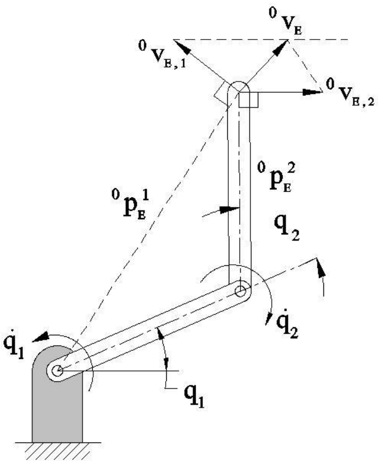
$$^0\mathbf{v}_E = \left[\begin{array}{c|c} ^0\mathbf{R}_E & \mathbf{0} \\ \hline \mathbf{0} & ^0\mathbf{R}_E \end{array}\right] {}^E\mathbf{v}_E.$$

But, $^E\mathbf{J}\dot{\mathbf{q}} = {}^E\mathbf{v}_E$, and $^0\mathbf{J}\dot{\mathbf{q}} = {}^0\mathbf{v}_E$, so

$$^0\mathbf{J} = \left[\begin{array}{c|c} ^0\mathbf{R}_E & \mathbf{0} \\ \hline \mathbf{0} & ^0\mathbf{R}_E \end{array}\right] {}^E\mathbf{J}.$$

Computing the Jacobian

Each column of the Jacobian is associated with one joint velocity. Consider the 2R planar platform we looked at: each column of the Jacobian maps a particular joint rate onto the velocity of the end-effector.



${}^0\mathbf{p}_E{}^i$ = Position vector of O_E with respect to frame $\{i\}$ but expressed in terms of frame $\{0\}$.

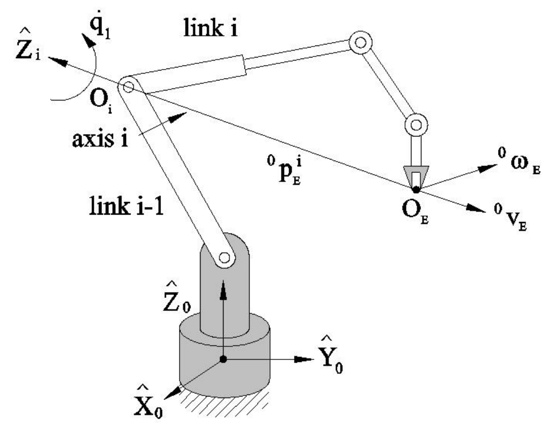
$$\begin{array}{rcl}
\text{vector} & & \text{scalar}\quad\text{vector} \\
\downarrow & & \downarrow \quad \swarrow \\
{}^0\mathbf{v}_{E,1} & = & \dot{q}_1\,\mathbf{J}_{L,1}, \\
{}^0\mathbf{v}_{E,2} & = & \dot{q}_2\,\mathbf{J}_{L,2}, \\
{}^0\mathbf{v}_{E} & = & {}^0\mathbf{v}_{E,1} + {}^0\mathbf{v}_{E,2}.
\end{array}$$

<u>General Derivation of $\mathbf{J}$:</u>

In the figure on the next page, $\begin{bmatrix} {}^0\mathbf{v}_E \\ {}^0\boldsymbol{\omega}_E \end{bmatrix} = {}^0\mathbf{J}\dot{\mathbf{q}}$. Find $\mathbf{J}$.

We construct the Jacobian by considering each joint rate starting from $i = 1$.

$$\begin{aligned}
{}^0\mathbf{v}_E &= \mathbf{J}_{L,1}\,\dot{q}_1 + \ldots + \mathbf{J}_{L,n}\,\dot{q}_n, \\
&= \begin{bmatrix} J_{L11} & J_{L12} & J_{L13} \\ J_{L21} & J_{L22} & J_{L23} \\ J_{L31} & J_{L32} & J_{L33} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}, \\
&= \begin{bmatrix} J_{L11} \\ J_{L21} \\ J_{L31} \end{bmatrix} \dot{q}_1 + \begin{bmatrix} J_{L12} \\ J_{L22} \\ J_{L32} \end{bmatrix} \dot{q}_2 + \begin{bmatrix} J_{L13} \\ J_{L23} \\ J_{L33} \end{bmatrix} \dot{q}_2, \\
{}^0\boldsymbol{\omega}_E &= \mathbf{J}_{A,1}\dot{q}_1 + \ldots + \mathbf{J}_{A,n}\dot{q}_n.
\end{aligned}$$



If the i^{th} joint is an R-pair:

$$\begin{aligned} {}^0\mathbf{v}_{E,i} &= ({}^0\hat{\mathbf{Z}}_i \times {}^0\mathbf{p}_E{}^i)\,\dot{q}_i \;=\; \mathbf{J}_{L,i}\,\dot{q}_i, \\ {}^0\boldsymbol{\omega}_{E,i} &= {}^0\hat{\mathbf{Z}}_i\,\dot{q}_0 \;=\; \mathbf{J}_{A,i}\,\dot{q}_i. \end{aligned}$$

Note that in the above equations, $\dot{q}_i$ is scalar, since it is the angular velocity imparted by the R-pair, $\dot{\theta}_i$.

If the i^{th} joint is a P-pair:

$$\begin{aligned} {}^0\mathbf{v}_{E,i} &= {}^0\hat{\mathbf{Z}}_i\,\dot{q}_i \;=\; \mathbf{J}_{L,i}\,\dot{q}_i, \\ {}^0\boldsymbol{\omega}_{E,i} &= \mathbf{J}_{A,i}\,\dot{q}_i. \end{aligned}$$

Note that in the above equations, $\dot{q}_i$ is scalar, since it is the linear velocity imparted by the P-pair, $\dot{d}_i$.

Assemble ${}^0\mathbf{J}$:

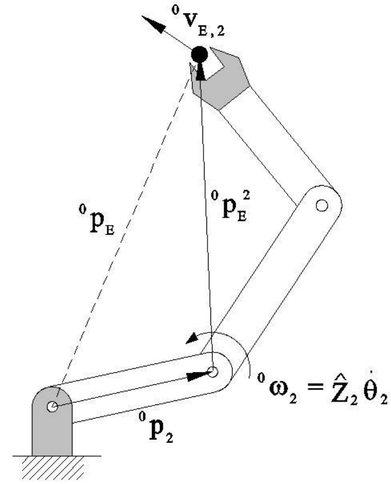
$$\begin{aligned} {}^0\mathbf{J}_{6\times n} &= \begin{bmatrix} \mathbf{J}_{L_1} & \mathbf{J}_{L_2} & \ldots & \mathbf{J}_{L_n} \\ \mathbf{J}_{A_1} & \mathbf{J}_{A_2} & \ldots & \mathbf{J}_{A_n} \end{bmatrix}, \\ &= \left[\begin{array}{c|c|c|c|c} {}^0\hat{\mathbf{Z}}_1 \times {}^0\mathbf{p}_E{}^1 & {}^0\hat{\mathbf{Z}}_2 \times {}^0\mathbf{p}_E{}^2 & {}^0\hat{\mathbf{Z}}_3 & \ldots & {}^0\hat{\mathbf{Z}}_n \times {}^0\mathbf{p}_E{}^n \\ \hline {}^0\hat{\mathbf{Z}}_1 & {}^0\hat{\mathbf{Z}}_2 & 0 & \ldots & {}^0\hat{\mathbf{Z}}_n \end{array}\right], \\ &\quad\;\; \begin{array}{ccccc} (\text{R-pair} & \text{R-pair} & \text{P-pair} & \ldots & \text{R-pair}) \end{array}. \end{aligned}$$

${}^0\hat{\mathbf{Z}}_i$ and ${}^0\mathbf{p}_E{}^i$ can be obtained directly from the homogeneous transformations, ${}^0\mathbf{T}_i$.

$${}^0\mathbf{T}_i = \left[\begin{array}{c|c|c|c} \mathbf{r}_i & \mathbf{r}_2 & \mathbf{r}_3 & {}^0\mathbf{p}_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

${}^0\hat{\mathbf{Z}}_i$: 3rd column of rotation matrix embedded in ${}^0\mathbf{T}_i$ ($\mathbf{r}_3$)
${}^0\mathbf{p}_i$: 4th column of ${}^0\mathbf{T}_i$ (ignore 4th row).

$$
\begin{aligned}
{}^0\mathbf{p}_E{}^i &= {}^0\mathbf{p}_E - {}^0\mathbf{p}_i, \\
&\Rightarrow \underbrace{\left[\begin{array}{cc|c} & & {}^0\mathbf{p}_E \\ \hline & & \end{array}\right]}_{{}^0\mathbf{T}_E} - \underbrace{\left[\begin{array}{cc|c} & & {}^0\mathbf{p}_i \\ \hline & & \end{array}\right]}_{{}^0\mathbf{T}_i},
\end{aligned}
$$



$$
\begin{aligned}
\text{Let } i = 2: \quad {}^0\mathbf{p}_E{}^2 + {}^0\mathbf{p}_2 &= {}^0\mathbf{p}_E, \\
{}^0\mathbf{p}_E{}^2 &= \mathbf{p}_E - {}^0\mathbf{p}_2, \\
{}^0\mathbf{v}_{E,2} &= \omega_2 \times {}^0\mathbf{p}_E{}^2, \\
&= ({}^0\hat{\mathbf{Z}}_2 \times {}^0\mathbf{p}_E{}^2)\dot{\theta}_2.
\end{aligned}
$$

Jacobian Example:

Velocity of interest: ${}^0\mathbf{v}_E = \begin{bmatrix} {}^0v_{Ex} \\ {}^0v_{Ey} \end{bmatrix}$. Determine ${}^0\mathbf{J}$ such that:

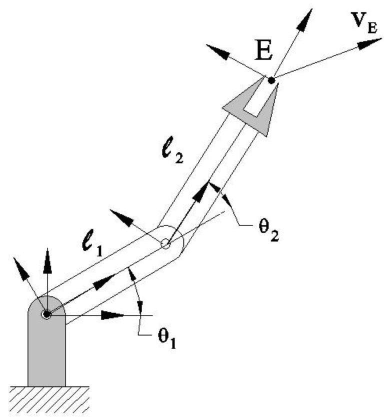


Figure 3.28: 2R planar manipulator.

$$\begin{bmatrix} {}^0v_{Ex} \\ {}^0v_{Ey} \end{bmatrix} = {}^0\mathbf{J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.$$

Solution:

1. To assemble ${}^0\mathbf{J} = [{}^0\hat{\mathbf{Z}}_1 \times {}^0\mathbf{p}_E{}^1 \mid {}^0\hat{\mathbf{Z}}_2 \times {}^0\mathbf{p}_E{}^2]$ we need ${}^0\mathbf{T}_1$, ${}^0\mathbf{T}_2$, ${}^0\mathbf{T}_E$.

$${}^0\mathbf{T}_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1\mathbf{T}_2 = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^2\mathbf{T}_E = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Compute } {}^0\mathbf{T}_E = {}^0\mathbf{T}_1\,{}^1\mathbf{T}_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 \\ -s_{12} & c_{12} & 0 & l_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Compute } {}^0\mathbf{T}_E = {}^0\mathbf{T}_2\,{}^2\mathbf{T}_3 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_2 \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Obtain ${}^0\hat{\mathbf{Z}}_1$, ${}^0\hat{\mathbf{Z}}_2$, ${}^0\mathbf{p}_E{}^1 = {}^0\mathbf{p}_E - {}^0\mathbf{p}_1$, ${}^0\mathbf{p}_E{}^2 = {}^0\mathbf{p}_E - {}^0\mathbf{p}_2$,

$$
\begin{aligned}
{}^0\hat{\mathbf{Z}}_1 &= {}^0\hat{\mathbf{Z}}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
{}^0\hat{\mathbf{Z}}_{1,2}{}^x &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \left(CPM = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \right), \\
{}^0\mathbf{p}_E{}^1 &= \begin{bmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \\ 0 \end{bmatrix}, \\
{}^0\mathbf{p}_E{}^2 &= \begin{bmatrix} l_2c_{12} \\ l_2s_{12} \\ 0 \end{bmatrix},
\end{aligned}
$$

3. Assemble ${}^0\mathbf{J}$:

$$
\begin{aligned}
{}^0\mathbf{J} &= \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} \\ l_1c_1 + l_2c_{12} & l_2c_{12} \end{bmatrix}, \\
{}^E\mathbf{J} &= \begin{bmatrix} l_1s_2 & 0 \\ l_1c_2 + l_2 & l_2 \end{bmatrix}, \\
\text{from: } {}^E\mathbf{J} &= {}^E\mathbf{R}_0\,{}^0\mathbf{J}.
\end{aligned}
$$

Compare this with the previous example.

3.4.3 Singularities

Using ${}^0\mathbf{J}$ we can compute ${}^0\mathbf{v}_E$ given any prescribed joint rate vector $\dot{\mathbf{q}}$. If we want to determine the joint rates for a given ${}^0\mathbf{v}_E$, we must determine the inverse of ${}^0\mathbf{J}$.

${}^0\mathbf{J}$ becomes non-invertible if it loses full rank (if it is square). The determinant of a rank-deficient matrix is always 0. Since $\mathbf{J}$ is a function of the joint variables, this means there may be some poses which cause $\mathrm{Det}(\mathbf{J}) = 0$. Such poses are called *singular poses* or *singularities*.

Without being too rigorous, we can classify singularities with two categories:

1. <u>Workspace Boundary Singularities:</u> These occur when the manipulator is fully stretched out, or folded back on itself to that the end-effector is at the boundary of the workspace.

2. <u>Workspace Interior Singularities:</u> These occur within the workspace. They can be due to many things, but are typically caused when two or more joint axes become coincident.

To obtain an expression for ${}^0\mathbf{J}$, we must transform ${}^3\mathbf{v}_3$ to the fixed frame $\{0\}$. This is accomplished by left-multiplying ${}^3\mathbf{v}_3$ by the rotation matrix embedded in ${}^0\mathbf{T}_3$.

$${}^0\mathbf{v}_3 = {}^0\mathbf{R}_3\ {}^3\mathbf{v}_3 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_1 c_2)\dot{\theta}_1 - l_2 c_1 s_2 \dot{\theta}_2 \\ (l_1 c_1 + l_2 c_1 c_2)\dot{\theta}_1 - l_2 s_1 s_2 \dot{\theta}_2 \\ -l_2 c_2 \dot{\theta}_2 \end{bmatrix}.$$

This system contains three equations and two unknowns, $\dot{\theta}_1$ and $\dot{\theta}_2$, given the linear velocity vector ${}^0\mathbf{v}_3$, as expected.

We can re-express the right hand side of the above equation as:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_1 s_2) & -l_1 c_1 s_2 \\ l_1 c_1 + l_2 c_1 c_2 & -l_2 s_1 s_2 \\ 0 & -l_2 c_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = {}^0\mathbf{J}\dot{\boldsymbol{\theta}}.$$

The 3×2 matrix is not immediately invertible, but to obtain the joint rates in terms of ${}^0\mathbf{v}_3$, we must invert it. This can be accomplished with the Moore-Penrose generalized inverse. We have an over-determined system of equations: three equations and two unknowns. In general, no exact solutions exist to such a system, and we find approximate solutions that minimize the error. This can be done with a *least squares* approach. This is exactly what the Moore-Penrose generalized inverse does!

It is defined to be:

$$\mathbf{J}^T = (\mathbf{J}^T\ \mathbf{J})^{-1}\ \mathbf{J}^T.$$

In our case, this gives:

$$\mathbf{J}^T = \begin{bmatrix} -\frac{s_1}{l_1 + l_2 c_2} & \frac{c_1}{l_1 + l_2 c_2} & 0 \\ -\frac{c_1 s_2}{l_2} & -\frac{s_1 s_2}{l_2} & -\frac{c_2}{l_2} \end{bmatrix}.$$

When a serial robot is in a singular configuration, it loses one or more DOF. There is some direction(s), or subspace in Cartesian space in which the end-effector can not move, regardless of what the joint rates are. Clearly, this happens at the boundary of the workspace.

Example: Determine an expression describing the singularities of the 2R robot from the previous example (Figure 3.28).

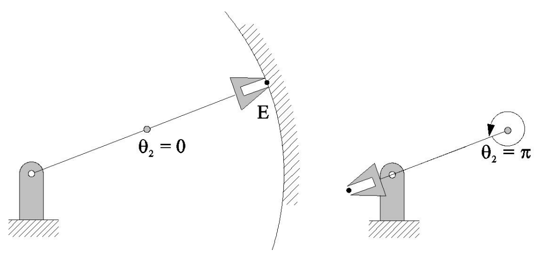
Solution: We have two Jacobians. ${}^0\mathbf{J}$ and ${}^E\mathbf{J}$. If a matrix is rank-deficient, the deficiency won't change if it is multiplied by any other matrix. This is a property of determinants:

$$\begin{aligned} \mathrm{Det}(\mathbf{AB}) &= \mathrm{Det}(\mathbf{A})\,\mathrm{Det}(\mathbf{B}), \\ \text{If } \mathrm{Det}(\mathbf{B}) &= 0, \ \Rightarrow \ \mathrm{Det}\left(\begin{bmatrix} {}^0\mathbf{R}_E & \mathbf{0} \\ \mathbf{0} & {}^0\mathbf{R}_E \end{bmatrix} {}^E\mathbf{J} \right) \ = \ 0, \\ \text{In general } \mathrm{Det}({}^E\mathbf{J}) &= 1\,\mathrm{Det}({}^0\mathbf{J}), \\ \Rightarrow \ \mathrm{Det}({}^E\mathbf{J}) &= \mathrm{Det}({}^0\mathbf{J}), \\ \Rightarrow \ \text{if } \mathrm{Det}({}^E\mathbf{J}) &= 0, \text{ then } \mathrm{Det}({}^0\mathbf{J}) \ = \ 0. \end{aligned}$$

(Note, in general, $\mathrm{Det}({}^E\mathbf{J}) = \mathrm{Det}({}^n\mathbf{J})$, since $\mathrm{Det}(\mathbf{R}) = 1$.) It's easier to compute $\mathrm{Det}({}^E\mathbf{J})$:

$$\mathrm{Det}\begin{pmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{pmatrix} = l_1 l_2 s_2.$$

The robot is in a singular configuration when $l_1 l_2 s_2 = 0$. This happens if l_1 or $l_2 = 0$, or $s_2 = 0$. $\sin\theta_2 = 0$ when $\theta_2 = 0, \pi$.



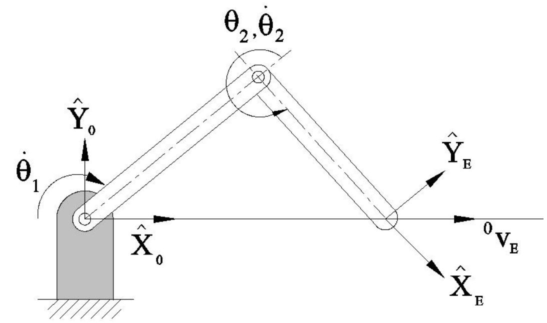
- $\theta_2 = 0 \ \ \Rightarrow$ boundary E can only move perpendicular to the line of $l_1 + l_2$.
- $\theta_2 = \pi \ \ \Rightarrow$ boundary same condition on motion of E.

Singularities should be avoided. Also, using the inverse Jacobian in the control system may be a bad idea. If $\mathbf{J}$ becomes singular, the joint rates become infinite! For 2×2 matrices:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \ \mathbf{A}^{-1} = \frac{1}{ad + bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{J}^{-1}\mathbf{v}, \\ &= \frac{\mathrm{Adj}(\mathbf{J})}{\mathrm{Det}(\mathbf{J})}\mathbf{v}, \\ &= \frac{\mathrm{Adj}(\mathbf{J})}{0}\mathbf{v} \ = \ \infty. \end{aligned}$$

Example: Determine the joint rates of the 2R manipulator in the figure below, so the end-effector has a constant velocity of 1 m/s along $\hat{\mathbf{X}}_0$. What happens when the end-effector is close to the workspace boundary?



$$\begin{aligned}
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
&= \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_2 l_1 c_1 - l_2 c_{12} \end{bmatrix}.
\end{aligned}$$

$$\text{As } \theta_2 \to 360^o (0^o),\ s_2 \to 0,\ \dot{\theta}_1, \dot{\theta}_2 \to \infty.$$

Obtaining the Jacobian by Differentiating the Closure Equations

Assume the kinematic closure equations exist, i.e.

$$^0\mathbf{p}_E =^0 \mathbf{T}_E\ ^E\mathbf{p}_E,$$

with

$$^0\mathbf{T}_E = \left[\begin{array}{ccc|c} & & & ^0x_E \\ & \mathbf{R}_k(\theta) & & ^0y_E \\ & & & ^0z_E \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

Here it is normal to select Euler angles to describe the orientation, although the fixed angles could also be used. However, it seems that unit quaternion or element-angle-axis representations could be difficult to use.

Let's choose Euler angles, it doesn't matter which. So

$$^0\mathbf{R}_E = F(\alpha_E, \beta_E, \gamma_E) = F(\theta_{E1}, \theta_{E2}, \theta_{E3}).$$

So $^0\mathbf{T}_E$ gives the position vector, and we can extract the Euler Angles from $\mathbf{R}$.

So we have:

$$\left.\begin{array}{rl} {}^0x_E = & f_x(q) \\ {}^0y_E = & f_y(q) \\ {}^0z_E = & f_z(q) \\ \alpha_E = & f_{\alpha E}(q) \\ \beta_E = & f_{\beta E}(q) \\ \gamma_E = & f_{\gamma E}(q) \end{array}\right\} \Rightarrow \begin{array}{rl} {}^0\dot{x}_E = & \frac{\delta f_x}{\delta q_1}\dot{q}_1 + \ldots + \frac{\delta f_x}{\delta q_n}\dot{q}_n \\ {}^0\dot{y}_E = & \frac{\delta f_y}{\delta q_1}\dot{q}_1 + \ldots + \frac{\delta f_y}{\delta q_n}\dot{q}_n \\ {}^0\dot{z}_E = & \frac{\delta f_z}{\delta q_1}\dot{q}_1 + \ldots + \frac{\delta f_z}{\delta q_n}\dot{q}_n \\ \dot{\alpha}_E = & \frac{\delta f_\alpha}{\delta q_1}\dot{q}_1 + \ldots + \frac{\delta f_\alpha}{\delta q_n}\dot{q}_n \\ \dot{\beta}_E = & \frac{\delta f_\beta}{\delta q_1}\dot{q}_1 + \ldots + \frac{\delta f_\beta}{\delta q_n}\dot{q}_n \\ \dot{\gamma}_E = & \frac{\delta f_\gamma}{\delta q_1}\dot{q}_1 + \ldots + \frac{\delta f_\gamma}{\delta q_n}\dot{q}_n \end{array}.$$

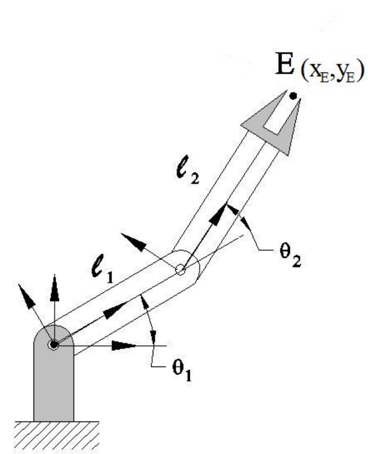
Note that α_E, β_E, γ_E depend on the Euler angle representation.

We define:

$$\begin{aligned} {}^0\dot{\mathbf{x}}_E &= \frac{d}{dt}\begin{bmatrix} {}^0x_E \\ \vdots \\ {}^0\gamma_E \end{bmatrix}, \\ &= \begin{bmatrix} {}^0\mathbf{v}_E \\ \dot{\boldsymbol{\theta}}_E \end{bmatrix}, \\ &= \begin{bmatrix} \frac{\delta f_x}{\delta q_1} & \cdots & \frac{\delta f_x}{\delta q_n} \\ \vdots & & \vdots \\ \frac{\delta f_\gamma}{\delta q_1} & \cdots & \frac{\delta f_\gamma}{\delta q_n} \end{bmatrix}\dot{\mathbf{q}}, \\ &= \mathbf{J}_D\,\dot{\mathbf{q}}. \end{aligned}$$

Where ${}^0\mathbf{v}_E$ and ${}^0\boldsymbol{\theta}_E$ are the *Euler rates*, and the subscript$D$ in $\mathbf{J}_D$ stands for differentiation.

<u>2R Example:</u>



$$
\begin{aligned}
{}^0x_E &= l_1c_1 + l_2c_{12}, \\
{}^0y_E &= l_1s_1 + l_2s_{12}, \\
{}^0\dot{x}_E &= -l_1s_1\dot{\theta}_1 - l_2s_{12}(\dot{\theta}_1 + \dot{\theta}_2), \\
{}^0\dot{y}_E &= l_1c_1\dot{\theta}_1 + l_2c_{12}(\dot{\theta}_1 + \dot{\theta}_2).
\end{aligned}
$$

Here,

$$
{}^0\mathbf{J}_D \;=\; {}^0\mathbf{J} = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} \\ l_1c_1 + l_2c_{12} & l_2c_{12} \end{bmatrix}.
$$

This is always true in the plane. This may be a better way to obtain the Jacobian, depending on the complexity of the closure equations.

Relationship Between Angular Velocities

We want the relationship ${}^0\boldsymbol{\omega}_E \;=\; {}^0\mathbf{J}_E\dot{\boldsymbol{\theta}}_E$ for any Euler (or fixed angle) set. Recall that:

$$
\dot{\mathbf{R}}\mathbf{R}^T \;=\; \mathbf{\Omega} \;=\; \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{3.27}
$$

From the matrix in Equation (3.27), we easily obtain, with $\mathbf{R}$ in terms of the appropriate angle set:

$$
\begin{aligned}
\omega_x &= \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23}, \\
\omega_y &= \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33}, \\
\omega_z &= \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13}.
\end{aligned}
$$

But this is a lot of computation. Consider the following geometric derivation. Suppose we are using Z-Y-Z Euler angles (α, β, γ). The first Euler rate, $\dot{\alpha}$, is an angular velocity of the end-effector about the $\hat{\mathbf{Z}}_0$ axis. The next, $\dot{\beta}$, is about the $\hat{\mathbf{Y}}_0{}'$ axis that has been transformed by the rotation matrix $\mathbf{R}_{\hat{\mathbf{Z}}_0}(\alpha)$. The third, $\dot{\gamma}$ is about the $\hat{\mathbf{Z}}_0{}''$ axis that has been transformed by the rotation matrix $\mathbf{R}_{\hat{\mathbf{Y}}_0{}'}(\beta)$, which in turn had been transformed by $\mathbf{R}_{\hat{\mathbf{Z}}_0}(\alpha)$. So we can write ${}^0\boldsymbol{\omega}_E$ in terms of $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ as:

$$
{}^0\boldsymbol{\omega}_E \;=\; \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix} + \mathbf{R}_{\hat{\mathbf{Z}}_0}(\alpha) \begin{bmatrix} 0 \\ \dot{\beta} \\ 0 \end{bmatrix} + \mathbf{R}_{\hat{\mathbf{Z}}_0}(\alpha)\mathbf{R}_{\hat{\mathbf{Y}}_0{}'}(\beta) \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix}, \tag{3.28}
$$

with:

$$\mathbf{R}_{\hat{\mathbf{Z}}_0}(\alpha) = \begin{bmatrix} \mathrm{c}\,\alpha & -\mathrm{s}\,\alpha & 0 \\ \mathrm{s}\,\alpha & \mathrm{c}\,\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{R}_{\hat{\mathbf{Y}}_0{}'}(\beta) = \begin{bmatrix} \mathrm{c}\,\beta & 0 & -\mathrm{s}\,\beta \\ 0 & 1 & 0 \\ -\mathrm{s}\,\beta & 0 & \mathrm{c}\,\beta \end{bmatrix},$$

$$\mathbf{R}_{\hat{\mathbf{Z}}_0}(\alpha)\mathbf{R}_{\hat{\mathbf{Y}}_0{}'}(\beta) = \begin{bmatrix} \mathrm{c}\,\alpha\,\mathrm{c}\,\beta & -\mathrm{s}\,\beta & \mathrm{c}\,\alpha\,\mathrm{s}\,\beta \\ \mathrm{s}\,\alpha\,\mathrm{c}\,\beta & \mathrm{c}\,\alpha & \mathrm{s}\,\alpha\,\mathrm{s}\,\beta \\ -\mathrm{s}\,\beta & 0 & \mathrm{c}\,\beta \end{bmatrix}.$$

Expanding (3.28), then rewriting it in matrix form operating on the vector of Euler rates in order $\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$, we obtain:

$$\begin{aligned} {}^0\boldsymbol{\omega}_E &= \begin{bmatrix} 0 & -\mathrm{s}\,\alpha & \mathrm{c}\,\alpha\,\mathrm{s}\,\beta \\ 0 & \mathrm{c}\,\alpha & \mathrm{s}\,\alpha\,\mathrm{s}\,\beta \\ 1 & 0 & \mathrm{c}\,\beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}, \\ &= \mathbf{J}_{E_{Z'Y'Z'}}(\alpha, \beta, \alpha) \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}. \end{aligned}$$

This gives exactly the same results as explicitly performing the differentiations of $\dot{\mathbf{R}}\mathbf{R}^T$, but is significantly easier to compute. In general we have:

$$\begin{aligned} \begin{bmatrix} {}^0\mathbf{v}_E \\ {}^0\boldsymbol{\omega}_E \end{bmatrix} &= \begin{bmatrix} {}^0\mathbf{v}_E \\ {}^0\mathbf{J}_E\dot{\boldsymbol{\theta}}_E \end{bmatrix}, \\ &= \left[\begin{array}{c|c} {}^0\mathbf{I}_{3\times 3} & \mathbf{0} \\ \hline \mathbf{0} & {}^0\mathbf{J}_E \end{array}\right] \begin{bmatrix} {}^0\mathbf{v}_E \\ {}^0\dot{\boldsymbol{\theta}}_E \end{bmatrix}, \\ &= \left[\begin{array}{c|c} {}^0\mathbf{I}_{3\times 3} & \mathbf{0} \\ \hline \mathbf{0} & {}^0\mathbf{J}_E \end{array}\right] {}^0\mathbf{J}_D\,\dot{\mathbf{q}}, \\ {}^0\mathbf{J}_D &= \left[\begin{array}{c|c} {}^0\mathbf{I}_{3\times 3} & \mathbf{0} \\ \hline \mathbf{0} & {}^0\mathbf{J}_E{}^{-1} \end{array}\right] {}^0\mathbf{J}. \end{aligned}$$

Note that ${}^0\mathbf{J}_E{}^{-1}$ may lead to representational singularities, therefore $\mathbf{J}_D$ may not be always obtainable, but $\mathbf{J}$ is.

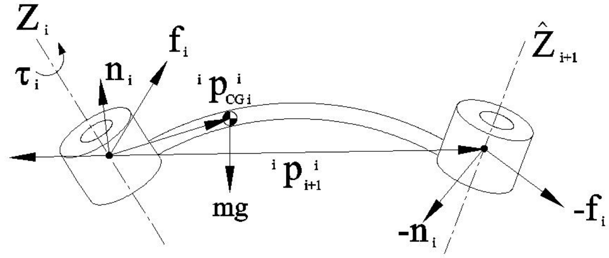
Chapter 4

Static Forces in Serial Robots

Statics is the study of force systems acting on stationary structures. There is no motion. When all the joints in a robot are locked, it should become a structure. If the task is to apply a force, or support a load while maintaining static equilibrium, the problem at hand is to compute the joint forces and torques which must be supplied in order to keep the system in static equilibrium.

When locked in a desired pose, we write a force-moment balance for each link in terms of the link frames. However, here we start where the force is applied: the end-effector. We compute the static torque acting about the joint axis to maintain equilibrium. We need the following definitions:

$$\begin{aligned}
\mathbf{f}_i &= \text{force exerted on link } i \text{ by link } i-1. \\
\mathbf{n}_i &= \text{torque exerted on link } i \text{ by link } i-1. \\
\mathbf{p}_{\mathrm{CG}_i}{}^i &= \text{position vector of center of gravity of link } i \text{ with respect to } i. \\
\mathbf{p}_{i+1}{}^i &= \text{position vector of } O_{i+1} \text{ with respect to } i.
\end{aligned}$$



The above figure shows the free body diagram of the i^{th} link. $\mathbf{f}_i$ is the force exerted on i by $i-1$. $\mathbf{f}_{i+1}$ is the reaction experienced by i due to the force it exerts on $i+1$. The same applies for $\mathbf{n}_i$ and $\mathbf{n}_{i+1}$.

Force Balance:

$$^i\mathbf{f}_i - {}^i\mathbf{f}_{i+1} + m_i\mathbf{g} = \mathbf{0}.$$

Condition for Force Equilibrium:

$$^i\mathbf{f}_i \quad = \quad {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{f}_{i+1} - m_i\mathbf{g}. \tag{4.1}$$

The recursion is arranged to go from the end-effector to lower numbered links to the base.

Moment Balance With Respect to the Center of Gravity (CG):
(no mg contribution)

$$^i\mathbf{n}_i \;=\; {}^i\mathbf{n}_{i+1} + {}^0\mathbf{p}_{\mathrm{CG}_i}{}^i \times {}^i\mathbf{f}_i + ({}^i\mathbf{p}_{i+1}{}^i - {}^i\mathbf{p}_{\mathrm{CG}_i}{}^i) \times \mathbf{f}_{i+1}.$$

Conditions for Moment Equilibrium:
With respect to CG:

$$^i\mathbf{n}_i \quad = \quad {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{n}_{i+1} + {}^i\mathbf{p}_{\mathrm{CG}_i}{}^i \times {}^i\mathbf{f}_i + ({}^i\mathbf{p}_{i+1}{}^i - {}^i\mathbf{p}_{\mathrm{CG}_i}{}^i) \times {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{f}_{i+1}. \tag{4.2}$$

With respect to O_i:

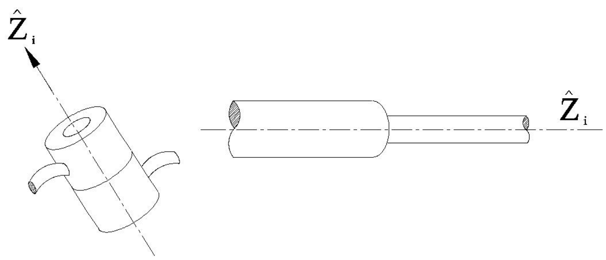
$$^i\mathbf{n}_i \quad = \quad {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{n}_{i+1} - n_i({}^i\mathbf{p}_{\mathrm{CG}_i}{}^i \times {}^i\mathbf{g}) + {}^i\mathbf{p}_{i+1}{}^i \times {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{f}_{i+1}. \tag{4.3}$$

If we start with a (known) description of the resultant force and moment, due to the actions contributed by each link from n to 0, i.e. from the end-effector to the base. To do this, we iteratively apply Equation (4.1) and either (4.2) or (4.3) from higher to lower numbered links.

Note: ${}^i\mathbf{f}_i$ and ${}^i\mathbf{n}_i$ are the total forces and moment that the i^{th} link experiences due to the other bodies connected to it, including the effects of gravity. It does not explicitly include the motor torques required to balance the ${}^i\mathbf{f}_i$ and ${}^i\mathbf{n}_i$ to maintain static equilibrium.

4.1 Actuator Torques (Forces)

All components of force and moment vectors are resisted by the structure of the robot itself, except for those components in the direction of the $\hat{\mathbf{Z}}_i$, the joint axes. To determine the joint torque, or force, τ_i, (we'll use the same symbol for both) required to maintain static equilibrium, we only need the projection of



the resultant moment, or force vector acting on that joint onto the joint axes. We generally only care about the magnitude of the torque, so all we need are the following *dot* products:

$$\begin{aligned} \text{For R-Pairs: } \tau_i &= {}^i\mathbf{n}_i{}^T\, {}^i\hat{\mathbf{Z}}_i. && (4.4)\\ \text{For P-Pairs: } \tau_i &= {}^i\mathbf{f}_i\, {}^i\hat{\mathbf{Z}}_i. && (4.5)\end{aligned}$$

4.2 Ignoring the Effect of Gravity on Link Mass

If the applied end-effector forces and link reactions are far in excess of the link weights, we may ignore the gravity terms. Furthermore, a moment balance about the CG does nothing for us.

Equations (4.1) and (4.3) simplify to:

$$\begin{aligned} {}^i\mathbf{f}_i &= {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{f}_{i+1} && (4.6)\\ {}^i\mathbf{n}_i &= {}^i\mathbf{p}_{i+1}{}^i \times {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{f}_{i+1} + {}^i\mathbf{R}_{i+1}\, {}^{i+1}\mathbf{n}_{i+1}. && (4.7)\end{aligned}$$

Note: Equation (4.6) simply states that ${}^i\mathbf{f}_i = {}^i\mathbf{f}_{i+1}$. Equations (4.6) and (4.7) can be rewritten as:

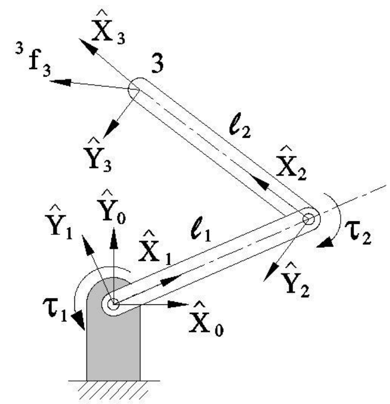
$$\begin{bmatrix} {}^i\mathbf{f}_i \\ {}^i\mathbf{n}_i \end{bmatrix} = \begin{bmatrix} {}^i\mathbf{R}_{i+1} & 0 \\ ({}^i\mathbf{p}_{i+1}{}^i) \times {}^i\mathbf{R}_{i+1} & {}^i\mathbf{R}_{i+1} \end{bmatrix}. \qquad (4.8)$$

4.3 Jacobians in the Force Domain

When a force acts on a mechanism, work is done on the mechanism by the force in proportion to the amount of displacement the mechanism has undergone. Work is defined as the scalar product of the force and displacement magnitudes. Work has units of energy: J = N·m.

$$\begin{aligned} dW &= \mathbf{F}\cdot d\mathbf{s},\\ \Rightarrow\ W &= \int \mathbf{F}\cdot d\mathbf{s}.\end{aligned}$$

2R Planar Example:
The manipulator applies $\mathbf{f}_3$ to its environment. Determine the joint torque required to maintain static equilibrium. Ignore gravity effects.



Solution 1: Direct application of

$$\begin{bmatrix} {}^i\mathbf{f}_i \\ {}^i\mathbf{n}_i \end{bmatrix} = \begin{bmatrix} {}^i\mathbf{R}_{i+1} & 0 \\ {}^i\mathbf{p}_{i+1}{}^i \times {}^i\mathbf{R}_{i+1} & {}^i\mathbf{R}_{i+1} \end{bmatrix} \begin{bmatrix} {}^{i+1}\mathbf{f}_{i+1} \\ {}^{i+1}\mathbf{n}_{i+1} \end{bmatrix}.$$

$$\begin{aligned}
{}^2\mathbf{R}_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\Rightarrow \quad {}^w\mathbf{f}_2 &= \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}, \\
{}^2\mathbf{n}_2 &= \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}, \\
{}^1\mathbf{f}_1 &= {}^1\mathbf{R}_2\,{}^2\mathbf{f}_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix}, \\
{}^1\mathbf{n}_1 &= {}^1\mathbf{p}_2 \times {}^1\mathbf{R}_2\,{}^2\mathbf{f}_2 + {}^1\mathbf{R}_2\,{}^2\mathbf{n}_2, \\
&= \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}, \\
&= \begin{bmatrix} 0 \\ 0 \\ l_1 s_1 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix}.
\end{aligned}$$

Because we have a 2R linkage, $\tau_i = {}^i\mathbf{n}_i \cdot {}^i\hat{\mathbf{Z}}_i$:

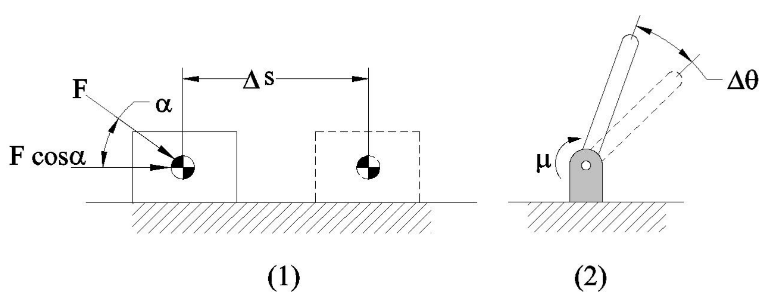
$$\left.\begin{array}{rc} \tau_1 = & l_1 s_2 f_x + (l_1 c_2 + l_2) f_y \\ \tau_2 = & l_2 f_y \end{array}\right\} \Rightarrow \boldsymbol{\tau} = \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

<u>Solution 2:</u> We could also use the Jacobian $\boldsymbol{\tau} =^0 \mathbf{J}^T\ {}^0\mathbf{F}$

$$\begin{aligned} \begin{bmatrix} \tau_i \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} {}^0\mathbf{T}_3 \begin{bmatrix} {}^3f_{3x} \\ {}^3f_{3y} \end{bmatrix}, \\ &= \begin{bmatrix} k_1 s_2 & l_2 + l_1 c_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} {}^3f_{3x} \\ {}^3f_{3y} \end{bmatrix}. \end{aligned}$$

4.4 Virtual Work

A mechanical system is in a state of static equilibrium if the virtual work vanishes for arbitrary virtual displacements which conform to geometric constraints.



Any assumed small displacement δs away from the *steady-state* pose determined by constraints and forces acting on a body in static equilibrium is called a *virtual displacement*. The term *virtual* is used to indicate the displacement does not really exist. It is used to compare various possible equilibrium positions in the process of determining the true one, to within a desired error. The difference is (for (1) in the above figure):

$$\delta W = \mathbf{F} \cdot \delta \mathbf{s} = F \delta s \cos \alpha.$$

Where α is the angle between $\mathbf{F}$ and $\delta\mathbf{s}$. Whereas $d\mathbf{s}$ represents a real infinitesimal change in actual position and can be integrated, $\delta\mathbf{s}$ is a virtual quantity and cannot be integrated.

We may also have virtual rotations and moments (i.e. in (2)):

$$\delta W = N \delta \theta.$$

For an arbitrary manipulator, we have:

$$
\begin{aligned}
0 &= \delta W : \tau_i \delta q_i + \ldots + \tau_n \delta q_n - \mathbf{f}^T \delta \mathbf{x}_{EE} - \mathbf{n}^T \delta \boldsymbol{\theta}_E \hat{\mathbf{k}}, \qquad (4.9)\\
\text{Let } \boldsymbol{\tau} &= \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix},\\
\delta \mathbf{q} &= \delta \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix},\\
\mathbf{F} &= \begin{bmatrix} \mathbf{f} \\ \mathbf{n} \end{bmatrix},\\
\delta \mathbf{x} &= \delta \begin{bmatrix} \mathbf{x}_E \\ \boldsymbol{\theta}_E \end{bmatrix}.
\end{aligned}
$$

We can write (4.9) as:

$$
\begin{aligned}
\boldsymbol{\tau}^T \delta \mathbf{q} - \mathbf{F} \delta \mathbf{x} &= 0,\\
\Rightarrow \ \mathbf{F}^T \delta \mathbf{x} &= \boldsymbol{\tau}^T \delta \mathbf{q}. \qquad (4.10)
\end{aligned}
$$

By definition, the Jacobian is:

$$
\delta \mathbf{x} = \mathbf{J} \delta \boldsymbol{\theta}. \qquad (4.11)
$$

Note ($\delta \boldsymbol{\theta} = \delta \mathbf{q}$).
Substitute (4.11) into (4.10)

$$
\mathbf{F}^T \mathbf{J} \delta \boldsymbol{\theta} = \boldsymbol{\tau}^T \delta \boldsymbol{\theta}. \qquad (4.12)
$$

Equation (4.12) is valid for all $\delta \boldsymbol{\theta}$ due to the fact that $\mathbf{J}$ is a continuous function, or can be regarded that way. We can factor out $\delta \boldsymbol{\theta}$, giving:

$$
\mathbf{F}^T \mathbf{J} = \boldsymbol{\tau}^T. \qquad (4.13)
$$

Transposing both sides of (4.13) (<u>when $\mathbf{g}$ is ignored</u>) gives:

$$
\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}, \qquad (4.14)
$$

where $\boldsymbol{\tau}$ is the restoring actuator torques and forces, and $\mathbf{F}$ is the forces and moments applied *to* the environment.

The transpose of the Jacobian maps the Cartesian forces and moments applied by the end-effector into the joint torques and/or forces required to maintain the system in static equilibrium. To transform ${}^0\mathbf{F}$, we also require ${}^0\mathbf{J}$, to give

$$
\boldsymbol{\tau} = {}^0\mathbf{J}^T \, {}^0\mathbf{F}.
$$

When the Jacobian is singular, along with the inability of the manipulator to move in a certain way, there are also directions in which the end-effector cannot exert static force. That is, $\mathbf{F}$ could be increased or decreased in directions defining the *null-space* of $\mathbf{J}^T$ with no effect on the calculated $\boldsymbol{\tau}$.

Thus, singularities can manifest themselves in force and velocity domains. This is why kinematics and statics are *dual* in nature.

Chapter 5

Manipulator Dynamics

So far, we have examined position, velocity, and static forces and torques in manipulators. Next we consider motions and the forces causing the motions: dynamics. The study of the dynamics of mechanisms is a huge field. We can't even approach the surface to scratch it in the course of these lectures. But, several formulations for obtaining the equations of motion are well suited to our study of serial manipulators.

As for the position and velocity level dynamics, we have two main problems of interest:

1. Inverse Dynamics: Given the vectors of joint angles (positions), joint rates, and joint accelerations, determine the joint torques required (useful for control).

$$\left.\begin{array}{c}\mathbf{q}\\ \dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{array}\right\} \Rightarrow \boldsymbol{\tau}$$

2. Forward Dynamics: Given $\boldsymbol{\tau}$, determine the motion $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ (useful for simulation).

$$\boldsymbol{\tau} \Rightarrow \left\{\begin{array}{c}\mathbf{q}\\ \dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{array}\right.$$

5.1 Rigid Body Acceleration

Before we look at dynamics, we must first look at the acceleration level of kinematics. As with velocities, the frame of differentiation, in general, is frame

$\{0\}$ unless otherwise indicated:

$$\begin{aligned} {}^0\mathbf{v}_1 &= {}^0\mathbf{v}_1{}^0, \\ \text{and } {}^0\frac{d}{dt}{}^0\mathbf{v}_1 = {}^0\dot{\mathbf{v}}_1 &= {}^0\mathbf{a}_1 = {}^0\mathbf{a}_1{}^0, \\ \text{and } {}^0\frac{d}{dt}{}^0\boldsymbol{\omega}_1 = {}^0\dot{\boldsymbol{\omega}}_1 &= {}^0\boldsymbol{\alpha}_1 = {}^0\boldsymbol{\alpha}_1{}^0. \end{aligned}$$

5.1.1 Linear Acceleration

The velocity of a point P, described by position vector ${}^1\mathbf{p}$ in $\{1\}$ as seen from frame $\{0\}$. Recall,

$${}^0\mathbf{v}_P = {}^0\mathbf{v}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1 + {}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1,$$

where

$$\begin{aligned} {}^0\mathbf{v}_P &= \text{relative velocity of } P \text{ with respect to } \{0\}. \\ {}^0\mathbf{v}_{1_{\mathrm{ORG}}} &= \text{velocity of } O_1 \text{ in frame } \{0\}. \\ {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1 &= \text{linear velocity of } P \text{ with respect to } \{1\} \text{ in frame } \{0\}. \\ {}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1 &= \text{due to the angular velocity of frame } \{1\} \text{ in } \{0\}. \end{aligned}$$

The relative acceleration of P in $\{1\}$ with respect to $\{0\}$ due to its own acceleration and the acceleration of the moving axes $\{1\}$ is thus obtained by differentiating the relative velocity equation:

$$\begin{aligned} \frac{d}{dt}{}^0\mathbf{v}_P &= \frac{d}{dt}{}^0\mathbf{v}_{1_{\mathrm{ORG}}} + \frac{d}{dt}({}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1) + \frac{d}{dt}({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1), \\ \Rightarrow\ {}^0\mathbf{a}_P &= {}^0\mathbf{a}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{a}_P{}^1 + {}^0\dot{\mathbf{R}}_1\,{}^1\mathbf{v}_P{}^1 + {}^0\boldsymbol{\alpha}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{v}^1 \qquad (5.1) \\ &\quad + {}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P + {}^0\boldsymbol{\omega}_1 \times {}^0\dot{\mathbf{R}}_1\,{}^1\mathbf{p}^1. \end{aligned}$$

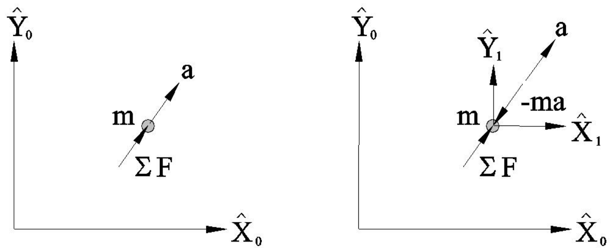
Recall that ${}^0\dot{\mathbf{R}}_1 = {}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1$. This gives:

$$\begin{aligned} {}^0\mathbf{a}_P &= {}^0\mathbf{a}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{a}_P{}^1 + {}^0\boldsymbol{\alpha}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1 + {}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1 + {}^0\boldsymbol{\omega}_1 \times {}^0\dot{\mathbf{R}}_1\,{}^1\mathbf{p}^1, \\ &\quad + {}^0\boldsymbol{\omega}_1 \times ({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1), \\ &= {}^0\mathbf{a}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\,{}^1\mathbf{a}_P{}^1 + {}^0\boldsymbol{\alpha}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1 + 2({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1), \\ &\quad + {}^0\boldsymbol{\omega}_1 \times ({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1). \end{aligned}$$

Where:

$$\begin{aligned} {}^0\mathbf{a}_{1_{\mathrm{ORG}}} &= \text{linear acceleration of } 0_1 \text{ in } \{0\}. \\ {}^0\mathbf{R}_1\,{}^1\mathbf{a}_P{}^1 &= \text{relative acceleration of } \mathbf{p} \text{ with respect to } \{1\} \text{ expressed in } \{0\}. \\ {}^0\boldsymbol{\alpha}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{p}^1 &= \text{due to angular acceleration of } \{1\} \text{ in } \{0\} \text{ perpendicular to both } \alpha \\ &\quad\ \text{and } P. \text{ (i.e. tangent to circle centered at } O_1 \text{ with radius } {}^1\mathbf{p}^1). \\ 2({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1) &= \text{Coriolis acceleration.} \\ {}^0\boldsymbol{\omega}_1 \times ({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1{}^1\,{}^1\mathbf{p}^1) &= \text{centripetal acceleration: towards axis of } \boldsymbol{\omega}. \end{aligned}$$

Whenever there is angular velocity there is centripetal acceleration (not to be confused with centrifugal acceleration). This comes from D'Alembert's Principle: When a particle is observed from fixed frame $\{0\}$ its absolute acceleration



is obtained from $\sum {}^0\mathbf{F} = m\,{}^0\mathbf{a}$. When the particle moves with O_1, and is observed in $\{1\}$, the particle appears to be at rest, or in a state of static equilibrium in $\{1\}$. The observer, who is accelerating with $\{1\}$, concludes that a force $-m^1\mathbf{a}$ acts on the particle to balance $\sum {}^1\mathbf{F}$. This perspective allows a dynamics problem to be treated by the method of statics. An important philosophical contribution contained in D'Alembert's *Traité de Dynamique*, published in 1743, simply amounts to rewriting:

$$\begin{aligned} \sum \mathbf{F} &= m\mathbf{a}, \\ \text{as: } \sum \mathbf{F} - m\mathbf{a} &= 0, \end{aligned}$$

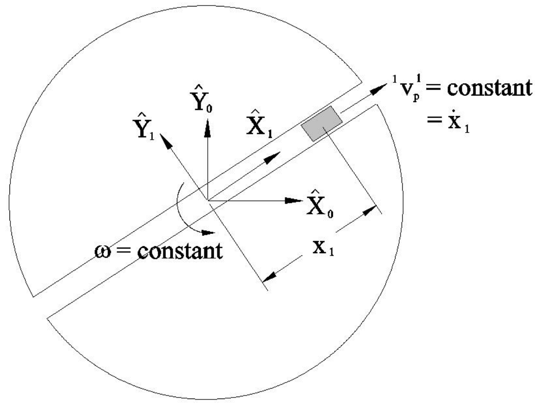
which is a force balance if $-m\mathbf{a}$ is considered as a "force". This "imaginary" force, or "pseudo" force, or "virtual" force, is known as the *inertia force* and the acceleration as the *centrifugal acceleration.* The centrifugal acceleration has the same magnitude, but opposite sense as the centripetal acceleration. The artificial state of equilibrium is called *dynamic equilibrium* and is central to the *Kane's Equations of Motion* approach to multi-body dynamics (more on that later).

5.1.2 Coriolis Acceleration

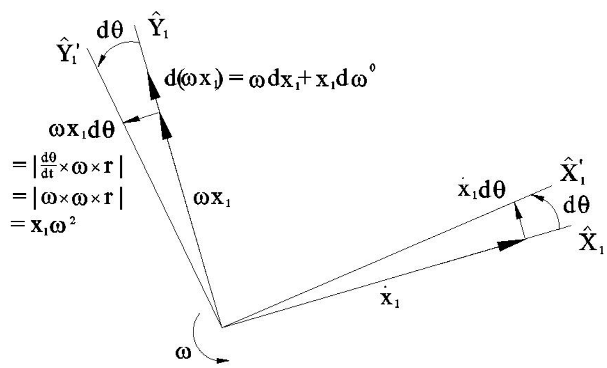
This term represents the difference between the acceleration of P measured with respect to $\{1\}$ expressed in $\{0\}$, and the acceleration of P measured with respect to $\{1\}$ but expressed in $\{1\}$. Its direction is perpendicular to ${}^0\mathbf{v}_P{}^1$ and ${}^0\boldsymbol{\omega}_1$.

It is difficult to imagine Coriolis acceleration because it is a combination of two physical properties of the motion (that accounts for the 2 in $2({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\,{}^1\mathbf{v}_P{}^1)$).

Consider a particle P constrained to move in the radial slot of a rotating disk. Let ${}^0\boldsymbol{\omega}_1 = \text{constant}$ and ${}^1\mathbf{v}_P{}^1 = \text{constant}$. The velocity of P has components due to $\dot{x}_1$ and to ωx_1, i.e. due to the velocity along the slot and to the angular velocity of the disk. Now, let's consider the *changes* in these velocity components after time dt.



The changes in $\dot{x}_1$ due to rotation after an infinitesimal rotation $\omega dt = d\theta$ which cause the axes to rotate through angle $d\theta$ to a new set $\hat{\mathbf{X}}_1{}^1 - \hat{\mathbf{Y}}_1{}^1$.



The velocity increment due to the change in direction of $^1\mathbf{v}_P{}^1$ is $\dot{x}_1 d\theta$. The change due to the magnitude of $x_1\omega$ is ωdx, both being in the $\hat{\mathbf{Y}}_1$ direction. We get after dividing by dt:

$$\dot{x}_1 \frac{d\theta}{dt} + \omega \frac{dx_1}{dt} \;=\; \dot{x}_1\omega + \dot{x}_1\omega \;=\; 2\dot{x}_1\omega.$$

This gives the magnitude of the Coriolis acceleration

$$|2({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\, {}^1\mathbf{v}_P{}^1)| \;=\; 2\dot{x}_1\omega.$$

5.1.3 Link-to-link Propagation of Linear Acceleration

Relative acceleration between links:

$$\begin{aligned} {}^0\mathbf{a}_P &= {}^0\mathbf{a}_{1_{\mathrm{ORG}}} + {}^0\mathbf{R}_1\, {}^1\mathbf{a}_P{}^1 + {}^0\boldsymbol{\alpha}_1 \times {}^0\mathbf{R}_1\, {}^1\mathbf{p}^1 + 2({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\, {}^1\mathbf{v}_P{}^1), \\ &\quad + {}^0\boldsymbol{\omega}_1 \times ({}^0\boldsymbol{\omega}_1 \times {}^0\mathbf{R}_1\, {}^1\mathbf{p}^1). \end{aligned} \tag{5.2}$$

Find ${}^{i+1}\mathbf{a}_{i+1}$. To do this, use the above equation with subscripts $0 = i+1$, $P = 0_{i+1} = i+1$, and $1 = i$. This gives:

$$\begin{aligned} {}^{i+1}\mathbf{a}_{i+1} \;&=\; {}^{i+1}\mathbf{a}_1 + {}^{i+1}\mathbf{R}_i\; {}^{i}\mathbf{a}_{i+1}{}^{i} + {}^{i+1}\boldsymbol{\alpha}_i \times {}^{i+1}\mathbf{R}_i\; {}^{i}\mathbf{p}_{i+1}{}^{i} + 2({}^{i+1}\boldsymbol{\omega}_i \times {}^{i+1}\mathbf{R}_i\; {}^{i}\mathbf{v}_{i+1}{}^{i}), \\ &\quad + {}^{i+1}\boldsymbol{\omega}_i \times ({}^{i+1}\boldsymbol{\omega}_i \times {}^{i+1}\mathbf{R}_i\; {}^{i}\mathbf{p}^{i}). \end{aligned}$$

$$\begin{aligned} {}^{i}\mathbf{a}_{i+1} \;&=\; 0 \text{ for R-pairs.} \\ &=\; {}^{i}\hat{\mathbf{Z}}_{i+1}\ddot{d}_{i+1} \text{ for P-pairs.} \\ &=\; \text{relative acceleration of } O_{i+1} \text{ with respect to} \\ &\quad\;\; \{i\} \text{ expressed in } \{i\}. \\ {}^{i}\mathbf{v}_{i+1}{}^{i} \;&=\; 0 \text{ for R-pairs.} \\ &=\; {}^{i}\hat{\mathbf{Z}}_{i+1}\dot{d}_{i+1} \text{ for P-pairs.} \\ &=\; \text{relative velocity of } O_{i+1} \text{ with respect to} \\ &\quad\;\; \{i\} \text{ expressed in } \{i\}. \end{aligned}$$

For R-Pairs:

$$ {}^{i+1}\mathbf{a}_{i+1} \;=\; {}^{i+1}\mathbf{R}_i[{}^{i}\mathbf{a}_i + {}^{i}\boldsymbol{\alpha}_i \times {}^{i}\mathbf{p}_{i+1}{}^{i} + {}^{i}\boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{p}_{i+1}{}^{i})] =^{i+1} \mathbf{a}_{i+1,\text{R-PAIR}}. $$

For P-pairs:

$$ {}^{i+1}\mathbf{a}_{i+1} \;=\; {}^{i+1}\mathbf{a}_{i+1,\text{R-PAIR}} + 2({}^{i+1}\mathbf{R}_I\; {}^{i}\boldsymbol{\omega}_i \times {}^{i+1}\hat{\mathbf{Z}}_{i+1}\,\dot{d}_{i+1}) + {}^{i+1}\hat{\mathbf{Z}}_{i+1}\,\ddot{d}_{i+1}. $$

5.1.4 Angular Acceleration

Suppose we know ${}^{0}\boldsymbol{\omega}_i$ and ${}^{i}\boldsymbol{\omega}_{i+1}{}^{i}$,but want ${}^{0}\boldsymbol{\omega}_{i+1}$. We can write:

$$ {}^{0}\boldsymbol{\omega}_{i+1}{}^{i} \;=\; {}^{0}\boldsymbol{\omega}_i + {}^{0}\mathbf{R}_i\; {}^{i}\boldsymbol{\omega}_{i+1}{}^{i}. \tag{5.3} $$

The relative angular velocity is simply

$$\begin{aligned} {}^{i}\boldsymbol{\omega}_{i+1}{}^{i} \;&=\; {}^{i}\hat{\mathbf{Z}}_{i+1}\,\dot{\theta}_{i+1} \text{ for R-pairs.} \\ &=\; 0 \text{ for P-pairs.} \end{aligned}$$

Now, differentiate (5.3) with respect to time, together with the relative angular velocity, and express in $\{i+1\}$, we get, after setting superscript to $0 = i+1$:
R-pairs:

$$ {}^{i+1}\dot{\boldsymbol{\omega}} \;=\; {}^{i+1}\mathbf{R}_i\; {}^{i}\dot{\boldsymbol{\omega}}_i + {}^{i+1}\mathbf{R}_i\; {}^{i}\hat{\mathbf{Z}}_{i+1}\,\ddot{\theta}_{i+1} + {}^{i+1}\mathbf{R}_i({}^{i}\boldsymbol{\omega}_i \times {}^{i}\hat{\mathbf{Z}}_{i+1}\dot{\theta}_{i+1}), $$

Which is: $$ {}^{i+1}\boldsymbol{\alpha}_{i+1} \;=\; {}^{i+1}\mathbf{R}_i\; {}^{i}\boldsymbol{\alpha}_i + {}_{i+1}\hat{\mathbf{Z}}_{i+1}\ddot{\theta}_{i+1} + (({}^{i+1}\mathbf{R}_i\; {}^{i}\boldsymbol{\omega}_i) \times {}^{i+1}\hat{\mathbf{Z}}_{i+1}\dot{\theta}_{i+1}). $$

P-pairs:

$$ {}^{i+1}\boldsymbol{\alpha}_{i+1} \;=\; {}^{i+1}\mathbf{R}_i\; {}^{i}\boldsymbol{\alpha}_i. $$

5.2 Distribution of Mass: The Inertia Tensor

The linear equation of motion (Newton's equation) states that the resultant of all forces acting on a body equals the time rate of change of linear momentum. Since the linear velocity is the same for all points of a rigid body, we can express linear momentum as:

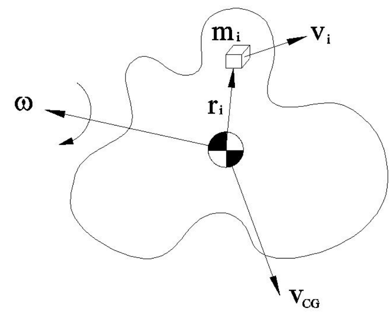


Figure 5.1: Space potato.

$$\begin{aligned}
\mathbf{G} &= m\mathbf{v}_{\mathrm{CG}} \quad \left[\frac{\text{kg m}}{\text{s}}\right] = [\text{Ns}], \\
&= \sum \mathbf{G}_i, \\
&= \sum m_i \mathbf{v}_i.
\end{aligned}$$

Without loss of generality, we can consider all the elements of mass m_i to be collected at the center of gravity.
We can write: Newton's Equation (for a single body)

$$\sum \mathbf{F} = \frac{d}{dt}\mathbf{G} = m\frac{d\mathbf{v}_{\mathrm{CG}}}{dt} = m\mathbf{a}_{\mathrm{CG}}.$$

Just as the mass m of a rigid body is a measure of its resistance to linear acceleration, the moment of inertia is a measure of the resistance of the rigid body to angular acceleration.

The rotational equations of motion (Euler's Equation) states that the moment about a fixed point of all forces acting on a rigid body equals the time rate of change of angular momentum of the body about the point. If the reference point is the center of gravity (CG), we have:

$$\text{Angular momentum} = \mathbf{H}^{\mathrm{CG}} = \int_m (\mathbf{r}_i \times \mathbf{v}_i)dm \quad \left[\frac{\text{kg m}^2}{\text{s}}\right] = [\text{Nms}].$$

Euler's Equation, for the rotational equation of motion with respect to the CG is:

$$\frac{d}{dt}\mathbf{H}^{\mathrm{CG}} = \frac{d}{dt}\int_m (\mathbf{r}_i \times \mathbf{v}_i)dm.$$

If the body is rigid, then at STP (standard temperature and pressure) its mass properties will be constant. Since $\mathbf{v}_i \;=\; \boldsymbol{\omega} \times \mathbf{r}_i$, we can write for each element of mass:

$$\mathbf{r}_i \times \mathbf{v}_i \;=\; \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i). \tag{5.4}$$

There is a useful identity for the *triple cross product*:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}),$$

so we may rewrite the right hand side of (5.4) as:

$$\begin{aligned}
\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) &= \boldsymbol{\omega}(\mathbf{r}_i \cdot \mathbf{r}_i) - \mathbf{r}_i(\mathbf{r}_i \cdot \boldsymbol{\omega}), \\
&= \boldsymbol{\omega}\mathbf{r}_i{}^T \mathbf{r}_i - \mathbf{r}_i\mathbf{r}_i{}^T \boldsymbol{\omega}), \\
&= (\mathbf{r}_i{}^T \mathbf{r}_i - \mathbf{r}_i\mathbf{r}_i{}^T)\boldsymbol{\omega}.
\end{aligned}$$

Note: $\mathbf{r}_i{}^T \mathbf{r}_i$ is a scalar, while $\mathbf{r}_i\mathbf{r}_i{}^T$ is a 3×3 matrix.

Hence,

$$\mathbf{H}^{\mathrm{CG}} \;=\; \left[\int_m (\mathbf{r}_i{}^T \mathbf{r}_i \, \mathbf{I}_{3\times 3} - \mathbf{r}_i\mathbf{r}_i{}^T) dm\right] \boldsymbol{\omega}.$$

We can take the angular velocity vector out of the integrand because it is the same for every line in the rigid body. The sum of all mass elements multiplied by a matrix (or tensor) function of each from the reference point, (here, CG) is called the *inertia tensor*. It is obtained by expanding the integrand:

$$\begin{aligned}
\mathbf{I}^{\mathrm{CG}} &= \int_m \left(\begin{bmatrix} r_x{}^2 + r_y{}^2 + r_z{}^2 & 0 & 0 \\ 0 & r_x{}^2 + r_y{}^2 + r_z{}^2 & 0 \\ 0 & 0 & r_x{}^2 + r_y{}^2 + r_z{}^2 \end{bmatrix}, \right. \\
&\qquad \left. - \begin{bmatrix} r_x{}^2 & r_x r_y & r_x r_z \\ r_y r_x & r_y{}^2 & r_y r_z \\ r_z r_x & r_z r_y & r_z{}^2 \end{bmatrix} \right) dm, \\
&= \int_m \begin{bmatrix} r_y{}^2 + r_z{}^2 & -r_x r_y & -r_x r_z \\ -r_y r_x & r_y{}^2 + r_z{}^2 & -r_y r_z \\ -r_z r_x & -r_z r_y & r_y{}^2 + r_z{}^2 \end{bmatrix} dm, \\
&= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & I_{zy} & I_{zz} \end{bmatrix}.
\end{aligned}$$

Note: in the above equation, clearly $r_z r_y = r_y r_z$, $r_x r_z = r_z r_x$, and $r_y r_z = r_z r_y$.

If the density is constant throughout, then $dm = \rho dV$, and $\mathbf{I}^{\mathrm{CG}}$ defines a geometric property of the rigid body since the integral depends on radii and volume. With each element defined as:

Mass Moments of Inertia:

$$\begin{aligned}
I_{xx} &= \iiint_V ({r_y}^2 + {r_z}^2)\rho dV, \\
I_{yy} &= \iiint_V ({r_x}^2 + {r_z}^2)\rho dV, \\
I_{zz} &= \iiint_V ({r_x}^2 + {r_y}^2)\rho dV.
\end{aligned}$$

Mass Products of Inertia:

$$\begin{aligned}
I_{xy} = I_{yx} &= \iiint_V (r_x r_y)\rho dV, \\
I_{yz} = I_{zy} &= \iiint_V (r_y r_z)\rho dV, \\
I_{xz} = I_{zx} &= \iiint_V (r_x r_z)\rho dV.
\end{aligned}$$

Now we can write Euler's Equation (for a single body) as:

$$\begin{aligned}
\sum \mathbf{N}^{\mathrm{CG}} = \frac{d}{dt}\mathbf{H}^{\mathrm{CG}} &= \mathbf{I}^{\mathrm{CG}}\boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{I}^{\mathrm{CG}}\boldsymbol{\omega}, \\
&= \mathbf{I}^{\mathrm{CG}}\boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{H}^{\mathrm{CG}}.
\end{aligned}$$

Notes:

1. For planar systems, the second term vanishes.
2. If we assume mass is concentrated in one point, then $\mathbf{I} = 0$ because $\rho dV = 0$ for a point.

5.2.1 Principal Axes and Principal Moments of Inertia

The inertia tensor elements are reference frame dependent. If we are free to choose, there is in general, one orientation of the reference frame such that the products of inertia, the off-diagonal elements in the inertia tensor vanish. When so aligned, the axes of the reference frame are called the *principal axes of inertia*. The corresponding mass moments are called the *principal moments of inertia*.

Important Facts of I

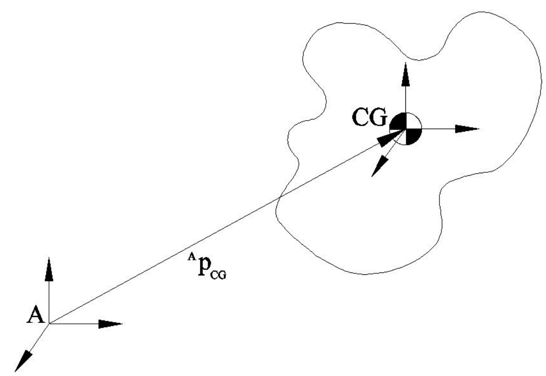
1. The eigenvalues of an arbitrary inertia tensor are the principal moments for the rigid body. The associated eigenvectors are the principal axes. Owing to the symmetry of the inertia tensor, it may always be diagonalized.
2. Moments of inertia are always positive. Products of inertia may be positive or negative.
3. The sum of the three moments of inertia of an arbitrary inertia tensor are invariant under rotations of the reference frame.

5.2.2 Parallel Axis Theorem

The inertia tensor depends on the location and orientation of the reference frame. The parallel axis theorem allows us to compute the change in **I** under a translation of the reference frame. It relates **I** in a frame with origin at CG to the corresponding **I** with respect to a translated reference frame.

Let ${}^{A}\mathbf{p}_{\mathrm{CG}}$ be the position vector of CG in the translated frame $\{A\}$. Then

$$\mathbf{I}^{A} \;=\; m({}^{A}\mathbf{p}_{\mathrm{CG}}{}^{T}\, {}^{A}\mathbf{p}_{\mathrm{CG}}\, \mathbf{I}_{3\times 3} - {}^{A}\mathbf{p}_{\mathrm{CG}}\, {}^{A}\mathbf{p}_{\mathrm{CG}}{}^{T}) + \mathbf{I}^{\mathrm{CG}}.$$



5.2.3 Rotating Axes

We may also see how **I** is affected by pure rotations of the reference frame, say from $\{1\}$ to $\{0\}$. That is, we change axes of $\mathbf{I}^{\mathrm{CG}}$ from being parallel with those of $\{1\}$ to those of $\{0\}$.

$${}^{0}\mathbf{I}_{1}{}^{\mathrm{CG}} \;=\; {}^{0}\mathbf{R}_{1}\, {}^{1}\mathbf{I}_{1}{}^{\mathrm{CG}}\, {}^{0}\mathbf{R}_{1}{}^{T}.$$

The above equation is important for Lagrange dynamics.

Most robots have links with relatively complicated shapes, so analytically determining **I** is difficult at best. Usually **I** is measured for each link with measurement instruments such as an inertia torsion pendulum. There are many examples of how to calculate **I** in every engineering mechanics textbook, in particular, *Engineering Mechanics, Vol 2: Dynamics* By Meriam and Kraige Wiley. So we'll stop here with **I**.

5.3 Iterative Newton-Euler Dynamics (Inverse Dynamics)

This technique allows us to compute the joint torques and forces necessary to generate a desired trajectory. We assume the joint position, velocity, and acceleration variables $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ are known. This, together with kinematics and mass data, is enough information to compute the joint torques.

- Equations are written in successive frames.
- Constraint forces are propagated.
- Efficient: the number of computations increases linearly with the number of DOF.
- Well suited to programming.
- Two forms:
  - Numerical: plug in values.
  - Closed form: leave symbolic.

Step 1: Outward Iterations to Compute Velocities and Accelerations

We start at the base, $i = 0$, and move outward to the end-effector, link by link. The goal is to compute the inertial force and torque (i.e. related to mass properties) acting at the center of gravity of each link. Recall the Newton-Euler Equations for link $i+1$.

$$\begin{aligned}
{}^{i+1}\mathbf{F}_{i+1} &= m_{i+1}\,{}^{i+1}\mathbf{a}_{\mathrm{CG}_{i+1}}, \\
{}^{i+1}\mathbf{N}_{i+1} &= {}^{i+1}\mathbf{I}_{i+1}{}^{\mathrm{CG}_{i+1}}\,{}^{i+1}\boldsymbol{\alpha}_{i+1} + {}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\mathbf{I}_{i+1}{}^{\mathrm{CG}_{i+1}}\,{}^{i+1}\boldsymbol{\omega}_{i+1}.
\end{aligned}$$

For n links we go from $i = 0$ to $i = n-1$. Additionally, we need the linear acceleration of the CG:

$${}^{i+1}\mathbf{a}_{\mathrm{CG}_{i+1}} = {}^{i+1}\boldsymbol{\alpha}_{i+1} \times {}^{i+1}\mathbf{p}_{\mathrm{CG}_{i+1}}{}^{i+1} + {}^{i+1}\boldsymbol{\omega}_{i+1}({}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\mathbf{p}_{\mathrm{CG}_{i+1}}{}^{i+1}) + {}^{i+1}\mathbf{a}_{i+1}.$$

We also need in general:

$$\begin{aligned}
{}^{i+1}\boldsymbol{\omega}_{i+1} &= {}^{i+1}\mathbf{R}_i\,{}^{i}\boldsymbol{\omega}_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{\mathbf{Z}}_{i+1}, \\
{}^{i+1}\boldsymbol{\alpha}_{i+1} &= {}^{i+1}\mathbf{R}_i\,{}^{i}\boldsymbol{\alpha}_i + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{\mathbf{Z}}_{i+1} + {}^{i+1}\mathbf{R}_i\,{}^{i}\boldsymbol{\omega}_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{\mathbf{Z}}_{i+1}, \\
{}^{i+1}\mathbf{a}_{i+1} &= {}^{i+1}\mathbf{R}_i[{}^{i}\boldsymbol{\alpha}_i \times {}^{i}\mathbf{p}_{i+1}{}^{i} + {}^{i}\boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{p}_{i+1}{}^{i})] + 2({}^{i+1}\mathbf{R}_i\,{}^{i}\boldsymbol{\omega}_i, \\
&\quad +\dot{d}_{i+1}\,{}^{i+1}\hat{\mathbf{Z}}_{i+1}) + \ddot{d}_{i+1}\,{}^{i+1}\hat{\mathbf{Z}}_{i+1}.
\end{aligned}$$

We need these to compute the internal forces and torques from the Newton-Euler Equations.

Initial Conditions

The internal effects of gravity can be included in the *initial conditions.* That is, set ${}^0\mathbf{a}_0 = -\mathbf{g}$. That is, assume the base accelerates upward at 1g. This "virtual" acceleration causes exactly the same effect on each link as including a gravity vector in the propagation equations, but with minimal computation!

So we have:

$$\begin{aligned} {}^0\boldsymbol{\omega}_0 &= {}^0\boldsymbol{\alpha}_0 = {}^0\mathbf{v}_0 = \mathbf{0}, \\ \text{and } {}^0\mathbf{a}_0 &= -\mathbf{g}. \end{aligned}$$

Now use the Newton-Euler Equations to compute ${}^{i+1}\mathbf{F}_{i+1}$ and ${}^{i+1}\mathbf{N}_{i+1}$, the inertial force and torque acting at the center of mass of each link.

Step 2: Inward Iterations to Compute Joint Forces and Torques

Now that we have computed the internal forces and torques acting on each link, we must determine the joint forces and torques which cause the inertial link forces and torques. This is done by writing a force and moment balance on the free-body diagram of each link.

Each link has forces exerted on it by neighbors and experiences its own inertial forces and torques caused by its motion.

Inertial Forces and Torques:

$$\mathbf{F}_i,\ \mathbf{N}_i.$$

Forces and Torques Exerted by Neighbors:

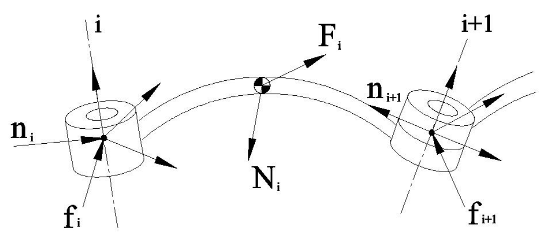


Figure 5.2: For a FBD, $\mathbf{f}_{i+1}$ is replaced with the reaction link i experiences in causing $\mathbf{f}_{i+1}$, i.e., $-\mathbf{f}_{i+1}$.

$$\begin{aligned} \mathbf{f}_i &= \text{force exerted on link } i \text{ by link } i-1. \\ \mathbf{n}_i &= \text{torque exerted on link } i \text{ by link } i-1. \end{aligned}$$

Summing the forces acting on link i:

$$ {}^i\mathbf{F}_i = m_i\, {}^i\mathbf{a}_{\mathrm{CG}_i} = {}^i\mathbf{f}_i - \mathbf{R}_{i+1}\, {}^{i+1}\mathbf{f}_{i+1}. \tag{5.5}$$

Summing the moments about CG:

$$\begin{aligned}
{}^i\mathbf{N}_i &= {}^i\mathbf{I}^{\mathrm{CG}_i}\,{}^i\boldsymbol{\alpha}_i + {}^i\boldsymbol{\omega}_i \times {}^i\mathbf{I}^{\mathrm{CG}_i}\,{}^i\boldsymbol{\omega}_i, \\
&= {}^i\mathbf{n}_i - {}^i\mathbf{n}_{i+1} - {}^i\mathbf{p}_{\mathrm{CG}_i}{}^i \times {}^i\mathbf{F}_i - {}^i\,\mathbf{p}_{i+1}{}^i \times {}^i\mathbf{R}_{i+1}\,{}^{i+1}\mathbf{f}_{i+1}. \qquad (5.6)
\end{aligned}$$

Now we can rearrange (5.5) and (5.6), after including some additional rotations to adjust the superscripts, so that they are recursive from higher to lover numbered links. We get:

$$\begin{aligned}
{}^i\mathbf{f}_i &= {}^i\mathbf{F}_i + {}^i\mathbf{R}_{i+1}\,{}^{i+1}\mathbf{f}_{i+1}, \qquad (5.7)\\
{}^i\mathbf{n}_i &= {}^i\mathbf{N}_i + {}^i\mathbf{R}_{i+1}\,{}^{i+1}\mathbf{n}_{i+1} + ({}^i\mathbf{p}_{\mathrm{CG}_i}{}^i \times {}^i\mathbf{F}_i) + ({}^i\mathbf{p}_{i+1}{}^i \times {}^i\mathbf{R}_{i+1}\,{}^{i+1}\mathbf{f}_{i+1}). \qquad (5.8)
\end{aligned}$$

In order to move the links they must be accelerated and decelerated. These are equations of motion because they equate forces required to cause the motion in terms of acceleration.

For the end-effector, link n, we know ${}^{n+1}\mathbf{f}_{n+1} = {}^{n+1}\mathbf{n}_{n+1} = 0$ if the robot is moving free in space. If the end-effector has contact with the environment, they are non-zero in value, but are known (measured with load cell, for example).

Step 3

The structure of the robot resists all components of the above forces and moments with the exception of those in the direction of the joint axes. These forces and torques must be supplied by the actuators. So, as in the static force case, the required joint torques are the $\hat{\mathbf{Z}}$ components of the torque or force applied by the neighboring link.

<u>For R-pairs:</u>

$$\boldsymbol{\tau}_i = {}^i\mathbf{n}_i{}^T\,{}^i\hat{\mathbf{Z}}_i = ({}^i\mathbf{n}_i)_Z.$$

<u>For P-pairs:</u>

$$\boldsymbol{\tau}_i = {}^i\mathbf{f}_i{}^T\,{}^i\hat{\mathbf{Z}}_i = ({}^i\mathbf{f}_i)_Z.$$

Observations

The Newton-Euler formulation for serial robot dynamics is numerically very efficient. The computational complexity is the same for each link. The number of equations grows linearly with each increase in DOF. The equations apply to any serial manipulator.

They are also useful for writing the equations of motion in closed form. This is done by applying the Newton-Euler formulation symbolically, instead of using numerical values for m, $\mathbf{I}$, ${}^i\mathbf{p}_{\mathrm{CG}_i}{}^i$, ${}^{i+1}\mathbf{R}_i$, etc.

5.3.1 The State-space Representation

When the Newton-Euler Equations are evaluated symbolically we can collect the terms as follows:

$$\boldsymbol{\tau} \; = \; \mathbf{M}(q)\ddot{\mathbf{q}} + \mathbf{V}(q,\dot{q}) + \mathbf{G}(q) + \mathbf{N}(q), \text{or} \;\; \boldsymbol{\tau} - \mathbf{N}(q) \; = \; \mathbf{M}(q)\ddot{\mathbf{q}} + \mathbf{V}(q,\dot{q}) + \mathbf{G}(q).$$

Where

- $\mathbf{M}(q)$ is a square, symmetric, positive definite matrix (i.e. all its eigenvalues are positive). It is called the *mass matrix*, and is composed of terms which multiply $\ddot{q}_i$. The mass matrix depends on position q_i.
- $\mathbf{V}(q,\dot{q})$ is an $n \times 1$ vector of centrifugal and coriolis force terms, i.e. its elements are those terms multiplying $\dot{q_i}^{\,2}$ and $\dot{q}_i\dot{q}_j$. It depends on position and velocity.
- $\mathbf{G}(q)$ is an $n \times 1$ vector of gravity terms. It contains all those terms in which the gravitational constant, g appears. It is also dependent on position.
- $\mathbf{N}(q)$ are terms due to the applied forces. They are, in this case, what's left over. They are equal to $\mathbf{J}^T \; \mathbf{F}_{\text{applied}}$, where $\mathbf{F}_{\text{applied}}$ are forces and moments applied to the environment, or by changing, forces and moments applied to the end-effector by the environment.

5.3.2 Forward Dynamics

Given $\boldsymbol{\tau}$, compute $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$. This problem is important for simulation of robot arms. We write the dynamics in closed form and solve the equations for $\ddot{\mathbf{q}}$ with *symbolic computer algebra* software.

$$\ddot{\mathbf{q}} \;\; = \;\; \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{V} - \mathbf{G} - \ldots). \tag{5.9}$$

Then if we want a computer simulation of the dynamics, we can numerically integrate $\ddot{\mathbf{q}}$ to obtain $\dot{\mathbf{q}}$ and $\mathbf{q}$, given a set of initial conditions.

$$\begin{aligned} \mathbf{q}(t=0) &= \mathbf{q}(0) \; = \; q_0, \\ \dot{\mathbf{q}}(t=0) &= \dot{\mathbf{q}}(0) \; = \; 0. \end{aligned}$$

Then, numerically integrate (5.9) in time steps Δt We can use *Euler integration* starting at $t = 0$ and iteratively compute

$$\begin{aligned} \dot{\mathbf{q}}(t+\Delta t) &= \dot{\mathbf{q}}(t) + \ddot{\mathbf{q}}(t)\Delta t, \\ \mathbf{q}(t+\Delta t) &= \mathbf{q}(t) + \dot{\mathbf{q}}\Delta t + \frac{1}{2}\ddot{\mathbf{q}}(t)\Delta t^2. \end{aligned}$$

At each iteration, $\ddot{\mathbf{q}}$ is computed from (5.9). Thus, position, velocity, and acceleration of the end-effector caused by certain input torque can be computed numerically.

5.4 Lagrangian Formulation of Dynamics

The Newton-Euler formulation is based on force balance to determine the equations of motion. It is said to be *force based*. The Lagrangian formulation depends on the energy balance of the manipulator and is said to be *energy based*. Both formulations must yield identical equations of motion for a given manipulator.

To use the Lagrange equations, positions must be described by a set of independent generalized coordinates. They must uniquely specify a pose. They can be translational or rotational displacement variables. Additionally, in general, a set of generalized coordinates for a mechanical system need not be unique, but the set must uniquely describe the pose. The set of joint angles and offsets for a serial manipulator uniquely specify a pose and are thus a set of generalized coordinates.

Let T be the kinetic energy of a mechanical system and let U be its potential energy. Lagrange's Equation of motion for each link can be written as:

$$\frac{d}{dt}\left(\frac{\delta T_i}{\delta \dot{q}_i}\right) - \frac{\delta T_i}{\delta q_i} + \frac{\delta U_i}{\delta q_i} \ = \ Q_i, \quad i = 1, \ldots, n,$$

where

- q_i = generalized coordinates.
- Q_i = generalized forces. These account for all forces acting *on* the system except for inertial $[(F = mg)$ and $(N = I\alpha + \omega \times I\omega)]$ forces and gravity forces. Thus,

  $$\begin{aligned} \mathbf{Q} \ &= \ \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}} - \boldsymbol{\tau}_{\text{friction}} \ = \ \boldsymbol{\tau} + {}^0\mathbf{J}^T \, \mathbf{F}_{\text{ext}} - \boldsymbol{\tau}_{\text{friction}}, \\ &= \ \boldsymbol{\tau} - \boldsymbol{\tau}_{\text{app}} - \boldsymbol{\tau}_{\text{friction}} \ = \ \boldsymbol{\tau} - {}^0\mathbf{J}^T \, \mathbf{F}_{\text{app}} - \boldsymbol{\tau}_{\text{friction}}, \end{aligned}$$

  where $\boldsymbol{\tau}$ are the torques required to cause the motion, $\boldsymbol{\tau}_{\text{app}}$ are torques required to apply forces and moments to the environment, $\boldsymbol{\tau}_{\text{friction}}$ are joint torques required to overcome coulomb and viscous friction.

There are other effects in $\mathbf{Q}$ that are being neglected. For instance, link deformations, bearing and gear eccentricity all have to be accounted for to obtain a tremendously accurate model, but are extremely difficult to model. So we settle for a reasonably accurate model.

Kinetic Energy (T) of a Manipulator

The kinetic energy of the i^{th} link can be expressed as

$$T_i \ = \ \frac{1}{2}[m_i \, {}^0\mathbf{v}_{\text{CG}_i}{}^T \, {}^0\mathbf{v}_{\text{CG}_i} + {}^i\boldsymbol{\omega}_i \, {}^i\mathbf{I}_i{}^{\text{CG}_i} \, {}^i\boldsymbol{\omega}_i].$$

Also the total kinetic energy of the manipulator is the sum of the kinetic energy of each link:

$$T = \sum_{i=1}^{n} T_i.$$

But recall we can express velocities of the links in terms of joint rates using Jacobians:

$$
\begin{aligned}
{}^0\mathbf{v}_{\mathrm{CG}_i} &= {}^0\mathbf{J}_L{}^{(i)}\dot{\mathbf{q}}, \\
{}^0\boldsymbol{\omega}_{\mathrm{CG}_i} &= {}^0\mathbf{J}_A{}^{(i)}\dot{\mathbf{q}}.
\end{aligned}
$$

Recall that

$$
{}^0\mathbf{J} = \left[\begin{array}{c} {}^0\mathbf{J}_L \\ \hline {}^0\mathbf{J}_A \end{array}\right] = \left[\begin{array}{cccc} {}^0\mathbf{J}_{L,1} & {}^0\mathbf{J}_{L,2} & \dots & {}^0\mathbf{J}_{L,n} \\ \hline {}^0\mathbf{J}_{A,1} & {}^0\mathbf{J}_{A,2} & \dots & {}^0\mathbf{J}_{A,n} \end{array}\right].
$$

The Jacobian for the i^{th} link is not affected by the $i+1 \ \dots \ n$ links. So:

$$
{}^0\mathbf{J}_L{}^{(i)} = \left[\begin{array}{cccccccc} {}^0\mathbf{J}_{L,1}{}^{(i)} & {}^0\mathbf{J}_{L,2}{}^{(i)} & \dots & {}^0\mathbf{J}_{L,i}{}^{(i)} & \mathbf{0}^{i+1} & \mathbf{0}^{i+2} & \dots & \mathbf{0}^{n} \\ \hline {}^0\mathbf{J}_{A,1}{}^{(i)} & {}^0\mathbf{J}_{A,2}{}^{(i)} & \dots & {}^0\mathbf{J}_{A,i}{}^{(i)} & \mathbf{0}^{i+1} & \mathbf{0}^{i+2} & \dots & \mathbf{0}^{n} \end{array}\right].
$$

Further recall: (for the i^{th} link and j^{th} column)

$$
\begin{aligned}
{}^0\mathbf{J}_{L,j}{}^{(i)} &= \begin{cases} {}^0\hat{\mathbf{Z}}_j \times {}^0\mathbf{p}_{\mathrm{CG}_i}{}^{j}, & j = i, \dots, i \ \text{ For R-pairs.} \\ {}^0\hat{\mathbf{Z}}_j, & j = 1, \dots, i \ \text{ For P-pairs.} \end{cases} \\
{}^0\mathbf{J}_{A,j}{}^{(i)} &= \begin{cases} {}^0\hat{\mathbf{Z}}_j, & j = i, \dots, i \ \text{ For R-pairs.} \\ 0, & j = 1, \dots, i \ \text{ For P-pairs.} \end{cases}
\end{aligned}
$$

So we can rewrite the kinetic energy of the i^{th} link as:

$$
T_i = \frac{1}{2}\,\dot{\mathbf{q}}^T[m_i\,({}^0\mathbf{J}_L{}^{(i)})^T\,{}^0\mathbf{J}_L{}^{(i)} + ({}^0\mathbf{J}_A{}^{(i)})^T\,{}^0\mathbf{I}_i{}^{\mathrm{CG}_i}\,{}^0\mathbf{J}_A{}^{(i)}]\,\dot{\mathbf{q}}.
$$

Since T is a function of $\dot{q}$, the time derivative of $\frac{\delta T}{\delta \dot{q}}$ will depend on $\ddot{q}$. Since a mass or inertia is always associated with each $\ddot{q}$, we can write:

$$
\frac{d}{dt}\left(\frac{\delta T_i}{\delta \dot{q}_i}\right) = \sum_{j=1}^{n}\mathbf{M}_{ij}\ddot{q}_j \ \Rightarrow \ \mathbf{M}(q)\ddot{\mathbf{q}}.
$$

Mass Matrix:

$$
\mathbf{M}(q) = \sum_{i=1}^{n}\{m_i\,({}^0\mathbf{J}_L{}^{(i)})\,{}^0\mathbf{J}_L{}^{(i)} + ({}^0\mathbf{J}_A{}^{(i)})^T\,{}^0\mathbf{I}_i{}^{\mathrm{CG}_i}\,{}^0\mathbf{J}_A{}^{(i)}\}.
$$

We can also write:

$$
T = \frac{1}{2}\,\dot{\mathbf{q}}\mathbf{M}(q)\dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{M}_{ij}\dot{q}_i\,\dot{q}_j.
$$

Additionally:

$$
\frac{\delta T}{\delta q} = \sum_{j=1}^{n}\sum_{k=1}^{n}m_{ijk}\,\dot{q}_k\,\dot{q}_j,
$$

where:

$$m_{ijk} \;=\; \frac{\delta \mathbf{M}_{ij}}{\delta q_k} - \frac{1}{2}\frac{\delta \mathbf{M}_{jk}}{\delta q_i}$$

Because of their dependency on $\dot{q}^2$ or $\dot{q}\dot{q}$, the $\frac{\delta T}{\delta q}$ terms equate to the velocity terms:

$$\mathbf{V}(q,\dot{q})_i \;=\; \sum_{j=1}^{n}\sum_{k=1}^{n} m_{ijk}\dot{q}_k\;\dot{q}_j, \quad i = 1,\ldots,n.$$

The gravity terms are:

$$\mathbf{G}_i \;=\; -\mathbf{g}^T \;\sum_{j=1}^{n} m_j \;{}^0\mathbf{J}_{L,i}{}^{(j)}.$$

These come from $\frac{d}{dt}\left(\frac{\delta T_i}{\delta \dot{q}_i}\right)$ and $\frac{\delta U_i}{\delta q_i}$, and also note that $\Delta U_i \;=\; m_i g \Delta h_i$. Assembling everything, we get:

$$\begin{aligned}
\mathbf{Q}_i \;&=\; \frac{d}{dt}\left(\frac{\delta T_i}{\delta \dot{q}_i}\right) - \frac{\delta T_i}{\delta q_i} + \frac{\delta U_i}{\delta q_i}, \quad i = 1,\ldots,n, \\
&=\; \sum_{j=1}^{n} \mathbf{M}_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} m_{ijk}\;\dot{q}_k\;\dot{q}_j - \mathbf{g}^T \sum_{j=1}^{n} m_j \;{}^0\mathbf{J}_{L,i}{}^{(j)}.
\end{aligned}$$

Since $\mathbf{Q} = \sum \mathbf{Q}_i$, we finally get:

$$\mathbf{M}(q)\ddot{\mathbf{q}} + \mathbf{V}(q,\dot{q}) + \mathbf{G}(q) \;=\; \boldsymbol{\tau}_{\text{joint}} - \boldsymbol{\tau}_{\text{applied}} - \boldsymbol{\tau}_{\text{friction}}.$$

Newton-Euler Example:

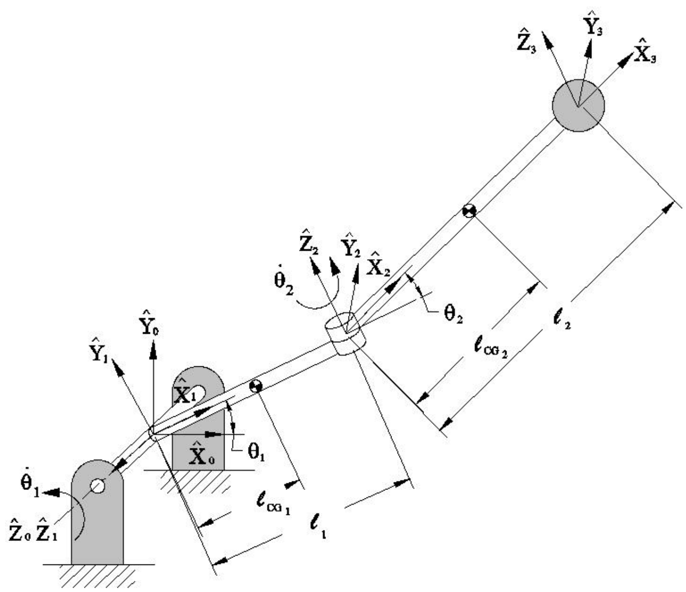


Table 5.1: DH Parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	${}^0\theta_1$
2	-90	l_1	0	${}^0\theta_2$
3	0	l_2	0	0

$$
\begin{aligned}
{}^0\mathbf{R}_1 &= \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
{}^0\mathbf{R}_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\,\alpha_1 & -\mathrm{s}\,\alpha_1 \\ 0 & \mathrm{s}\,\alpha_1 & \mathrm{c}\,\alpha_1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\,\theta_2 & -\mathrm{s}\,\theta_2 & 0 \\ \mathrm{s}\,\theta_2 & \mathrm{c}\,\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{c}\,\theta_2 & -\mathrm{s}\,\theta_2 & 0 \\ \mathrm{s}\,\theta_2 & \mathrm{c}\,\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\,\theta_2 & -\mathrm{s}\,\theta_2 & 0 \\ 0 & 0 & 1 \\ 0-\mathrm{s}\,\theta_2 & -\mathrm{c}\,\theta_2 & 0 \end{bmatrix}, \\
{}^0\mathbf{T}_1 &= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^1\mathbf{T}_2 &= \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^2\mathbf{T}_3 &= \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

$$
{}^0\mathbf{p}_1{}^0 = \text{Position vector of } O_1 \text{ measured from } O_0 \text{ expressed in } \{0\} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
$$

$$
{}^1\mathbf{p}_2{}^1 = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix},\ {}^2\mathbf{p}_3{}^2 = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}, {}^1\mathbf{p}_{\mathrm{CG}_1}{}^1 = \begin{bmatrix} l_{\mathrm{CG}_1} \\ 0 \\ 0 \end{bmatrix},\ {}^2\mathbf{p}_{\mathrm{CG}_2}{}^2 = \begin{bmatrix} l_{\mathrm{CG}_2} \\ 0 \\ 0 \end{bmatrix},
$$

$$
{}^0\mathbf{a}_0 = {}^0\mathbf{g} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}.
$$

$$
\begin{aligned}
{}^1\mathbf{I}^{\mathrm{CG}_1} &= \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix}, \\
{}^2\mathbf{I}^{\mathrm{CG}_2} &= \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix}.
\end{aligned}
$$

Initial Conditions:

$$ {}^{0}\mathbf{a}_0 = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}, \quad {}^{0}\boldsymbol{\omega}_0 = {}^{0}\dot{\boldsymbol{\omega}}_0 = {}^{0}\mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. $$

$$ \begin{aligned} {}^{4}\mathbf{f}_4 &= \begin{bmatrix} {}^{4}f_{4x} \\ {}^{4}f_{4y} \\ {}^{4}f_{4z} \end{bmatrix} = \text{Force exerted on environment by link 3}, \\ {}^{4}\mathbf{n}_4 &= \begin{bmatrix} {}^{4}n_{4x} \\ {}^{4}n_{4y} \\ {}^{4}n_{4z} \end{bmatrix} = \text{Moment exerted on environment by link 3}. \end{aligned} $$

Outward Iterations to Compute Velocity and Acceleration:
All joints are R-pairs, so:

$$ \begin{aligned} {}^{i+1}\mathbf{a}_{i+1} &= {}^{i+1}\mathbf{R}_i[{}^{i}\mathbf{a}_i + {}^{i}\boldsymbol{\alpha}_i \times {}^{i}\mathbf{p}_{i+1} + \boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{p}_{i+1}{}^{i})], \\ {}^{i+1}\boldsymbol{\alpha}_{i+1} &= {}^{i+1}\mathbf{R}_i\, {}^{i}\mathbf{a}_i + {}^{i+1}\hat{\mathbf{Z}}_{i+1}\ddot{\theta}_{i+1} + ({}^{i+1}\mathbf{R}_i\, {}^{i}\boldsymbol{\omega}_i) \times {}^{i+1}\hat{\mathbf{Z}}_{i+1}\dot{\theta}_{i+1}, \\ {}^{i+1}\boldsymbol{\omega}_{i+1} &= {}^{i+1}\hat{\mathbf{Z}}_{i+1}\dot{\theta}_{i+1} + {}^{i+1}\mathbf{R}_i\, {}^{i}\boldsymbol{\omega}_i. \end{aligned} $$

Joint 1: $(i = 0)$

$$ \begin{aligned} {}^{1}\boldsymbol{\omega}_1 &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ {}^{1}\mathbf{R}_0 = {}^{0}\mathbf{R}_1{}^{T} &= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ {}^{0}\mathbf{a}_0 &= \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}, \\ {}^{0}\boldsymbol{\alpha}_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\ {}^{0}\boldsymbol{\omega}_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\ {}^{1}\mathbf{a}_1 &= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1 g \\ -c_1 g \\ 0 \end{bmatrix}, \\ {}^{1}\boldsymbol{\alpha}_1 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}. \end{aligned} $$

The mass matrix is:

$$\mathbf{M}(\theta)\ddot{\boldsymbol{\theta}} \;=\; \begin{bmatrix} m_2[l_1(l_1 - l_{\mathrm{CG}_2}) - l_{\mathrm{CG}_2}c_2(l_1 l_{\mathrm{CG}_2})] & 0 \\ +l_{\mathrm{CG}_1}{}^2 m_1 + I_{zz_1} - I_{yy_2}c_2 & \\ & \\ 0 & l_{\mathrm{CG}_2}{}^2 m_2 + I_{zz_2} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}.$$

The velocity vector of centrifugal and Coriolis forces terms depends only on velocities: their squares and products:

$$\mathbf{V}(\theta,\dot{\theta}) \;=\; \begin{bmatrix} [s_2(I_{yy_2} - I_{zz_2} - l_1 l_{\mathrm{CG}_2}m_2) - c_2 s_2(I_{yy_2} + l_{\mathrm{CG}_2}{}^2 m_2)]\dot{\theta}_1\,\dot{\theta}_2 - I_{xx_2}s_2\dot{\theta}_1{}^2 \\ l_1 l_{\mathrm{CG}_2}m_2 s_2\dot{\theta}_1{}^2 \end{bmatrix}.$$

The gravity vector, $G(\theta)$, contains all terms in which the gravitational constant appears:

$$\mathbf{G}(\theta) \;=\; \begin{bmatrix} -g[(l_{\mathrm{CG}_1}m_1 + l_1 m_2)c_2 + l_{\mathrm{CG}_2}m_2 c_1 c_2] \\ g l_{\mathrm{CG}_2}m_2 s_1 s_2 \end{bmatrix}.$$

The remaining terms are moments caused by the applied force and moment at the end-effector. They are collected in:

$$\mathbf{N} \;=\; \begin{bmatrix} (l_1 + l_2 c_2)^3 f_{3z} - {}^3n_{3x}s_2 - {}^3n_{3y}c_2 \\ l_2\,{}^3f_{3y} + {}^3n_{3z} \end{bmatrix}.$$

For Lagrange

The Jacobian that relates end-effector forces to joint torques is:

$${}^0\mathbf{J} \;=\; \left[\begin{array}{c|c} {}^0\hat{\mathbf{Z}}_1 \times {}^0\mathbf{p}_E{}^1 & {}^0\hat{\mathbf{Z}}_2 \times {}^0\mathbf{p}_E{}^2 \\ \hline {}^0\hat{\mathbf{Z}}_1 & {}^0\hat{\mathbf{Z}}_2 \end{array}\right] \;=\; \left[\begin{array}{c|c} {}^0\mathbf{J}_{L,1} & {}^0\mathbf{J}_{L,2} \\ \hline {}^0\mathbf{J}_{A,1} & {}^0\mathbf{J}_{A,2} \end{array}\right].$$

Since the force vector is given with respect to frame $\{3\}$, which is also frame $\{E\}$, we need ${}^E\mathbf{J}$. Recall:

$${}^E\mathbf{J} \;=\; \left[\begin{array}{c|c} {}^E\mathbf{R}_0 & \mathbf{0} \\ \hline \mathbf{0} & {}^E\mathbf{R}_0 \end{array}\right],$$

then

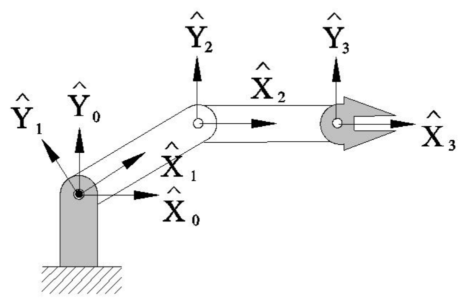
$${}^3\mathbf{N}_3 \;=\; {}^E\mathbf{J}\;{}^3\mathbf{F}_3.$$

Recall

$${}^0\mathbf{p}_E{}^i \;=\; {}^0\mathbf{p}_E - {}^0\mathbf{p}_i$$

Newton-Euler and Lagrange Example

The iterative Newton-Euler Equations are employed to compute the closed-form equations of motion of the planar 2R manipulator.



Initial Conditions:
The angular velocity and angular acceleration of the fixed base is:

$$^{0}\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad ^{0}\boldsymbol{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The forces and moments that link 2 (at the end-effector reference point, the origin of $\{3\}$) applies to the environment are:

$$^{3}\mathbf{f}_3 = \begin{bmatrix} ^{3}f_{3x} \\ ^{3}f_{3y} \\ ^{3}f_{3z} \end{bmatrix}, \quad ^{3}\mathbf{n}_3 = \begin{bmatrix} ^{3}n_{3x} \\ ^{3}n_{3y} \\ ^{3}n_{3z} \end{bmatrix}.$$

All of the Z-axes in their respective frames are described by the unit vector

$$\hat{\mathbf{Z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

. The magnitude of the angular velocity is ω. So, in the direction of the Z-axis it is:

$$\omega_{\hat{\mathbf{Z}}} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}.$$

The magnitude of the angular acceleration is α. So in the direction of the Z-axis it is:

$$\omega_{\hat{\mathbf{Z}}} = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}.$$

Position vectors locating relative positions of reference frame origins:

$$ {}^{0}\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{1}\mathbf{p}_2 = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{2}\mathbf{p}_3 = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}. $$

Position vectors locating relative locations of centers of gravity:

$$ {}^{1}\mathbf{p}_{1_{\mathrm{CG}}} = \begin{bmatrix} l_{1_{\mathrm{CG}}} \\ 0 \\ 0 \end{bmatrix}, \quad {}^{2}\mathbf{p}_{2_{\mathrm{CG}}} = \begin{bmatrix} l_{2_{\mathrm{CG}}} \\ 0 \\ 0 \end{bmatrix}. $$

The effects of gravity on each link is accounted for by assuming an imaginary acceleration of the base upwards by an amount equal to the magnitude $-g$:

$$ {}^{0}\mathbf{a}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}. $$

The inertia tensors assume that the coordinate reference frames are also the principal axes:

$$ \begin{aligned} {}^{1}\mathbf{I}^{\mathrm{CG}_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l_{zz1} \end{bmatrix}, \\ {}^{2}\mathbf{I}^{\mathrm{CG}_2} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l_{zz2} \end{bmatrix}. \end{aligned} $$

Rotation matrices:

$$ \begin{aligned} {}^{0}\mathbf{R}_1 &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ {}^{1}\mathbf{R}_2 &= \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ {}^{2}\mathbf{R}_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} $$

Application of Iterative Newton-Euler Equations:
Outward iterations to compute velocities and accelerations use the following recursive equations:

Frame 1 ($i = 0$):

$$
\begin{aligned}
{}^1\mathbf{a}_1 &= {}^0\mathbf{R}_1{}^T({}^0\mathbf{a}_0 + ({}^0\boldsymbol{\alpha}_0 \times {}^0\mathbf{p}_1) + ({}^0\boldsymbol{\omega}_0 \times ({}^0\boldsymbol{\omega}_0 \times {}^0\mathbf{p}_1))), \\
&= \begin{bmatrix} \sin(\theta_1)g \\ \cos(\theta_1)g \\ 0 \end{bmatrix}, \\
{}^1\boldsymbol{\omega}_1 &= \begin{bmatrix} 0 \\ 0 \\ \omega_1 \end{bmatrix} +{}^0 \mathbf{R}_1\,{}^0\boldsymbol{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ \omega_1 \end{bmatrix}, \\
{}^1\boldsymbol{\alpha}_1 &= {}^0\mathbf{R}_1\,{}^0\boldsymbol{\alpha}_0 + \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \end{bmatrix} + ({}^0\mathbf{R}_1\,{}^1\boldsymbol{\omega}_1) \times \begin{bmatrix} 0 \\ 0 \\ \omega_1 \end{bmatrix}, \\
&= \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \end{bmatrix}, \\
{}^1\mathbf{a}_{\mathrm{CG}_1} &= ({}^1\boldsymbol{\alpha}_1 \times {}^1\mathbf{p}_{\mathrm{CG}_1}) + ({}^1\boldsymbol{\omega}_1 \times ({}^1\boldsymbol{\omega}_1 \times {}^1\mathbf{p}_{\mathrm{CG}_1})) + {}^1\mathbf{a}_1, \\
&= \begin{bmatrix} -{\omega_1}^2 l_{\mathrm{CG}_1} + \sin(\theta_1)g \\ -{\alpha_1}^2 l_{\mathrm{CG}_1} + \cos(\theta_1)g \\ 0 \end{bmatrix}.
\end{aligned}
$$

Frame 2 ($i = 1$):

$$
\begin{aligned}
{}^2\mathbf{a}_2 &= {}^1\mathbf{R}_2{}^T({}^1\mathbf{a}_1 + ({}^1\boldsymbol{\alpha}_1 \times {}^1\mathbf{p}_2) + ({}^1\boldsymbol{\omega}_1 \times ({}^1\boldsymbol{\omega}_1 \times {}^1\mathbf{p}_2))), \\
&= \begin{bmatrix} \cos(\theta_2)(\sin(\theta_1)g - {\omega_1}^2\, l_1) + \sin(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1) \\ -\sin(\theta_2)(\sin(\theta_1)g - {\omega_1}^2\, l_1) + \cos(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1) \\ 0 \end{bmatrix}, \\
{}^2\boldsymbol{\omega}_2 &= \begin{bmatrix} 0 \\ 0 \\ \omega_2 \end{bmatrix} +{}^1 \mathbf{R}_2\,{}^1\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \omega_1 + \omega_2 \end{bmatrix}, \\
{}^2\boldsymbol{\alpha}_2 &= {}^1\mathbf{R}_2\,{}^1\boldsymbol{\alpha}_2 + \begin{bmatrix} 0 \\ 0 \\ \alpha_2 \end{bmatrix} + \left(({}^1\mathbf{R}_2\,{}^2\boldsymbol{\omega}_2) \times \begin{bmatrix} 0 \\ 0 \\ \omega_2 \end{bmatrix}\right), \\
&= \begin{bmatrix} 0 \\ 0 \\ \alpha_1 + \alpha_2 \end{bmatrix}, \\
{}^2\mathbf{a}_{\mathrm{CG}_2} &= ({}^2\boldsymbol{\alpha}_2 \times {}^2\mathbf{p}_{\mathrm{CG}_2}) + ({}^2\boldsymbol{\omega}_2 \times ({}^2\boldsymbol{\omega}_2 \times {}^2\mathbf{p}_{\mathrm{CG}_2})) + {}^2\mathbf{a}_2, \\
&= \begin{bmatrix} -(\omega_2 + \omega_1)^2\, l_{\mathrm{CG}_2} + \cos(\theta_2)(\sin(\theta_1)g - {\omega_1}^2\, l_1) + \sin(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1) \\ (\alpha_1 + \alpha_2) l_{\mathrm{CG}_2} - \sin(\theta_2)(\sin(\theta_1)g - {\omega_1}^2\, l_1) + \cos(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1) \\ 0 \end{bmatrix}.
\end{aligned}
$$

Frame 3 ($i=2$):

$$\begin{aligned}
{}^{3}\mathbf{a}_3 &= {}^{2}\mathbf{R}_3{}^{T}({}^{2}\mathbf{a}_2 + ({}^{2}\boldsymbol{\alpha}_2 \times {}^{2}\mathbf{p}_3) + ({}^{2}\boldsymbol{\omega}_2 \times ({}^{2}\boldsymbol{\omega}_2 \times {}^{2}\mathbf{p}_3))),\\
{}^{3}\boldsymbol{\omega}_3 &= {}^{2}\boldsymbol{\omega}_2,\\
{}^{3}\boldsymbol{\alpha}_3 &= {}^{2}\boldsymbol{\alpha}_2,\\
{}^{3}\mathbf{a}_{\mathrm{CG}_3} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Inward iterations to compute joint forces and torques use the following recursive equations:

Frame 3 ($i=2$): ${}^{3}\mathbf{f}_3$ and ${}^{3}\mathbf{n}_3$ are given as initial conditions.

Frame 2 ($i=1$):

$$\begin{aligned}
{}^{2}\mathbf{F}_2 &= {}^{2}\mathbf{a}_{\mathrm{CG}_2} \cdot m_2,\\
&= \begin{bmatrix} m_2[-(\omega_2+\omega_1)^2\, l_{\mathrm{CG}_2} + \cos(\theta_2)(\sin(\theta_1)g - \omega_1{}^2\, l_1) + \sin(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1)] \\ m_2[(\alpha_1+\alpha_2)l_{\mathrm{CG}_2} - \sin(\theta_2)(\sin(\theta_1)g - \omega_1{}^2\, l_1) + \cos(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1)] \\ 0 \end{bmatrix},\\
{}^{2}\mathbf{f}_2 &= {}^{2}\mathbf{F}_2 + {}^{2}\mathbf{R}_3\,{}^{3}\mathbf{f}_3,\\
&= \begin{bmatrix} m_2[-(\omega_2+\omega_1)^2\, l_{\mathrm{CG}_2} + \cos(\theta_2)(\sin(\theta_1)g - \omega_1{}^2\, l_1) + \sin(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1)] + {}^{3}f_{3x} \\ m_2[(\alpha_1+\alpha_2)l_{\mathrm{CG}_2} - \sin(\theta_2)(\sin(\theta_1)g - \omega_1{}^2\, l_1) + \cos(\theta_2)(\cos(\theta_1)g + \alpha_1\, l_1)] + {}^{3}f_{3y} \\ {}^{3}f_{3z} \end{bmatrix},\\
{}^{2}\mathbf{N}_2 &= {}^{2}\mathbf{I}^{\mathrm{CG}_2}\,{}^{2}\boldsymbol{\alpha}_2 + ({}^{2}\boldsymbol{\omega}_2 \times ({}^{2}\mathbf{I}^{\mathrm{CG}_2}\,{}^{2}\boldsymbol{\omega}_2)),\\
&= \begin{bmatrix} 0 \\ 0 \\ I_{zz2}(\alpha_1+\alpha_2) \end{bmatrix},\\
{}^{2}\mathbf{n}_2 &= {}^{2}\mathbf{N}_2 + {}^{3}\mathbf{R}_2\,{}^{3}\mathbf{n}_3 + ({}^{2}\mathbf{p}_{\mathrm{CG}_2} \times {}^{2}\mathbf{F}_2) + ({}^{2}\mathbf{p}_3 \times {}^{2}\mathbf{R}_3\,{}^{3}\mathbf{f}_3),\\
&= \begin{bmatrix} {}^{3}n_{3x} \\[1em] {}^{3}n_{3y} - l_2\,{}^{3}f_{3z} \\[1em] (I_{zz2}(\alpha_1+\alpha_2) + {}^{3}n_{3z} + l_{\mathrm{CG}_2}m_2((\alpha_1+\alpha_2)l_{\mathrm{CG}_2} \\ -\,\mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1\, g - \omega_1{}^2\, l_1) + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\, g + \alpha_1\, l_1)) + l_2\,{}^{3}f_{3y}) \end{bmatrix},\\
\boldsymbol{\tau}_2 &= {}^{2}\mathbf{n}_2[3],\\
&= I_{zz2}(\alpha_1+\alpha_2) + {}^{3}n_{3z} + l_{\mathrm{CG}_2}m_2((\alpha_1+\alpha_2)l_{\mathrm{CG}_2} - \mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1\, g - \omega_1{}^2\, l_1)\\
&\quad + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\, g + \alpha_1\, l_1)) + l_2\,{}^{3}f_{3y}.
\end{aligned}$$

Frame 1, $(i = 0)$:

$$
\begin{aligned}
{}^{1}\mathbf{F}_1 &= {}^{1}\mathbf{a}_{\mathrm{CG}_1} m_1, \\
&= \begin{bmatrix} m_1[-\omega_1{}^2 l_{\mathrm{CG}_1} + \sin(\theta_1) g] \\ m_2[-\alpha_1{}^2 l_{\mathrm{CG}_1} + \cos(\theta_1) g] \\ 0 \end{bmatrix}, \\
{}^{1}\mathbf{f}_1 &= {}^{1}\mathbf{F}_1 + {}^{1}\mathbf{R}_2\, {}^{2}\mathbf{f}_2, \\
&= \begin{bmatrix} (m_1(-\omega_1{}^2\, l_{\mathrm{CG}_1} + \mathrm{s}\,\theta_1\ g) + \mathrm{c}\,\theta_2(m_2[-(\omega_2+\omega_1)^2\, l_{\mathrm{CG}_2} \\ + \mathrm{c}\,\theta_2(\mathrm{s}\,\theta_1\ g - \omega_1{}^2\, l_1) + \mathrm{s}\,\theta_2(\mathrm{c}\,\theta_1\ g + \alpha_1\, l_1)] + {}^{3}f_{3x}) - \mathrm{s}\,\theta_2\,(m_2[(\alpha_1+\alpha_2) l_{\mathrm{CG}_2} \\ - \mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1 g - \omega_1{}^2\, l_1) + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\ g + \alpha_1\, l_1)] + {}^{3}f_{3y})) \\ \\ (m_1(-\alpha_1\, l_{\mathrm{CG}_1} + \mathrm{c}\,\theta_1\ g) + \mathrm{s}\,\theta_2(m_2[-(\omega_2+\omega_1)^2\, l_{\mathrm{CG}_2} \\ + \mathrm{c}\,\theta_2(\mathrm{s}\,\theta_1\ g - \omega_1{}^2\, l_1) + \mathrm{s}\,\theta_2(\mathrm{c}\,\theta_1\ g + \alpha_1\, l_1)] + {}^{3}f_{3x}) - \mathrm{c}\,\theta_2\,(m_2[(\alpha_1+\alpha_2) l_{\mathrm{CG}_2} \\ - \mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1\ g - \omega_1{}^2\, l_1) + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\ g + \alpha_1\, l_1)] + {}^{3}f_{3y})) \\ \\ {}^{3}f_{3z} \end{bmatrix}, \\
{}^{1}\mathbf{N}_1 &= ({}^{1}\mathbf{I}^{\mathrm{CG}_1}\, {}^{1}\boldsymbol{\alpha}_1) + ({}^{1}\boldsymbol{\omega}_1 \times ({}^{1}\mathbf{I}^{\mathrm{CG}_1}\, {}^{1}\boldsymbol{\omega}_1)), \\
&= \begin{bmatrix} 0 \\ 0 \\ I_{zz1}\,\alpha_1 \end{bmatrix}, \\
{}^{1}\mathbf{n}_1 &= {}^{1}\mathbf{N}_1 + ({}^{1}\mathbf{R}_2\, {}^{2}\mathbf{n}_2) + ({}^{1}\mathbf{p}_{\mathrm{CG}_1} \times {}^{1}\mathbf{F}_1) + ({}^{1}\mathbf{p}_2 \times {}^{1}\mathbf{R}_2\, {}^{2}\mathbf{f}_2), \\
&= \begin{bmatrix} \mathrm{c}\,\theta_2\, {}^{3}n_{3x} - \mathrm{s}\,\theta_2({}^{3}n_{3y} - l_2\, {}^{3}f_{3z}) \\ \\ \mathrm{s}\,\theta_2\, {}^{3}n_{3x} + \mathrm{c}\,\theta_2({}^{3}n_{3y} - l_2\, {}^{3}f_{3z}) - l_1\, {}^{3}f_{3z} \\ \\ (I_{zz1}\alpha_1 + I_{zz2}(\alpha_1+\alpha_2) + {}^{3}n_{3z} + l_{\mathrm{CG}_2} m_2((\alpha_1+\alpha_2) l_{\mathrm{CG}_2} \\ - \mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1\ g - \omega_1{}^2\, l_1) + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\ g + \alpha_1 l_1)) + l_2\, {}^{3}f_{3y} + l_{\mathrm{CG}_1} m_1(\alpha_1\ l_{\mathrm{CG}_1} + \mathrm{c}\,\theta_1\ g) \\ + l_1\,\mathrm{s}\,\theta_2(m_2(-(\omega_2+\omega_1)^2\, l_{\mathrm{CG}_2} + \mathrm{c}\,\theta_2(\mathrm{s}\,\theta_1\ g - \omega_1{}^2\, l_1) + \mathrm{s}\,\theta_2(\mathrm{c}\,\theta_1\ g + \alpha_1\, l_1)) + {}^{3}f_{3x}) \\ + \mathrm{c}\,\theta_2(m_2((\alpha_1+\alpha_2) l_{\mathrm{CG}_2} - \mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1\ g - \omega_1{}^2\, l_1) + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\ g + \alpha_1\, l_1)) + {}^{3}f_{3y})) \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\tau_1 &= {}^1\mathbf{n}_1[3], \\
&= I_{zz1}\alpha_1 + I_{zz2}(\alpha_1+\alpha_2) + {}^3n_{3z} + l_{\mathrm{CG}_2}m_2((\alpha_1+\alpha_2)l_{\mathrm{CG}_2} \\
&\quad - \mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1\, g - \omega_1{}^2\, l_1) + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\, g + \alpha_1 l_1)) + l_2\,{}^3f_{3y} + l_{\mathrm{CG}_1}m_1(\alpha_1\, l_{\mathrm{CG}_1} + \mathrm{c}\,\theta_1\, g) \\
&\quad + l_1\,\mathrm{s}\,\theta_2(m_2(-(\omega_2+\omega_1)^2\, l_{\mathrm{CG}_2} + \mathrm{c}\,\theta_2(\mathrm{s}\,\theta_1\, g - \omega_1{}^2\, l_1) + \mathrm{s}\,\theta_2(\mathrm{c}\,\theta_1\, g + \alpha_1\, l_1)) + {}^3f_{3x}) \\
&\quad + \mathrm{c}\,\theta_2(m_2((\alpha_1+\alpha_2)l_{\mathrm{CG}_2} - \mathrm{s}\,\theta_2(\mathrm{s}\,\theta_1\, g - \omega_1{}^2\, l_1) + \mathrm{c}\,\theta_2(\mathrm{c}\,\theta_1\, g + \alpha_1\, l_1)) + {}^3f_{3y}), \\
{}^1\boldsymbol{\tau}_1 &= (I_{zz1} + I_{zz2} + l_{\mathrm{CG}_1}{}^2\, m_1 + l_{\mathrm{CG}_2}m_2(l_{\mathrm{CG}_2} + \mathrm{c}\,\theta_2\, l_1) + \\
&\quad l_1\,(m_2\,\sin\theta_2{}^2\, l_1 - \mathrm{c}\,\theta_2\, m_2(l_{\mathrm{CG}_2} + \mathrm{c}\,\theta_2\, l_1)))\alpha_1 - l_1\,\mathrm{s}\,\theta_2\, l_{\mathrm{CG}_2}m_2\,\omega_2{}^2 - \\
&\quad 2l_1\,\mathrm{s}\,\theta_2\, m_2\omega_1 l_{\mathrm{CG}_2}\omega_2 + (l_1\,\mathrm{s}\,\theta_2\, l_{\mathrm{CG}_2}m_2 + l_1(\mathrm{s}\,\theta_2\, m_2\,(-l_{\mathrm{CG}_2} - \mathrm{c}\,\theta_2\, l_1) + \mathrm{c}\,\theta_2\, m_2\,\mathrm{s}\,\theta_2\, l_1))\omega_1{}^2 \\
&\quad + l_{\mathrm{CG}_2}\, m_2(\alpha_2\, l_{\mathrm{CG}_2} - \mathrm{s}\,\theta_2\;\mathrm{s}\,\theta_1\, g + \mathrm{c}\,\theta_2\;\mathrm{c}\,\theta_1\, g) \\
&\quad + l_1(\mathrm{s}\,\theta_2(m_2(\mathrm{c}\,\theta_2\;\mathrm{s}\,\theta_1\, g + \mathrm{s}\,\theta_2\;\mathrm{c}\,\theta_1\, g) + {}^3f_{3x}) + \\
&\quad \mathrm{c}\,\theta_2(m_2(\alpha_2\, l_{\mathrm{CG}_2} - \mathrm{s}\,\theta_2\;\mathrm{s}\,\theta_1\, g + \mathrm{c}\,\theta_2\;\mathrm{c}\,\theta_1\, g) + {}^3f_{3y})) \\
&\quad + I_{zz2}\alpha_2 + l_2\,{}^3f_{3y} +{}^3\, n_{3z} + l_{\mathrm{CG}_1}\, m_1\;\mathrm{c}\,\theta_1\, g, \\
{}^2\boldsymbol{\tau}_2 &= (l_{\mathrm{CG}_2}{}^2\, m_2 + I_{zz2})\alpha_2 + (I_{zz2} + l_{\mathrm{CG}_2}\, m_2(l_{\mathrm{CG}_2} - \mathrm{c}\,\theta_2\, l_1))\alpha_1 \\
&\quad + l_1\,\mathrm{s}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2\,\omega_1{}^2 + l_2\,{}^3f_{3y} + {}^3n_{3z} + l_{\mathrm{CG}_2}\, m_2(-\mathrm{s}\,\theta_2\;\mathrm{s}\,\theta_1\, g + \mathrm{c}\,\theta_2\;\mathrm{c}\,\theta_1\, g).
\end{aligned}
$$

State-space representation:

$$
\begin{aligned}
\mathbf{M} &= \begin{bmatrix}
\begin{array}{c}(I_{zz1} + I_{zz2} + l_{\mathrm{CG}_1}{}^2\, m_1 + l_{\mathrm{CG}_2}{}^2\, m_2 \\ +2l_1\,\mathrm{c}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2 + m_2 l_1{}^2)\end{array} & \begin{array}{c}(l_{\mathrm{CG}_2}{}^2\, m_2 + l_1\,\mathrm{c}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2 \\ +I_{zz2})\end{array} \\
l_{\mathrm{CG}_2}{}^2\, m_2 + l_1\,\mathrm{c}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2 + I_{zz2} & l_{\mathrm{CG}_2}{}^2\, m_2 + I_{zz2}
\end{bmatrix}. \\
\mathbf{V} &= \begin{bmatrix}
\begin{array}{c}((l_1\;\mathrm{s}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2 + l_1(\mathrm{s}\,\theta_2\, m_2(-l_{\mathrm{CG}_2} - \mathrm{c}\,\theta_2\, l_1) \\ +\mathrm{c}\,\theta_2\, m_2\;\mathrm{s}\,\theta_2\, l_1))\omega_1{}^2 - l_1\;\mathrm{s}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2\,\omega_2{}^2 \\ +(l_1\;\mathrm{s}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2 + l_1(\mathrm{s}\,\theta_2\, m_2(-l_{\mathrm{CG}_2} - \mathrm{c}\,\theta_2\, l_1) \\ +\mathrm{c}\,\theta_2\, m_2\;\mathrm{s}\,\theta_2\, l_1))\omega_1\,\omega_2)\end{array} \\
l_1\;\mathrm{s}\,\theta_2\, l_{\mathrm{CG}_2}\, m_2\,\omega_1{}^2
\end{bmatrix}. \\
\mathbf{G} &= \begin{bmatrix}
\begin{array}{c}(l_{\mathrm{CG}_2}\, m_2(-\mathrm{s}\,\theta_2\;\mathrm{s}\,\theta_1 + \mathrm{c}\,\theta_2\;\mathrm{c}\,\theta_1) + l_1(\mathrm{s}\,\theta_2\, m_2(\mathrm{c}\,\theta_2\;\mathrm{s}\,\theta_1 \\ +\mathrm{s}\,\theta_2\;\mathrm{c}\,\theta_1) + \mathrm{c}\,\theta_2\, m_2(-\mathrm{s}\,\theta_2\;\mathrm{s}\,\theta_1 + \mathrm{c}\,\theta_2\;\mathrm{c}\,\theta_1))) + l_{\mathrm{CG}_1}\, m_1\;\mathrm{c}\,\theta_1\, g\end{array} \\
l_{\mathrm{CG}_2}\, m_2(-\mathrm{s}\,\theta_2\;\mathrm{s}\,\theta_1 + \mathrm{c}\,\theta_2\;\mathrm{c}\,\theta_1)
\end{bmatrix}. \\
\mathbf{N} &= {}^1\boldsymbol{\tau}_1\;{}^2\boldsymbol{\tau}_2 - \mathbf{M}\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} - \mathbf{V} - \mathbf{G} \\
&= \begin{bmatrix}
\begin{array}{c}(l_1\;\mathrm{s}\,\theta_2\,{}^3f_{3x} + l_1\;\mathrm{c}\,\theta_2\,{}^3f_{3y} \\ +l_2\,{}^3f_{3y} + {}^3n_{3z} - 2l_1\;\mathrm{s}\,\theta_2\, m_2\,\omega_1\, l_{\mathrm{CG}_2}\,\omega_2)\end{array} \\
l_2\,{}^3f_{3y} +{}^3\, n_{3z}
\end{bmatrix}.
\end{aligned}
$$

Now if we let $l_1 = 2$, $l_2 = 2$, $l_{\mathrm{CG}_1} = 1$, $l_{\mathrm{CG}_2} = 1$, $m_1 = 2$, $m_2 = 1$, $I_{xx1} = 1$,

$I_{yy1} = 2$, $I_{zz1} = 3$, $I_{xx2} = 3$, $I_{yy2} = 2$, $I_{zz2} = 1$, $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{3}$, then:

$$\mathbf{M} = \begin{bmatrix} 13 & 3 \\ 3 & 2 \end{bmatrix}.$$

And the eigenvectors are (using Matlab's eigenvectors(M) command):

$$\begin{bmatrix} -0.9689931823 \\ -0.2470874613 \end{bmatrix}, \begin{bmatrix} 0.2470874613 \\ -0.9689931823 \end{bmatrix}.$$

<u>Or, using Lagrange:</u>

$$\begin{aligned}
{}^0\mathbf{T}_1 &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^1\mathbf{T}_2 &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^0\mathbf{T}_2 &= {}^0\mathbf{T}_1\, {}^1\mathbf{T}_2, \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & \cos\theta_1\, l_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & \sin\theta_1\, l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^0\mathbf{R}_1 &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
{}^0\mathbf{R}_2 &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

The inertia tensors assume that the coordinate reference frames are aligned with principal axes:

$$\begin{aligned}
{}^1\mathbf{I}^{\mathrm{CG}_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}, \\
{}^1\mathbf{I}^{\mathrm{CG}_2} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}.
\end{aligned}$$

Now these inertia tensors must be rotated to align with the base frame:

$$\begin{aligned}
{}^{0}\mathbf{I}^{\mathrm{CG}_1} &= {}^{0}\mathbf{R}_1\ {}^{1}\mathbf{I}^{\mathrm{CG}_1}\ {}^{0}\mathbf{R}_1{}^{T}, \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}, \\
{}^{0}\mathbf{I}^{\mathrm{CG}_2} &= {}^{0}\mathbf{R}_2\ {}^{2}\mathbf{I}^{\mathrm{CG}_2}\ {}^{0}\mathbf{R}_2{}^{T}, \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
{}^{0}\hat{\mathbf{Z}}_1 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
{}^{0}\hat{\mathbf{Z}}_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
\mathbf{p}_{\mathrm{CG}_1} &= \begin{bmatrix} l_{\mathrm{CG}_1} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\
{}^{0}\mathbf{p}_{\mathrm{CG}_1} &= {}^{0}\mathbf{T}_1\ \mathbf{p}_{\mathrm{CG}_1}, \\
&= \begin{bmatrix} \cos\theta_1\ l_{\mathrm{CG}_1} \\ \sin\theta_1\ l_{\mathrm{CG}_1} \\ 0 \\ 1 \end{bmatrix}, \\
\mathbf{p}_{\mathrm{CG}_2} &= \begin{bmatrix} l_{\mathrm{CG}_2} \\ 0 \\ 0 \end{bmatrix}, \\
{}^{0}\mathbf{p}_{\mathrm{CG}_2}{}^{2} &= {}^{0}\mathbf{R}_2\ \mathbf{p}_{\mathrm{CG}_2}, \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) l_{\mathrm{CG}_2} \\ \sin(\theta_1+\theta_2) l_{\mathrm{CG}_2} \\ 0 \end{bmatrix}, \\
\mathbf{p}_{\mathrm{CG}_2}{}^{1} &= \begin{bmatrix} l_1 + l_{\mathrm{CG}_2}\cos(\theta_2) \\ l_{\mathrm{CG}_2}\sin(\theta_2) \\ 0 \\ 1 \end{bmatrix}, \\
{}^{0}\mathbf{p}_{\mathrm{CG}_2}{}^{1} &= \begin{bmatrix} \cos\theta_1\ l_1 + \cos(\theta_1+\theta_2) l_{\mathrm{CG}_2} \\ \sin\theta_1 + \sin(\theta_1+\theta_2) l_{\mathrm{CG}_2} \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Jacobians:

$$
\begin{aligned}
{}^{1}\mathbf{J}_{L_1} &= \begin{bmatrix} \sin\theta_1 \; l_{\mathrm{CG}_1} \\ \cos\theta_1 \; l_{\mathrm{CG}_1} \\ 0 \end{bmatrix}, \\
\mathbf{J}_{L_1} &= \begin{bmatrix} \sin\theta_1 \; l_{\mathrm{CG}_1} & 0 \\ \cos\theta_1 \; l_{\mathrm{CG}_1} & 0 \\ 0 & 0 \end{bmatrix}, \\
\mathbf{J}_{A_1} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
{}^{1}\mathbf{J}_{L_2} &= \begin{bmatrix} -\sin\theta_1 \; l_1 - \sin(\theta_1 + \theta + 2) l_{\mathrm{CG}_2} \\ \cos\theta_1 \; l_1 + \cos(\theta_1 + \theta_2) l_{\mathrm{CG}_2} \\ 0 \end{bmatrix}, \\
\mathbf{J}_{L_2} &= \begin{bmatrix} \sin\theta_1 \; l_1 - \sin(\theta_1 + \theta_2) l_{\mathrm{CG}_2} & -\sin(\theta_1 + \theta_2) l_{\mathrm{CG}_2} \\ \cos\theta_1 \; l_1 - \cos(\theta_1 + \theta_2) l_{\mathrm{CG}_2} & -\cos(\theta_1 + \theta_2) l_{\mathrm{CG}_2} \\ 0 & 0 \end{bmatrix}, \\
\mathbf{J}_{A_2} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.
\end{aligned}
$$

Mass matrix $\mathbf{M}$:
For $i = 1$:

$$
\begin{aligned}
\mathbf{M}_{1L} &= \mathbf{J}_{L_1}{}^{T} \, \mathbf{J}_{L_1} \, m_1, \\
&= \begin{bmatrix} m_1 \, l_{\mathrm{CG}_1}{}^{2} & 0 \\ 0 & 0 \end{bmatrix}, \\
\mathbf{M}_{1A} &= \mathbf{J}_{A_1}{}^{T} \, {}^{0}\mathbf{I}^{\mathrm{CG}_1} \, \mathbf{J}_{A_1}, \\
&= \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

For $i = 2$:

$$
\begin{aligned}
\mathbf{M}_{2L} &= {\mathbf{J}_{L_2}}^T \, \mathbf{J}_{L_2} \, m_2, \\
&= \left[\begin{array}{cc}
\begin{array}{c} (m_2 \, {l_1}^2 + 2m_2 \, l_{\mathrm{CG}_2} \sin(\theta_1+\theta_2) \sin\theta 1 \, l_1 \\ +m_2 \, {l_{\mathrm{CG}_2}}^2 + 2m_2 \, l_{\mathrm{CG}_2} \cos(\theta_1+\theta_2) \cos\theta_1 \, l_1) \end{array} & \begin{array}{c} (m_2 \, l_{\mathrm{CG}_2} \sin(\theta_1+\theta_2) \sin\theta_1 \, l_1 \\ +m_2 \, l_{\mathrm{CG}_2} \cos(\theta_1+\theta_2) \cos\theta_1 \, l_1) \end{array} \\
\\
\begin{array}{c} m_2 \, l_{\mathrm{CG}_2} \sin(\theta_1+\theta_2) \sin\theta_1 \, l_1 + m_2 \, {l_{\mathrm{CG}_2}}^2 \\ +m_2 l_{\mathrm{CG}_2} \cos(\theta_1+\theta_2) \cos\theta_1 \, l_1 \end{array} & m_2 \, l_{\mathrm{CG}_2}
\end{array} \right], \\
\mathbf{M}_{2A} &= {\mathbf{J}_{A_2}}^T \, {}^0\mathbf{I}^{\mathrm{CG}_2} \, \mathbf{J}_{A_2}, \\
&= \left[\begin{array}{cc} I_{zz2} & I_{zz2} \\ I_{zz2} & I_{zz2} \end{array} \right], \\
\mathbf{M} &= \left[\begin{array}{cc}
\begin{array}{c} (m_1 {\mathbf{I}^{\mathrm{CG}_1}}^2 + I_{zz1} + m_2 \, {l_1}^2 \\ + 2m_2 \, l_{\mathrm{CG}_2} \, l_1 \, \cos\theta_1 + m_2 \, {l_{\mathrm{CG}_2}}^2 + I_{zz2}) \end{array} & m_2 \, l_{\mathrm{CG}_2} \, l_1 \, \cos\theta_2 + m_2 \, {l_{\mathrm{CG}_2}}^2 + I_{zz2} \\
\\
m_2 \, l_{\mathrm{CG}_2} \, l_1 \, \cos\theta_2 + m_2 \, l_{\mathrm{CG}_2}^2 + I_{zz2} & m_2 \, {l_{\mathrm{CG}_2}}^2 + I_{zz2}
\end{array} \right].
\end{aligned}
$$

And if we sub in the same values as before, we get:

$$\mathbf{M} = \left[\begin{array}{cc} 13 & 3 \\ 3 & 2 \end{array} \right].$$

And the same eigenvectors:

$$\left[\begin{array}{c} -0.9689931823 \\ -0.2470874613 \end{array} \right], \left[\begin{array}{c} 0.2470874613 \\ -0.9689931823 \end{array} \right].$$