CONTINUOUS APPROXIMATE SYNTHESIS OF PLANAR FUNCTION-GENERATORS MINIMISING THE DESIGN ERROR

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ABSTRACT

It has been observed in the literature that as the cardinality of the prescribed discrete input-output data set increases, the corresponding four-bar linkages that minimise the Euclidean norms of the design and structural errors tend to converge to the same linkage. The important implication is that minimising the Euclidean norm of the structural error can be accomplished implicitly by minimising that of the design error. The problem is that the approximate synthesis of a device that minimises the structural error is very computationally expensive compared to one that minimises the design error. Hence, the goal of this paper is to take the first step towards proving that as the cardinality of the data set tends towards infinity that observation is indeed true. This will be accomplished by integrating the synthesis equations in the range between minimum and maximum inputs, thereby reposing the discrete approximate synthesis problem as a continuous one. In this paper we prove that a lower bound of the Euclidean norm of the design error for a planar RRRR function-generating linkage exists and is attained with continuous approximate synthesis.

Keywords: approximate and continuous kinematic synthesis; design error; structural error; function-generating linkage.

SYNTHÈSE CINÉMATIQUE DES GÉNÉRATEURS DE FONCTION PLANE

RÉSUMÉ

Il a été observé dans la litérature que lorsque la cardinalité des données entrées-sorties discrètes prescrites augmentent, les liens des quadrilatères articulés correspondants qui minimisent les normes euclidiennes des erreurs de conception et des erreurs structurelles tendent à converger vers la même liaison. La conséquence importante est que la minimisation de la norme euclidienne de l'erreur structurale peut être accomplie en minimisant implicitement l'erreur de conception. Le problème est que la synthèse approximative d'une méthode qui minimise l'erreur de calcul de structure est très couteuse par rapport à celle qui minimise l'erreur de conception. Par conséquent, l'objectif de cet article est de démontrer que la cardinalité de l'ensemble de données tend vers l'infini. Ceci sera réalisé par l'intégration des équations de synthèse dans la gamme entre les entrées minimum et maximum, reposant ainsi le problème de synthèse approximatif discret comme un processus continu. Dans cet article, nous démontrons que la limite inférieure de la norme euclidienne de l'erreur de conception pour une liaison fonction génératrice plane RRRR existe et est atteint avec une synthèse approximative continue.

Mots-clés : la synthèse cinématique approximatives discrètes et continue; erreur de conception; erreur structurelle; liaison fonction génératrice.

1. INTRODUCTION

Design and structural errors are important performance indicators in the assessment and optimisation of function-generating linkages arising by means of approximate synthesis. The *design error* indicates the error residual incurred by a specific linkage in satisfying its synthesis equations. The *structural error*, in turn, is the difference between the prescribed linkage output and the actual generated output for a given input value [1]. From a design point of view it may be successfully argued that the structural error is the one that really matters, for it is directly related to the performance of the linkage.

It was shown in [2] that as the cardinality of the prescribed discrete input-output (I/O) data-set increases, the corresponding linkages that minimise the Euclidean norms of the design and structural errors tend to converge the same linkage. The important implication of this observation is that the minimisation of the Euclidean norm of the structural error can be accomplished indirectly via the minimisation of the corresponding norm of the design error, provided that a suitably large number of I/O pairs is prescribed. Note that the minimisation of the Euclidean norm of the design error leads to a linear least-squares problem whose solution can be obtained directly [3, 4], while the minimisation of the same norm of the structural error leads to a nonlinear least-squares problem, and hence, calls for an iterative solution [1].

Several issues have arisen in the design error minimisation for four-bar linkages. First, the condition number of the synthesis matrix may lead to design parameters that poorly approximate the prescribed function [5]. This problem can be addressed through careful selection of the I/O pairs used to generate the synthesis matrix. Otherwise, it has also been suggested to introduce dial zeros whose values are chosen to minimise the condition number of the synthesis matrix [6]. Second, the design parameters depend on the I/O set cardinality. However, some convergence has been observed as the number of I/O pairs grows. Hence, the I/O set cardinality might be fixed as soon as the minimal design error reaches some tolerance [2].

The goal of this paper is to take the first step towards proving that the convergence observed in [2] is true for planar four-bar function-generators. More precisely, a proof will be given for the design error that as the cardinality of the I/O data set increases from numbers of discrete pairs to infinity between minimum and maximum pairs that a lower bound for the 2-norm for the design error exists, and corresponds to the infinite I/O set, thereby changing the discrete approximate synthesis problem to a continuous approximate synthesis problem. To this end, the design error minimisation occurs in the space of a continuous function possessing a 2-norm defined later in this paper. However, our study is restricted to the planar RRRR function-generating linkage, where R denotes *revolute joint*, synthesized using the kinematic model defined in [7].

2. DESIGN ERROR MINIMISATION: THE DISCRETE APPROXIMATE APPROACH

The synthesis problem of planar four-bar function-generators consists of determining all relevant design parameters such that the mechanism can produce a prescribed set of m I/O pairs, $\{\psi_i, \phi_i\}_1^m$, where ψ_i and ϕ_i represent the i^{th} input and output variables, respectively, and m is the cardinality of the data-set. We define n to be the number of independent design parameters required to fully characterise the mechanism. For planar RRRR linkages, $n = 3$ [7]. If $m = n$, the problem is termed *exact synthesis* and may be considered a special case of approximate synthesis where $m > n$.

We consider the optimisation problem of planar four-bar function-generators as the approximate solution of an overdetermined linear system of equations with the minimum error. The synthesis equations that are used to establish the linear system of equations for a four-bar function generator that are used here are the *Freudenstein Equations* from [7]. Consider the mechanism in Figure 1(a). The i^{th} configuration is governed by:

$$k_1 + k_2 \cos(\phi_i) - k_3 \cos(\psi_i) \quad = \quad \cos(\psi_i - \phi_i), \tag{1}$$

where the k's are the *Freudenstein Parameters*, which are the following link length ratios:

$$k_1 = \frac{(a_1^2 + a_2^2 + a_4^2 - a_3^2)}{2a_2a_4}; \quad k_2 = \frac{a_1}{a_2}; \quad k_3 = \frac{a_1}{a_4}. \tag{2}$$

Given a set of three Freudenstein parameters, the corresponding set of link lengths, scaled by a_1, are:

$$\Rightarrow a_1 = 1; \quad a_2 = \frac{1}{k_2}; \quad a_4 = \frac{1}{k_3}; \quad a_3 = (a_1^2 + a_2^2 + a_4^2 - 2a_2a_4k_1)^{1/2}. \tag{3}$$

The set of I/O equations can be written in the following form, using Equation (1)

$$\mathbf{Sk} \;=\; \mathbf{b}, \tag{4}$$

where $\mathbf{S}$ is the $m \times 3$ *synthesis matrix*, whose i^{th} row is the 1×3 array $\mathbf{s}_i$, $\mathbf{b}$ is an m-dimensional vector, whereas $\mathbf{k}$ is the 3-dimensional vector of design variables called the *Freudenstein parameters* [7]. For the planar RRRR mechanism we have:

$$\mathbf{s}_i \;=\; \left[\; 1 \quad \cos\phi_i \quad -\cos\psi_i \;\right], \quad i = 1, ..., m, \tag{5}$$

$$b_i \;=\; \left[\; \cos(\psi_i - \phi_i) \;\right], \quad i = 1, ..., m, \tag{6}$$

$$\mathbf{k} \;=\; \left[\; k_1 \quad k_2 \quad k_3 \;\right]^T. \tag{7}$$

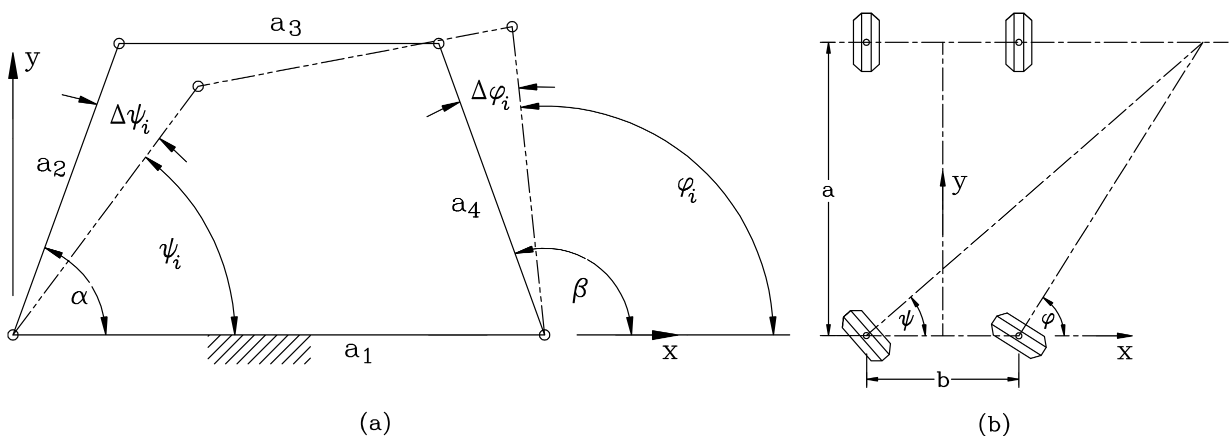

Fig. 1. (a) A four-bar linkage in two configurations. (b) Graphical illustration of the *Steering Condition*.

The synthesised linkage will only be capable of generating the desired function approximately. The design error is the algebraic difference of the left-hand side of Equation (4) less the right-hand side. Because we will be comparing errors associated with different cardinalities, we now include the cardinality m in the definition. The m-dimensional design error vector $\mathbf{d}_m$ for a set of m ($m > 3$) I/O pairs, $\{(\psi_i, \phi_i)_{i=1...m}\}$, is defined as:

$$\mathbf{d}_m = \mathbf{S}_m\mathbf{k} - \mathbf{b}_m. \tag{8}$$

If the outputs prescribed by the functional relationship, $\phi_{pres,i}$, correspond precisely to the outputs generated by the mechanism, i.e., $\phi_{gen,i}$, then, $\|\mathbf{d}_m\| = 0$. However, for a general prescribed function $\phi_{pres}(\psi)$, $\|\mathbf{d}_m\| \neq 0$ and we seek the Freudenstein parameter vector that minimises the norm of the design error vector. In general, the weighted Euclidian norm is used:

$$\|\mathbf{d}_m\|^2_{\mathbf{W}_m,2} = \frac{1}{2}\,\mathbf{d}_m^T\mathbf{W}_m\mathbf{d}_m, \tag{9}$$

where $\mathbf{W}_m$ is an $m \times m$ diagonal matrix with strictly positive elements. In a typical design problem, $\mathbf{W}_m$ is used to adjust the impact on the optimisation of specific I/O pairs. However, for the purposes of this work, $\mathbf{W}_m$ will be set to the identity matrix, $\mathbf{I}_m$. The optimal Freudenstein parameters $\mathbf{k}_m^*$ for this norm are:

$$\mathbf{k}_m^* = \mathbf{S}_m^+ \mathbf{b}_m, \tag{10}$$

where $\mathbf{S}_m^+$ is the Moore-Penrose generalized inverse of the synthesis matrix, and the corresponding minimal design error is:

$$\min_{\mathbf{k}} \|\mathbf{d}_m\|_2 = \|\mathbf{d}_m^*\|_2 = \|(\mathbf{I}_m - \mathbf{S}_m \mathbf{S}_m^+)\mathbf{b}_m\|_2. \tag{11}$$

For numerical stability considerations, it is always desirable to have a well-conditioned synthesis matrix, otherwise the numerical values of $\mathbf{S}_m^+$ may be significantly distorted by very small singular values, or singular values identically equal to zero, leading to optimised $\mathbf{k}$ that imply a mechanism which very poorly approximates the function. Hence, the *dial zeros* α and β have been introduced to minimise the condition number, κ, i.e. the ratio of the maximum to the minimum singular values:

$$\psi = \alpha + \Delta\psi; \quad \phi = \beta + \Delta\phi. \tag{12}$$

Including the dial zeros, the synthesis equation, Equation (1) becomes:

$$k_1 + k_2 \cos(\beta + \Delta\phi) - k_3 \cos(\alpha + \Delta\psi) = \cos(\alpha + \Delta\psi - \beta - \Delta\phi), \tag{13}$$

and, the I/O pairs are regarded as a set of incremental angular changes $\{(\Delta\psi_i, \Delta\phi_i)_{i=0..m}\}$. $\mathbf{d}_m^*$, $\mathbf{k}_m^*$ and $\mathbf{S}_m$ are now also functions of the dial zeros. With this modification, the design error minimisation problem can be efficiently solved in a least squares sense in two steps:

1. find the dial zeros to minimise the condition number, $\kappa_m(\alpha, \beta)$, of the synthesis matrix, $\mathbf{S}$;
2. find the corresponding optimal Freudenstein parameters using Equation (10).

3. DESIGN ERROR MINIMISATION: THE CONTINUOUS APPROXIMATE APPROACH

A major issue associated with the discrete approach to the design error minimisation is the appropriate choice for the cardinality of the I/O pair data set such that the minimisation of the structural error is implied. Indeed, the choice of m depends on the prescribed function $\Delta\phi(\Delta\psi)$ and m is generally fixed when some level of convergence is observed. For the example used in [2] $m = 40$ was observed to be a good choice. We now propose to evaluate the design error over the continuous range $[\Delta\psi_{min}, \Delta\psi_{max}]$ of the prescribed function. This requires the function to be continuous over $[\Delta\psi_{min}, \Delta\psi_{max}]$, and also requires a different vector space, denoted $\mathscr{C}^0([\Delta\psi_{min}, \Delta\psi_{max}])$, where upon the following 2-norm has been imposed:

$$\forall f \in \mathscr{C}^0([\Delta\psi_{min}, \Delta\psi_{max}]), \|f\|_2 = \left(\int_{\Delta\psi_{min}}^{\Delta\psi_{max}} |f|^2(x)dx \right)^{\frac{1}{2}}. \tag{14}$$

Assuming that the prescribed function belongs to $\mathscr{C}^0([\Delta\psi_{min}, \Delta\psi_{max}])$, the design error is:

$$\|\mathbf{d}(\alpha, \beta)\|_2 = \left(\int_{\Delta\psi_{min}}^{\Delta\psi_{max}} (k_1 + k_2 \cos(\beta + \Delta\phi) - k_3 \cos(\alpha + \Delta\psi) - \cos(\alpha + \Delta\psi - \beta - \Delta\phi))^2 d\Delta\psi \right)^{\frac{1}{2}}. \tag{15}$$

After some algebraic manipulation, it can be shown that Equation (15) is a quadratic function in terms of the Freudenstein parameters:

$$\|\mathbf{d}(\alpha, \beta)\|_2^2 = \mathbf{k}^T \mathbf{A}(\alpha, \beta)\mathbf{k} - 2\mathbf{e}(\alpha, \beta)^T \mathbf{k} + c(\alpha, \beta), \tag{16}$$

where $\mathbf{A}(\alpha,\beta)$ is a 3×3 symmetric matrix whose six distinct elements a_{ij} are:

$$
\begin{aligned}
a_{11} &= \int_{\Delta\psi_{min}}^{\Delta\psi_{max}} d\Delta\psi;\\
a_{12} &= \cos(\beta)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\phi)d\Delta\psi - \sin(\beta)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\sin(\Delta\phi)d\Delta\psi;\\
a_{13} &= -\cos(\alpha)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\psi)d\Delta\psi + \sin(\alpha)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\sin(\Delta\psi)d\Delta\psi;\\
a_{22} &= \cos(\beta)^2\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\phi)^2 d\Delta\psi - 2\cos(\beta)\sin(\beta)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\phi)\sin(\Delta\phi)d\Delta\psi\\
&\quad + \sin(\beta)^2\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\sin(\Delta\phi)^2 d\Delta\psi;\\
a_{23} &= -\cos(\alpha)\cos(\beta)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\psi)\cos(\Delta\phi)d\Delta\psi + \cos(\alpha)\sin(\beta)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\psi)\sin(\Delta\phi)d\Delta\psi\\
&\quad + \sin(\alpha)\cos(\beta)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\sin(\Delta\psi)\cos(\Delta\phi)d\Delta\psi - \sin(\alpha)\sin(\beta)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\sin(\Delta\psi)\sin(\Delta\phi)d\Delta\psi;\\
a_{33} &= \cos(\alpha)^2\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\psi)^2 d\Delta\psi - 2\cos(\alpha)\sin(\alpha)\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\Delta\psi)\sin(\Delta\psi)d\Delta\psi\\
&\quad + \sin(\alpha)^2\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\sin(\Delta\psi)^2 d\Delta\psi;
\end{aligned}
$$

$\mathbf{e}(\alpha,\beta)$ is a 3-dimensional vector whose elements are:

$$
\begin{aligned}
e_1 &= \int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\alpha+\Delta\psi-\beta-\Delta\phi)d\Delta\psi;\\
e_2 &= \int_{\Delta\psi_{min}}^{\Delta\psi_{max}}(\cos(\beta+\Delta\phi)\cos(\alpha+\Delta\psi-\beta-\Delta\phi))d\Delta\psi;\\
e_3 &= -\int_{\Delta\psi_{min}}^{\Delta\psi_{max}}(\cos(\alpha+\Delta\psi)\cos(\alpha+\Delta\psi-\beta-\Delta\phi))d\Delta\psi;
\end{aligned}
$$

and finally $c(\alpha,\beta)$ is a scalar having the form:

$$
c = \int_{\Delta\psi_{min}}^{\Delta\psi_{max}}\cos(\alpha+\Delta\psi-\beta-\Delta\phi)^2 d\Delta\psi.
$$

If $\mathbf{A}(\alpha,\beta)$ is positive definite, the optimal Freudenstein parameters $\mathbf{k}^*(\alpha,\beta)$ which minimise $\|\mathbf{d}(\alpha,\beta)\|_2^2$ (or equivalently $\|\mathbf{d}(\alpha,\beta)\|_2$) are:

$$
\mathbf{k}^*(\alpha,\beta) = \mathbf{A}^{-1}(\alpha,\beta)\mathbf{e}(\alpha,\beta), \tag{17}
$$

and the minimal design error is:

$$
\min_{\mathbf{k}}\|\mathbf{d}(\alpha,\beta)\|_2 = \|\mathbf{d}^*(\alpha,\beta)\|_2 = c(\alpha,\beta) - \mathbf{e}(\alpha,\beta)^T\mathbf{A}^{-1}(\alpha,\beta)\mathbf{e}(\alpha,\beta). \tag{18}
$$

The assumption of positive definiteness for $\mathbf{A}(\alpha,\beta)$ will be discussed in Section 4. However, a necessary condition for $\mathbf{A}(\alpha,\beta)$ to be positive definite is that it is non-singular. This justifies *a posteriori* why we use the dial zeros. As in Section 2, the design error minimisation problem is solved in two steps:

1. find the dial zeros to minimise the condition number $\kappa(\alpha,\beta)$ of $\mathbf{A}(\alpha,\beta)$;
2. find the corresponding optimal Freudenstein parameters using Equation (17).

Intuitively, the continuous approximate approach should correspond to the limit of the discrete approximate approach. This is made more clear in the next section.

4. THE DISCRETE APPROXIMATE APPROACH IS LOWER BOUNDED BY THE CONTINUOUS APPROXIMATE APPROACH

In this section, we assume that $\Delta\phi_{pres}(\Delta\psi)$ is a continuously differentiable function (note that Propositions 1, 2 and 3 only require continuity). With this assumption and using the notation introduced in the previous sections, the following propositions hold.

Proposition 1 $\mathbf{A}(\alpha,\beta)$ is semi-positive definite, and

$$\lim_{m\to\inf}\frac{1}{\kappa_m(\alpha,\beta)}=\frac{1}{\kappa(\alpha,\beta)}.$$

Proposition 2 if $\mathbf{A}(\alpha,\beta)$ possesses full rank, then,

$$\lim_{m\to\inf}\mathbf{k}_m^*(\alpha,\beta)=\mathbf{k}^*(\alpha,\beta).$$

Recall that $\mathbf{k}^*(\alpha,\beta)$ minimises the design error under the condition that $\mathbf{A}(\alpha,\beta)$ is positive definite. Now, from Proposition 1, we can claim that $\mathbf{A}(\alpha,\beta)$ is at least semi-positive definite. However, the positive definitiveness is not guaranteed and it justifies somehow the need of the assumption in Proposition 2.

Proposition 3 if $\mathbf{A}(\alpha,\beta)$ possesses full rank, then,

$$\lim_{m\to\inf}\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}\|\mathbf{d}_m^*(\alpha,\beta)\|_2=\|\mathbf{d}^*(\alpha,\beta)\|_2.$$

Proposition 4 if the optimal solution (α^*,β^*) is unique, then,

$$\lim_{m\to\inf}(\alpha_m^*,\beta_m^*)=(\alpha^*,\beta^*).$$

Proposition 5 if the optimal solution (α^*,β^*) is unique, then,

$$\lim_{m\to\inf}\frac{1}{\kappa_m(\alpha_m,\beta_m)}=\frac{1}{\kappa(\alpha^*,\beta^*)}.$$

Moreover, if $\mathbf{A}(\alpha^*,\beta^*)$ possesses full rank, then,

$$\lim_{m\to\inf}\mathbf{k}_m^*(\alpha_m,\beta_m)=\mathbf{k}^*(\alpha^*,\beta^*),$$

and

$$\lim_{m\to\inf}\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}\|\mathbf{d}_m^*(\alpha_m,\beta_m)\|_2=\|\mathbf{d}^*(\alpha^*,\beta^*)\|_2.$$

Proposition 5 is our main result. Basically, it states that the optimal Freudenstein parameters (and the minimal design error) for the discrete approach converge to the optimal Freudenstein parameters (and the minimal design error) for the continuous approach.

4.1. Proofs

Proof of Proposition 1: the proof of Proposition 1 requires two results.

Proposition 6 Let f be a continuous function on some interval $[a,b]$, then

$$\lim_{n\to+\inf}\sum_{i=0}^{n-1}\frac{b-a}{n}f(a+i\frac{b-a}{n})=\int_a^b f(x)dx.$$

Proposition 7 The eigenvalues of a matrix are continuous functions of its elements.

From Proposition 6, the elements of $\mathbf{A}_m(\alpha,\beta)=\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}S_m^T(\alpha,\beta)S_m(\alpha,\beta)$ converge to the elements of $\mathbf{A}(\alpha,\beta)$. Hence, from Proposition 7, the eigenvalues of $\mathbf{A}_m(\alpha,\beta)$ converge to the eigenvalues of $\mathbf{A}(\alpha,\beta)$. The eigenvalues of every matrix are necessarily positive because they are the squares of the corresponding singular values. Since the eigenvalues of $\mathbf{A}_m(\alpha,\beta)$ are necessarily positive, the eigenvalues of $\mathbf{A}(\alpha,\beta)$ are positive, which proves that $\mathbf{A}(\alpha,\beta)$ is semi-definite positive (but not necessarily positive definite).

The inverse of the condition number is defined as the ratio of the smallest and largest singular values. Since $\mathbf{A}_m(\alpha,\beta)$ and $\mathbf{A}(\alpha,\beta)$ are not identically equal to 0 (in other words, their largest eigenvalue greater than 0), the inverse of the condition number of $\mathbf{A}_m(\alpha,\beta)$ converges to the inverse of the condition number of $\mathbf{A}(\alpha,\beta)$. Or, the condition number of $\mathbf{A}_m(\alpha,\beta)$ is the square of the condition number of $\mathbf{S}_m(\alpha,\beta)$, which completes the proof.

Proof of Proposition 2: the proof of Proposition 2 requires the following proposition:

Proposition 8 If a sequence of matrices $\mathbf{M}_n$ converges to a matrix $\mathbf{M}$ and $\mathbf{M}$ is invertible then, $\mathbf{M}_n^{-1}$ converges to $\mathbf{M}^{-1}$.

From Proposition 1, $\mathbf{A}_m(\alpha,\beta)$ converges towards $\mathbf{A}(\alpha,\beta)$. $\mathbf{A}(\alpha,\beta)$ possesses full rank by hypothesis, then there must be some index m_0 such that $\forall\ m\geq\ m_0$ and $\mathbf{A}_m(\alpha,\beta)$ possesses full rank. Hence, $\forall\ m\geq\ m_0$ $\mathbf{S}_m(\alpha,\beta)$ possesses full rank and the pseudo-inverse $\mathbf{S}_m^+(\alpha,\beta)$ is:

$$\mathbf{S}_m^+(\alpha,\beta)=(\mathbf{S}_m^T(\alpha,\beta)\mathbf{S}_m(\alpha,\beta))^{-1}\mathbf{S}_m^T(\alpha,\beta)=\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}\mathbf{A}_m^{-1}(\alpha,\beta)\mathbf{S}_m^T(\alpha,\beta). \tag{19}$$

Equation (10) then becomes:

$$\mathbf{k}_m^*(\alpha,\beta)=\mathbf{A}_m^{-1}(\alpha,\beta)\left(\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}\mathbf{S}_m^T(\alpha,\beta)\mathbf{b}_m(\alpha,\beta))\right). \tag{20}$$

From Proposition 6, $\left(\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}\mathbf{S}_m^T(\alpha,\beta)\mathbf{b}_m(\alpha,\beta)\right)$ converges to $\mathbf{e}(\alpha,\beta)$. From Proposition 8, $\mathbf{A}_m^{-1}(\alpha,\beta)$ converges towards $\mathbf{A}^{-1}(\alpha,\beta)$, hence $\mathbf{k}_m^*(\alpha,\beta)$ converges towards $\mathbf{A}^{-1}(\alpha,\beta)\mathbf{e}(\alpha,\beta)$ which is equal to $\mathbf{k}^*(\alpha,\beta)$ (Equation (17)). This completes the proof.

Proof of Proposition 3: Equation (11) can be rewritten:

$$\|\mathbf{d}_m^*(\alpha,\beta)\|_2=\mathbf{b}_m^T(\alpha,\beta)\mathbf{b}_m(\alpha,\beta)-\left(\mathbf{S}_m^T(\alpha,\beta)\mathbf{b}_m(\alpha,\beta)\right)^T\mathbf{k}_m^*(\alpha,\beta), \tag{21}$$

Multiply Equation (21) by $\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}$. From Proposition 6, $\left(\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}\mathbf{S}_m^T(\alpha,\beta)\mathbf{b}_m(\alpha,\beta)\right)$ converges to $\mathbf{e}(\alpha,\beta)$ and $\left(\frac{\Delta\psi_{max}-\Delta\psi_{min}}{m}\mathbf{b}_m^T(\alpha,\beta)\mathbf{b}_m(\alpha,\beta)\right)$ converges to $\mathbf{c}(\alpha,\beta)$. From Proposition 2, $\mathbf{k}_m^*(\alpha,\beta)$ converges towards $\mathbf{k}^*(\alpha,\beta)$. This completes the proof.

Proof of Proposition 4: the proof of Proposition 4 requires the following proposition:

Proposition 9 Let f be a function continuously differentiable on $[a,b]$, then

$$\left| \int_a^b f(x)dx - \lim_{n\to+\inf} \sum_{i=0}^{n-1} \frac{b-a}{n} f(a+i\frac{b-a}{n}) \right| \leq \frac{(b-a)\max\{f'(x), x\in[a,b]\}}{n}.$$

The dial zeros belong to $K=[-\pi,\pi]\times[-\pi,\pi]$, which is a compact set. Hence, the maximum of the first derivative of any entry of $\mathbf{A}_m(\alpha,\beta)$ is bounded uniformly relatively to (α,β). From Proposition 9, it follows that the elements of $\mathbf{A}_m(\alpha,\beta)$ converge uniformly relatively to (α,β) towards the elements of $\mathbf{A}(\alpha,\beta)$.

The sequence (α_m^*,β_m^*) belongs to K. Hence, there exists a subsequent $(\alpha^*_{\varphi(m)},\beta^*_{\varphi(m)})$ which converges to some $(\alpha^*_\varphi,\beta^*_\varphi)$. From the uniform convergence of $\mathbf{A}_m(\alpha,\beta)$, it follows that the elements of $\mathbf{A}_{\varphi(m)}(\alpha^*_{\varphi(m)},\beta^*_{\varphi(m)})$ converge towards the elements of $\mathbf{A}(\alpha^*_\varphi,\beta^*_\varphi)$.

Following the same arguments used in the proof of Proposition 1, we get:

$$\lim_{m\to\inf} \frac{1}{\kappa_{\varphi(m)}(\alpha^*_{\varphi(m)},\beta^*_{\varphi(m)})^2} = \frac{1}{\kappa(\alpha^*_\varphi,\beta^*_\varphi)}, \tag{22}$$

or $(\alpha^*_{\varphi(m)},\beta^*_{\varphi(m)})$ maximises the inverse of the condition number of $\mathbf{A}_{\varphi(m)}(\alpha,\beta)$, hence:

$$\forall(\alpha,\beta)\in K, \frac{1}{\kappa_{\varphi(m)}(\alpha^*_{\varphi(m)},\beta^*_{\varphi(m)})} \geq \frac{1}{\kappa_{\varphi(m)}(\alpha,\beta)}.$$

From Equation (22) and Proposition 1, taking the limit on both sides of this inequality gives:

$$\forall(\alpha,\beta)\in K, \frac{1}{\kappa(\alpha^*_\varphi,\beta^*_\varphi)} \geq \frac{1}{\kappa(\alpha,\beta)}.$$

Hence, $(\alpha^*_\varphi,\beta^*_\varphi)$ maximises the inverse of the condition number of $\mathbf{A}(\alpha,\beta)$. In other words, each convergent (α_m^*,β_m^*) converges to a maximum of the inverse of the condition number of $\mathbf{A}(\alpha,\beta)$. By hypothesis, this maximum is unique. Hence, $\forall\varphi, (\alpha^*_\varphi,\beta^*_\varphi)=(\alpha^*,\beta^*)$ and the whole sequence (α_m^*,β_m^*) converges to (α^*,β^*). This completes the proof.

Proof of Proposition 5: the first statement of Proposition 5 has been proved in the proof of Proposition 4 (see Equation (22)). From the uniform convergence arising from Proposition 9 the convergence in Proposition 2 and Proposition 3 is in fact uniform. The last two statements of Proposition 5 follow. To be rigorous, Proposition 8 should be modified to uniform convergence, but this introduces no contradictions.

5. EXAMPLE

The preceding results for continuous approximate synthesis that minimises the design error are now illustrated with an example. Let the prescribed function be the Ackerman steering condition for terrestrial vehicles. The steering condition can be expressed as a trigonometric function whose variables are illustrated in Figure 1(b):

$$\sin(\Delta\phi_{pres}-\Delta\psi) - \rho\sin(\Delta\psi)\sin(\Delta\phi_{pres}) = 0, \tag{23}$$

with ρ denoting the length ratio b/a, where a is the distance between front and rear axles, and b the distance between the pivots of the wheel-carriers, which are coupled to the chassis. With the dial zeros, the expression for the steering condition becomes:

$$\sin(\beta+\Delta\phi_{pres}-\alpha-\Delta\psi) - \rho\sin(\alpha+\Delta\psi)\sin(\beta+\Delta\phi_{pres}) = 0. \tag{24}$$

For our example, $\rho = 0.5$ and $[\Delta\psi_{min}, \Delta\psi_{max}] = [30.00, 40.00]$, where angles are specified in degrees.With these values, the prescribed function, i.e. the steering condition, is continuously differentiable. Hence, Proposition 5 must apply.

5.1. Establishing the Optimal Dial Zeros and Freudenstein Parameters

The multi-dimensional Nelder-Mead downhill simplex algorithm [8] is employed to find the optimal values for the dial zeros. Table 1 lists (α_m^*, β_m^*) for different values of m, as well as (α^*, β^*).

m	α_m^*	β_m^*	α^*	β^*
10	-61.80	67.320	-	-
40	-62.17	68.73	-	-
100	-62.23	69.03	-	-
400	-62.26	69.17	-	-
1000	-62.27	69.20	-	-
∞	-	-	-62.27	69.22

Table 1. Optimal dial zeros.

From the optimal dial zeros obtained in Table 1, it is now possible to compute the optimal Freudenstein parameters. Table 2 lists the optimised Freudenstein parameters, k_i, synthesis matrix condition numbers κ_m, and design error norms which have been normalized by dividing by $\sqrt{m}$ for comparison for different values of m as well as the values using the continuous approach.

m	k_1	k_2	k_3	κ_m	κ^*	$\lVert\mathbf{d}_m\rVert_2$	$\lVert\mathbf{d}^*\rVert_2$
10	-0.993	0.412	-0.429	18.24	-	6.93×10^{-4}	-
40	-1.001	0.406	-0.425	20.79	-	6.44×10^{-4}	-
100	-1.003	0.405	-0.424	21.38	-	6.31×10^{-4}	-
400	-1.003	0.404	-0.424	21.69	-	6.24×10^{-4}	-
1000	-1.004	0.404	-0.424	21.75	-	6.23×10^{-4}	-
∞	-1.004	0.404	-0.424	-	475.03	-	6.23×10^{-4}

Table 2. Optimised Freudenstein parmeters, condition numbers, and normalised design errors.

Continuous approximate synthesis eliminates the problem of determining an appropriate cardinality for the data-set. Basically, it considers the case $m = \infty$. Hence, there is no need to search for some convergence in order to set the proper value of m, which eliminates a source of error. However, the continuous approach requires numerical integrations, which itself is a source of error. These errors are in fact of the same nature. Indeed, from the development of Section 4, it is clear that discrete approximate synthesis is essentially a numerical integration method itself: the composite rectangle rule. Hence, comparing the errors arising from the discrete approximate synthesis with continuous approximate synthesis is equivalent to comparing the error terms of two different numerical integration methods. The example presented above employed the Matlab function *quadl*, which employs recursive adaptive Lobatto quadrature [9].

6. CONCLUSIONS AND FUTURE WORK

In this paper a proof has been given that the design error of planar RRRR function-generating linkages synthesised using over-constrained systems of equations established with discrete I/O data sets is bounded by a minimum value established using continuous approximate synthesis between minimum and maximum I/O values. Evaluating the design error over the whole range of the function requires the use of a functional normed space, thereby changing the discrete approximate synthesis problem to a continuous approximate synthesis problem. Assuming that the prescribed function $\Delta\phi_{pres}(\Delta\psi)$ is continuously differentiable, it is shown that the dial zeros, the optimal Freudenstein parameters, and the minimal design error for discrete approximate synthesis converge towards the dial zeros, the optimal Freudenstein parameters and the minimal design error for continuous approximate synthesis. In other words, the continuous approach corresponds to the discrete approach after setting the cardinality of the I/O set to $m = \infty$.

The extension of this work is to investigate how the structural error as defined in [2] bounds the design error. First, it should be determined if the structural error minimisation problem can be formulated and, more importantly solved, using the continuous approach. Second, it should be investigated whether in this case too, the continuous approach corresponds to the discrete approach with $m = \infty$. This is certainly much more challenging due to increased complexity of the continuous structural error minimisation problem, which is a non-linear problem with equality constraints, compared to the continuous design error minimisation problem, which is a quadratic problem without any constraints. Finally, one might ask whether our developments could be applied to other mechanism topologies, such as planar mechanisms possessing prismatic joints, as well as spherical, or spatial linkages.

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