Integrated Type And Dimensional Synthesis of Planar Four-Bar Mechanisms

Tim J. Luu and M. John D. Hayes

Abstract A novel approach to integrated type and approximate dimensional synthesis of planar four-bar mechanisms (i.e. linkages comprised of any two of *RR*, *PR*, *RP*, and *PP* dyads) for rigid-body guidance is proposed. The essence is to correlate coordinates of the coupler attachment points in two different coordinate frames, thereby reducing the number of independent variables defining a suitable dyad for the desired rigid-body motion from five to two. After applying these geometric constraints, numerical methods are used to size link lengths, locate joint axes, and decide between *RR*, *PR*, *RP* and *PP* dyads that, when combined, guide a rigid body through the best approximation, in a least-squares sense, of n specified positions and orientations, where $n \geq 5$. No initial guesses of type or dimension are required. An example is presented illustrating the effectiveness and robustness of this new approach.

Key words: Approximate type and dimensional synthesis; planar four-bar mechanisms; rigid body guidance; singular value decomposition.

1 Introduction

Planar linkages contain either revolute (*R*-pairs), or prismatic (*P*-pairs). These kinematic pairs permit rotations about one axis, or translations parallel to one direction, respectively. In general, dimensional synthesis for rigid body guidance assumes a mechanism *type*: i.e., planar 4*R*; slider-crank; crank-slider; trammel, etc.. Our aim is to develop a completely general planar mechanism synthesis algorithm that in-

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tegrates both *type* and *dimensional* synthesis for n-position approximate synthesis for rigid body guidance. The pairing of the two types leads to four possible dyads: revolute-revolute (*RR*), prismatic-revolute (*PR*), revolute-prismatic (*RP*), and prismatic-prismatic (*PP*).

There is an extensive body of literature reporting research on approximate dimensional kinematic synthesis of planar four-bar mechanisms for rigid-body guidance, see for example [12, 1, 6, 5, 4, 9]. However, there are no methods reported in the substantial body of literature that successfully integrate both type and approximate dimensional synthesis of planar four-bar mechanisms for rigid body guidance, without a priori knowledge or initial guesses with the exception of two special cases reported in [3, 2]. In this paper a method for doing so is presented for the first time.

The minimization criteria of the algorithm presented in this paper is purely mathematical: the condition number of the synthesis matrix. The algorithm will be enhanced when the transmission angle is incorporated as an optimization objective. It would be additionally beneficial to examine the order and branch defect problems. It may be that advances made in [10] can be incorporated into the integrated type-dimensional synthesis algorithm to address these issues. These issues notwithstanding, the algorithm presented in this paper is a robust foundation upon which to build. The algorithm is being adapted for synthesis of spatial motion platforms.

2 Kinematic Constraints: Circular and Linear

The motion of the coupler link in a four-bar planar mechanism is determined by the relative displacements of all links in the kinematic chain. The relative displacement of two rigid bodies in the plane can be considered as the displacement of a Cartesian reference coordinate frame E attached to one of the bodies with respect to a Cartesian reference coordinate frame Σ attached to the other. Without loss of generality, Σ may be considered fixed with E free to move, see Figure 2. The homogeneous coordinates of points represented in E are given by the ratios $(x:y:z)$. Those of the same points represented in Σ are given by the ratios $(X:Y:Z)$. The mapping between the coordinates of points expressed in the two reference frames is given by the homogeneous coordinate transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{1}$$

where (a,b) are the $(\frac{X}{Z},\frac{Y}{Z})$ Cartesian coordinates of the origin of E with respect to Σ, and θ is the orientation of E relative to Σ. Any point $(x:y:z)$ in E can be mapped to $(X:Y:Z)$ in Σ using this transformation. For rigid body guidance, each pose is defined by the position and orientation of E with respect to Σ, which is specified by the ordered triple (a,b,θ). Dyads are connected through the coupler link at the coupler attachment points M_1 and M_2, see Figure 1.

There is a specific type of constrained motion corresponding to each one of the four types of planar lower-pair dyad. The ungrounded *R* pair in an *RR* dyad is constrained to move on a circle with a fixed centre. Because of this they are denoted *circular constraints*. *Linear constraints* result when *PR* and *RP* dyads are employed because the *R* pair attachment point is constrained to move on a line defined by the *P* pair translation direction. The *PP* dyad represents a planar constraint: the line of one *P* pair direction is constrained to translate on the direction line of the other.

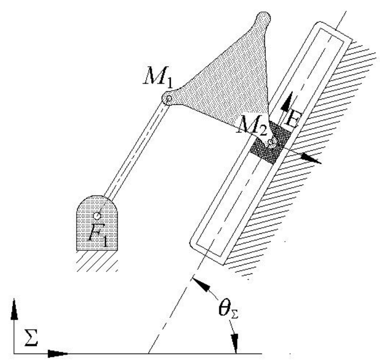


Fig. 1. Planar *RRRP* linkage.

It can be shown [2] that the model representing both circular and linear constraints for n Cartesian point coordinate pairs can be expressed in matrix form as

$$\mathbf{Ck} = \begin{bmatrix} X_j^2 + Y_j^2 & 2X_j & 2Y_j & 1 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \mathbf{0}, \tag{2}$$

where $\mathbf{C}$ is an $n \times 4$ dimensional array with $j \in \{1, 2, \ldots, n\}$, with X and Y being the Cartesian coordinates of points on either a circle or line, and the K_i are constant shape parameters determined by the constraint imposed by the dyad [2].

For circular constraints the K_i are defined as

$$K_0 = 1, \;\; K_1 = -X_c, \;\; K_2 = -Y_c, \;\; K_3 = K_1^2 + K_2^2 - r^2, \tag{3}$$

where (X_c, Y_c) are the Cartesian coordinates of the circle centre expressed in Σ and r is the circle radius.

Linear constraints require $K_0 = 0$ and the remaining K_i are proportional to line coordinates defined by

$$K_1 = -\frac{1}{2} F_{Z/\Sigma} \sin\theta_\Sigma, \;\; K_2 = \frac{1}{2} F_{Z/\Sigma} \cos\theta_\Sigma, \;\; K_3 = F_{X/\Sigma} \sin\theta_\Sigma - F_{Y/\Sigma} \cos\theta_\Sigma, \tag{4}$$

where $(F_{X/\Sigma} : F_{Y/\Sigma} : F_{Z/\Sigma})$ are homogeneous coordinates of a fixed point, expressed in Σ, on the line that makes an angle θ_Σ with the positive X-axis in Σ.

In the definitions of the K_i, the parameter K_0 acts as a binary switch between circular and linear constraints. When $K_0 = 1$ Equation (2) represents the implicit equation of points on a circle, and when $K_0 = 0$ the equation becomes that of a line. Nonetheless, K_0 is still an homogenizing parameter whose value is arbitrary. The K_i can be normalized by K_0, but only when K_0 is nonzero.

3 Integrating Type and Approximate Dimensional Synthesis

Equations (2), (3), and (4) are used to integrate type and approximate dimensional synthesis of planar for-bar mechanisms for rigid-body guidance. Constructing the required synthesis matrix **C** based on the prescribed poses is done by relating the position of the two rigid body attachment points M_1 and M_2 in both reference frames E and Σ, see Figure 1. Reference frames Σ and E are correlated in two ways:

1. Points M_1 and M_2 move on circles or lines in Σ;
2. Points M_1 and M_2 have constant coordinates in E.

Let (x,y) be the coordinates expressed in E of one of the coupler attachment points, M, and (X,Y) be the coordinates of the same point expressed in Σ. Carrying out the matrix multiplication in Equation (1) yields

$$\begin{aligned} X &= x\cos\theta - y\sin\theta + az, \\ Y &= x\sin\theta + y\cos\theta + bz, \\ Z &= z. \end{aligned} \tag{5}$$

Ignoring infinitely distant coupler attachment points, it is reasonable to set $z=1$ in Equation (5) and substituting the result into Equation (2), with $j \in \{1,2,\ldots,n\}$ yields

$$\mathbf{Ck} = \begin{bmatrix} (x\cos\theta_j - y\sin\theta_j + a_j)^2 + (x\sin\theta_j + y\cos\theta_j + b_j)^2 \\ 2\,(x\cos\theta_j - y\sin\theta_j + a_j) \\ 2\,(x\sin\theta_j + y\cos\theta_j + b_j) \\ 1 \end{bmatrix}^T \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0. \tag{6}$$

Prescribing $n > 5$ poses makes **C** an $n \times 4$ matrix. The parameters x and y possess constant values in E. The n-dimensional vector parameters **a**, **b**, and θ in **C** are all known a priori because they are the specified poses of E with respect to Σ.

The only unknown parameters in **C** are x and y. Determining the x and y that best satisfy Equation (6) will solve the problem. Once values for x and y are obtained **C** is fully determined, which allows the vector **k** to be identified. The problem is now a two dimensional search for x and y. However, at least two dyads are required to form a planar mechanism. This implies that there must be at least two distinct values for x and y in order for a complete solution to exist. The x and y are found such that they satisfy Equation (6). For equations of the form $\mathbf{Ck} = \mathbf{0}$ the only real **k** that satisfies the equation is the zero vector if **C** is not singular. In other words, for non-trivial **k** to exist, **C** must be rank deficient [11]. The task becomes finding values for x and y that make **C** rank deficient, or failing that, the most ill-conditioned.

The conditioning of a matrix is measured by the ratio of the largest and smallest singular values of the matrix, which is called the condition number κ. It is computationally more convenient to use is the inverse of the condition number, γ

$$\gamma \equiv \frac{1}{\kappa} = \frac{\sigma_{MIN}}{\sigma_{MAX}}, 0 \le \gamma \le 1, \tag{7}$$

because it is bounded both from above and below. A well conditioned matrix has $\gamma \approx 1$, while an ill-conditioned matrix has $\gamma \approx 0$. Therefore, selecting x and y that renders $\mathbf{C}$ the most rank deficient involves minimizing γ.

The Nelder-Mead *Downhill Simplex Method in Multidimensions* algorithm may be used for this minimization [7]. This method requires only function evaluations, not derivatives. It is not very efficient in terms of the evaluations it requires, but for the problem at hand the computational burden is relatively small. We will not discuss the convergence properties, because any optimization method may be employed.

Since the Nelder-Mead algorithm needs an initial guess, γ may be plotted in terms of x and y first, in the neighborhood of $(x, y) = (0, 0)$ up to a user-defined range of the maximum distance that the coupler attachment points can be from the moving frame E origin, denoted ε. As x and y represent the position of a coupler attachment point with respect to moving refernece frame E. The x and y parameters may then be selected approximately corresponding to the smallest value of γ. These points represent the local minima of the entire γ plot, that is, with $\varepsilon = \infty$. However, for practical reasons, with ε finite, these minima may be regarded as the global minima of the region of interest. At least two minima are required to obtain a planar four-bar mechanism, as each minimum corresponds to a single dyad. The Nelder-Mead algorithm is fed these approximate values as inputs, and converges to the values of x and y that minimize γ.

Once the values of x and y have been determined the matrix $\mathbf{C}$ in Equation (6) can be populated. The $\mathbf{k}$ parameters may then be estimated. We have elected to use singular value decomposition (SVD) because we are necessarily required to work with either singular, or numerically very-close-to-singular sets of equations. SVD decomposes any given $m \times n$ matrix $\mathbf{C}$ into the product of three matrix factors such that

$$\mathbf{C}_{m\times n} = \mathbf{U}_{m\times m}\mathbf{S}_{m\times n}\mathbf{V}^T_{n\times n}, \tag{8}$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, and $\mathbf{S}$ is a rectangular matrix whose only non-zero elements are on the diagonal of the upper $n \times n$ sub-matrix. These diagonal elements are the singular values of $\mathbf{C}$ arranged in descending order, lower bounded by zero [8]. SVD constructs orthonormal bases spanning the range of $\mathbf{C}$ in $\mathbf{U}$ and the nullspace of $\mathbf{C}$ in $\mathbf{V}$. If $\mathbf{C}$ is rank deficient, then the last $n-\text{rank}(\mathbf{C})$ singular values of $\mathbf{C}$ are zero. Furthermore, the corresponding columns of $\mathbf{V}$ are unit basis vectors that span the nullspace of $\mathbf{C}$. As such, any linear combination of these columns is a non-trivial solution that best satisfies the system $\mathbf{Ck} = \mathbf{0}$.

For overconstrained systems, where the $m \times n$ matrix $\mathbf{C}$ has $m > n$, in general no non-trivial exact solution exists, because in general an overconstrained synthesis matrix possesses full rank. In this case, the optimal approximate solution in a least-squares sense is last column of $\mathbf{V}$, corresponding to the smallest singular value of $\mathbf{C}$. Furthermore, the more ill-conditioned $\mathbf{C}$ is, the closer the optimal approximate solution is to being an exact solution. Because the K_i are homogeneous, the scaling posses no problem because $\mathbf{k}$ will be normalized by dividing through by K_0. In the case where $K_0 \approx 0$, or $K_0 = 0$ the linear definitions for K_1, K_2, and K_3 from

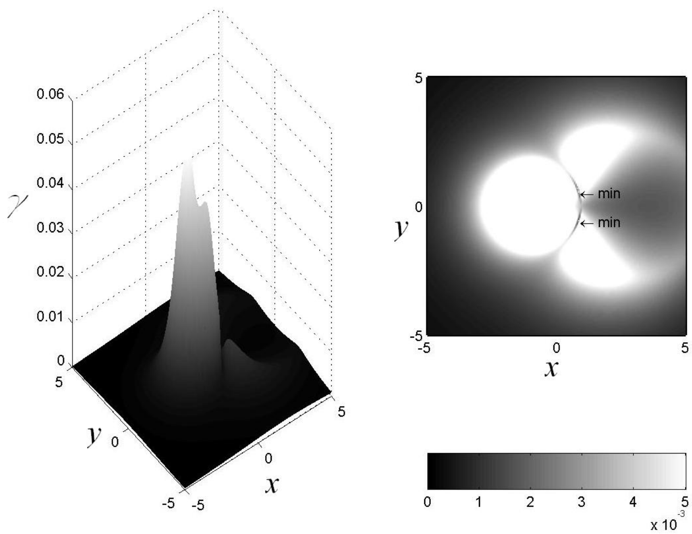


Fig. 2 The γ plot of the poses defining a square corner.

Equation (4) are used. The switching threshold for K_0 to represent either an *RR* or *PR* (or *RP*) dyad must be user defined based on the geometry of the problem.

Note that *PP* dyads are a special case. Two serial *P* pairs restricts the distal link from changing its orientation. For type synthesis, given any set of poses with non constant orientation, the *PP* dyad is immediately ruled out.

3.1 Example

Consider an example that requires completely general integrated type and approximate dimensional synthesis by defining poses that are impossible to generate exactly by any four-bar planar mechanism. The poses define a square corner. A point on a rigid body moves linearly between the Cartesian coordinates from (0, 1) to (1, 0) via (1,1). The orientation increases linearly from 0 to 90 degrees. The poses are listed in Table 1.

Pose	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
a	0.0	0.1	0.2	0.3	0.4	0.6	0.7	0.8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
b	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.8	0.7	0.6	0.4	0.3	0.2	0.1	0.0
$\theta°$	0.0	4.5	9.0	13.5	18.0	27.0	31.5	36.0	45.0	49.5	54.0	58.5	63.0	72.0	76.5	81.0	85.5	90.0

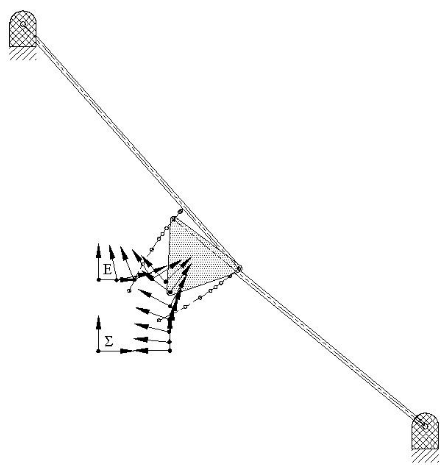
Table 1 Specified poses defining a square corner.

A planar four-bar mechanism cannot exactly replicate the motion defined above because points on the coupler generate either a 6^{th}, 4^{th}, or 2^{nd} order curve. The curve

$$x^n + y^n = 1 \tag{9}$$

approaches a square corner as $n \to \infty$. With $n \leq 6$ for planar four-bar mechanisms, it is impossible to exactly replicate the desired motion. Although a *PPPP* mechanism may be able to generate the desired point translation, the change in orientation rules out this type of mechanism.

The pose data are substituted into Equation (6) to populate $\mathbf{C}$. The γ of $\mathbf{C}$ are then plotted as a function of x and y and are shown in Figure 2. As can be seen in this figure, two distinct minima occur at approximately (0.8, 0.6) and (0.8, -0.6). Using the Nelder-Mead minimization and the pair of approximate x and y as initial guesses, the exact values of the two minima are found, and listed in Figure 3. These values are then substituted into Equation (6) to completely determine $\mathbf{C}$. SVD is then applied to $\mathbf{C}$ to find $\mathbf{k}$ corresponding to each minimum. The values of $\mathbf{k}$ thus determined are also listed in Figure 3. The resulting synthesized mechanism, illustrated in Figure 3, is composed of two *RR* dyads centred on (4.5843, -1.0539) and (-1.0539, 4.5843), both with links having length 1.7307.



	Dyad 1	Dyad 2
x	0.8413	0.8413
y	0.5706	-0.5706
K_0	0.2010	0.2010
K_0/K_0	1	1
K_1/K_0	-4.5843	1.0549
K_2/K_0	1.0539	-4.5843
K_3/K_0	1.2704	1.2704

Fig. 3 Identified *RRRR* mechanism and corresponding dyads.

4 Conclusions

In this paper a novel method was presented that integrates type and approximate dimensional synthesis of planar four-bar mechanisms used for rigid-body guidance. Coupler attachment points are correlated between moving frame E and fixed frame

Σ thereby reducing the number of independent variables defining a suitable dyad for the desired poses from five to two. Numerical methods are then used to determine both mechanism type and approximate dimensions. A numerical example was presented, illustrating the utility of the algorithm.

References

[1] Akhras, R., Angeles, J.: “Unconstrained Nonlinear Least-Square Optimization of Planar Linkages for Rigid-Body Guidance”. Mechanism and Machine Theory **25**(1), 97–118 (1990)

[2] Hayes, M.J.D., Luu, T., Chang, X.W.: “Kinematic Mapping Application to Approximate Type and Dimension Synthesis of Planar Mechanisms”. In: 9th Advances in Robotic Kinematics, *eds. Lenarčič, J. and Galletti, C., Kluwer Academic Publishers, Dordrecht, the Netherlands*, pp. 41–48 (June 28 - July 1, 2004)

[3] Hayes, M.J.D., Zsombor-Murray, P.J.: “Towards Integrated Type and Dimensional Synthesis of Mechanisms for Rigid Body Guidance”. In: Proceedings of the CSME Forum 2004, pp. 53–61. University of Western Ontario, London, ON, Canada (June 1-4, 2004)

[4] Kong, X.: “Approximate Kinematic Synthesis of Linkages Using Generalized Inverse Matrix and Continuation”. Mechanical Science and Technology **18**(1), 38–40 (1999)

[5] Larochelle, P.: “Approximate Motion Synthesis of Open and Closed Chains Via Parametric Constraint Manifold Fitting: Preliminary Results”. Proceedings of the ASME Design Engineering Technical Conferences and Computers and Information in Engineering Conference **2**, 1049–1057 (2003)

[6] Larochelle, P., McCarthy, J.M.: “Designing Planar Mechanisms Using a Bi-Invariant Metric in the Image Space of SO”. ASME, Design Engineering Division, 23rd Biennial Mechanisms Conference **70**(1), 221–228 (1994)

[7] Nelder, J.A., Mead, R.: “A Simplex Method for Function Minimization”. Computer Journal **7**, 308–313 (1965)

[8] Press, W., Teukolsky, S., Vetterling, W., Flannery, B.: Numerical Recipes in C, 2nd Edition. Cambridge University Press, Cambridge, England (1992)

[9] Ravani, B., Roth, B.: “Motion Synthesis Using Kinematic Mappings”. Journal of Mechanisms, Transmissions, and Automation in Design **105**(3), 460–467 (1983)

[10] Schröcker, H.P., Husty, M., McCarthy, J.M.: “Kinematic Mapping Based Evaluation of Assembly Modes for Planar Four-Bar Synthesis”. ASME, Journal of Mechanical Design **129**(9), 924–929 (2007)

[11] Strang, G.: Linear Algebra and its Applications. Academic Press (1980)

[12] Vallejo, J., Aviles, R., Hernandez, A., Amezua, E.: “Nonlinear Optimization of Planar Linkages for Kinematic Syntheses”. Mechanism and Machine Theory **30**(4), 501–518 (1995)