Velocity Level Kinematic Analysis of Serial *n*A-Chains

James D. Robinson and M. John D. Hayes

Abstract The algebraic screw pair, or A-pair, represents a new class of kinematic constraint that exploits the self-motions inherent to a specific configuration of Griffis-Duffy platform. The A-pair causes a sinusoidal coupling of rotation and translation between adjacent links in the kinematic chain. The resulting linkage is termed an A-chain. This paper presents a derivation of the manipulator Jacobian of *n*A-chains in general, and a specific 4 degree-of-freedom hybrid serial-parallel 4A-chain.

Key words: Algebraic screw pair; Griffis-Duffy platform; *n*A Jacobian.

1 Introduction

The algebraic screw pair [4], or A-pair, is a novel kinematic pair based on a specific configuration of parallel manipulator called the Griffis-Duffy platform (GDP) [1]. The GDP is a special configuration of the six legged, six degree-of-freedom (DOF) Stewart-Gough platform (SGP) that, in most configurations, is subject to self-motions regardless of the lengths of the actuated legs [2]. Kinematic chains composed of rigid links serially connected by A-pairs are denoted A-chains. The A-pair induces a sinusoidal coupling of rotation and translation between adjacent links. For this paper the derivation of the manipulator Jacobian of a 4A-chain, illustrated in Figure 1(a), is used to demonstrate the method. While the method does not fail for $n > 4$, the terms become inconveniently large to express explicitly.

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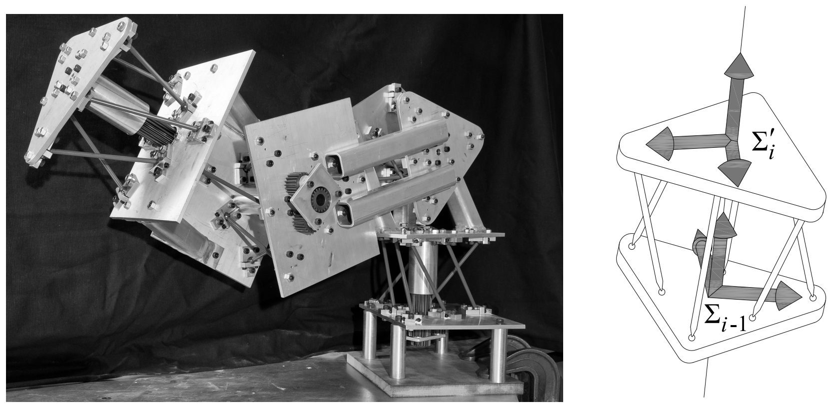

(a) (b)

Fig. 1 (a) Prototype 4A-chain. (b) Midline-to-vertex configuration GDP.

The A-pairs used in this paper are the midline-to-vertex GDP configuration, see Figure 1(b). They are constrained by: the fixed base and moving platform anchor point triangles are congruent equilateral triangles with each side of the triangles being of length a and the six legs are all of a fixed length, l, equal to the height, h, of the triangles as illustrated in Figure 2(a).

The value of l is

$$l = h = \frac{a\sqrt{3}}{2}. \tag{1}$$

It turns out that the self-motions of this GDP couple rotation about an axis passing through the geometric centres of both the fixed base and moving platform triangles with translation along that axis. Using the coordinate systems illustrated in Figure 2(b), it can be shown [2] that the separation of the fixed base and moving platform, d, is a function of the rotation angle, θ, about the axis common to both the fixed base and moving platform:

$$d = \rho \sin\left(\frac{\theta}{2}\right), \text{ where } \rho = \frac{a\sqrt{6}}{3}. \tag{2}$$

It is expected that A-chains will exhibit increased stiffness and positioning accuracy relative to R-chains. While we currently lack empirical proof, it appears to be true based on a visual comparison of the prototype manipulator with the first four R-pair joints in a Thermo CRS A645. The proposed actuation system consists of a central spline affixed to the moving platform that is constrained by three spur gears affixed to the base, all possessing identical pitch diameters. One of the spur gears is active, which rotates the spline. This arrangement allows the spline to translate along its axis of rotation.

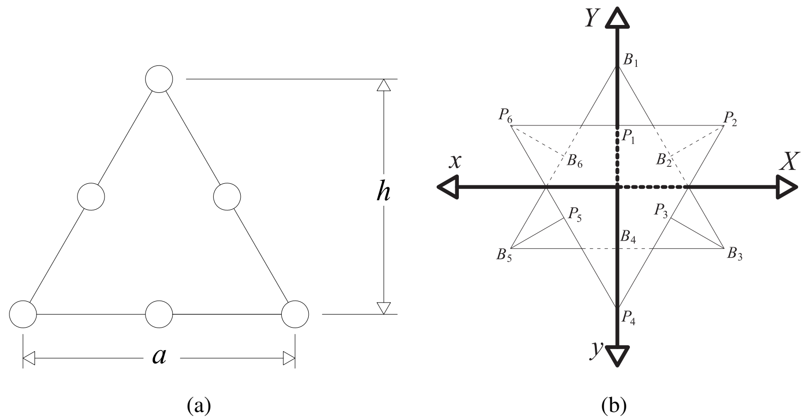

Fig. 2 (a) Platform shape parameters. (b) Coordinate systems and leg anchor point.

2 The Jacobian of a Single A-Pair

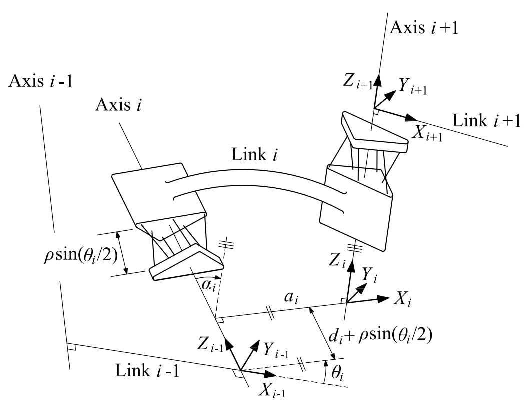

Fig. 3 DH-parameters of a link in an A-chain.

The Jacobian matrix of a manipulator maps its joint rates to the linear and angular velocities of its end effector (EE). Standard methods from the literature, see [5] for example, can be adapted to account for the coupled translation and rotation of the A-pair. The Jacobian matrix of a 1A-chain can be determined by examining the rotation and translation components of the coupled motion separately. The orientation of the joint is directly expressed by the joint variable θ_1. The translation component of the joint motion is a function of θ_1, and computed with Equation (2).

In a 1A-chain, the linear velocity of the EE induced by $\dot{\theta}_1$ has two components: one due to the rotation of the joint, perpendicular to the axis of rotation as with a revolute joint; the other is due to the translation coupled to the rotation, and is expressed by $p_{e_z} = d_1 + \rho \sin(\theta_1/2)$, where p_{e_z} is the $\hat{\mathbf{z}}_0$-component of the EE position vector, d_1 is the offset of the EE from the base along $\hat{\mathbf{z}}_0$ when $\theta_1 = 0$, and $\hat{\mathbf{z}}_0$ is axis of rotation. There is only one joint rate $\dot{q}_1 = \dot{\theta}_1$. The influence of the rotation of the joint on the linear velocity is found as if it were a revolute joint:

$$\dot{q}_1 \mathbf{J}_{\mathrm{P}_{1\mathrm{r}}} = z_0 \times (\mathbf{p}_e - \mathbf{p}_0)\dot{\theta}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -\dot{\theta}_1 a_1 \cos\theta_1 - 0 \\ -\dot{\theta}_1 a_1 \sin\theta_1 - 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 a_1 \sin\theta_1 \\ -\dot{\theta}_1 a_1 \cos\theta_1 \\ 0 \end{bmatrix}, \quad (3)$$

where, in general, $\mathbf{J}_{\mathrm{P}_{1\mathrm{r}}}$ is the is the 3×1 vector mapping the angular rate of joint i to its contribution to the linear velocity of the EE, $\mathbf{p}_e$ and $\mathbf{p}_0$ are the position vectors of the EE coordinate origin, and position vector of the joint coordinate system origin both expressed in the non moving frame, and a_1 is the DH-parameter for the link length of a link affixed to the moving platform of the single A-pair, illustrated in Figure 3. The time derivative of p_{e_z} yields the translation component of the Jacobian, $\mathbf{J}_{\mathrm{P}_1\mathrm{t}}$:

$$\dot{q}_1 \mathbf{J}_{\mathrm{P}_1\mathrm{t}} = \tfrac{d}{dt} \begin{bmatrix} 0 \\ 0 \\ d_1 + \rho \sin\left(\frac{\theta_1}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\dot{\theta}_1}{2} \rho \cos\left(\frac{\theta_1}{2}\right) \end{bmatrix}, \quad (4)$$

therefore $\mathbf{J}_{\mathrm{P}_1\mathrm{t}} = \left[0 \ 0 \ \frac{\rho}{2} \cos\left(\frac{\theta_1}{2}\right) \right]^{\mathrm{T}}$. Summing the two components yields the mapping from the joint rate $\dot{q}_1$ to the EE linear velocity:

$$\mathbf{J}_{\mathrm{P}_1} = \mathbf{J}_{\mathrm{P}_1\mathrm{r}} + \mathbf{J}_{\mathrm{P}_1\mathrm{t}} = \begin{bmatrix} a_1 \sin\theta_1 \\ -a_1 \cos\theta_1 \\ \frac{\rho}{2} \cos\left(\frac{\theta_1}{2}\right) \end{bmatrix}, \quad (5)$$

and $\dot{\mathbf{p}}_e = \mathbf{J}_{P_1}(\mathbf{q})\dot{q}_1$.

The translation that is coupled with the rotation of the A-pair does not have an effect on the orientation of the EE, thus the contribution of the A-pair actuation rate to the angular velocity of the EE, ω_e, is the same as that of a revolute joint:

$$\mathbf{J}_{\mathrm{O}_1} = \hat{\mathbf{z}}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (6)$$

and $\omega_e = \mathbf{J}_{O_1}(\mathbf{q})\dot{q}_1$. The full Jacobian is

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{P_1} \\ \mathbf{J}_{O_1} \end{bmatrix}. \quad (7)$$

However, in this A-pair the EE coordinate system origin is located at the geometric centre of the moving platform (the EE frame is coincident with a base frame located at the geometric centre of the fixed base when the A-pair is in the theoretical home position). Hence, the origin of the EE lies on the joint axis rendering $a_1 = 0$ and the velocity relations simplify to

$$\mathbf{J}_{P_1} = \begin{bmatrix} 0 \\ 0 \\ \frac{\rho}{2}\cos\left(\frac{\theta_1}{2}\right) \end{bmatrix}, \quad \mathbf{J}_{O_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

The Jacobian matrix for the single A-pair is always rank deficient, which is expected because motion in an arbitrary direction will never be possible with just a single A-pair. If one considers only the two achievable degrees of freedom of the single A-pair with the EE origin on the joint axis (motion along the joint axis and rotation about the same axis) a more useful analysis can be performed.

The mapping to EE angular velocity from the joint rate is one-to-one and independent of the joint state. This implies that, if joint limits are ignored, the angular velocity of the EE can always be controlled one-to-one. However, the mapping of linear velocity is dependant on the joint state and cannot be continuously controlled. When $\cos\left(\frac{\theta_1}{2}\right)$ goes to zero (*i.e.* when θ approaches 180°) the joint approaches a singular position. At the singularity the linear velocity of the EE is null and any rotation away from $\theta_1 = 180°$ in either direction will result in motion in the negative $\hat{\mathbf{z}}_0$-direction only. The singularity is also evident if the Jacobian is rearranged to solve for the joint rate required to achieve a certain velocity, v_1, along the $\hat{\mathbf{z}}_0$-axis:

$$\dot{\theta}_1 = \frac{2v_1}{\rho\cos\left(\frac{\theta_1}{2}\right)}. \tag{9}$$

As θ_1 approaches 180°, $\dot{\theta}_1$ approaches infinity.

3 The Jacobian of a 4A-Chain

Link i	a_i	α_i	d_i	θ_{fi}
1	0	90°	d_1	0°
2	a_2	180°	$-\rho$	−90°
3	0	−90°	$-\rho$	90°
4	0	0°	d_4	0°

Table 1 DH-Parameters.

The DH-parameters of the 4A-chain illustrated in Figure 1(a), using the DH-parameter convention in [5], are listed in Table 1. Figure 3 shows the assignment of DH-parameters to a link in an A-chain. The position vector of the base frame origin is $\mathbf{p}_0 = \mathbf{0}$. The relative pose of the EE, $\mathbf{p}_e$, is found by transforming the homogeneous coordinates of its origin to the non moving base coordinate reference system. The homogeneous transformation, obtained using the methods of [4], has the form:

$$^{0}\mathbf{T}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_{e_x} & -c_1c_{2-3}c_4 + s_1s_4 & c_1c_{2-3}s_4 + s_1c_4 & -c_1s_{2-3} \\ p_{e_y} & -s_1c_{2-3}c_4 - c_1s_4 & s_1c_{2-3}s_4 - c_1c_4 & -s_1s_{2-3} \\ p_{e_z} & -s_{2-3}c_4 & s_{2-3}s_4 & c_{2-3} \end{bmatrix}, \tag{10}$$

where

$$\mathbf{p}_e = \begin{bmatrix} p_{e_x} \\ p_{e_y} \\ p_{e_z} \end{bmatrix} = \begin{bmatrix} s_1\rho\, s\frac{\theta_2}{2} + c_1 s_2 a_2 - s_1 \rho\, s\frac{\theta_3}{2} - c_1 s_{2-3} \rho\, s\frac{\theta_4}{2} - c_1 s_{2-3} d_4 \\ -c_1 \rho\, s\frac{\theta_2}{2} + s_1 s_2 a_2 + c_1 \rho\, s\frac{\theta_3}{2} - s_1 s_{2-3} \rho\, s\frac{\theta_4}{2} - s_1 s_{2-3} d_4 \\ \rho\, s\frac{\theta_1}{2} + d_1 - c_2 a_2 + \rho\, c_{2-3} s\frac{\theta_4}{2} + d_4 c_{2-3} \end{bmatrix}$$

is the position vector of the EE origin, and c_1, s_1, *etc.* are abbreviations for $\cos\theta_1$, $\sin\theta_1$, *etc.*, respectively. In addition to the EE pose the transformation matrices describing the pose of each intermediate reference frame ($\Sigma_i, i = 1,2,3$) are important. The pose of Σ_1 is given by

$$^0\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_{1_x} & c_1 & 0 & s_1 \\ p_{1_y} & s_1 & 0 & -c_1 \\ p_{1_z} & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} p_{1_x} \\ p_{1_y} \\ p_{1_z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho\, s\frac{\theta_1}{2} + d_1 \end{bmatrix}. \tag{11}$$

The pose of Σ_2 is given by

$$^0\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_{2_x} & c_1 s_2 & -c_1 c_2 & -s_1 \\ p_{2_y} & s_1 s_2 & -s_1 c_2 & c_1 \\ p_{2_z} & -c_2 & -s_2 & 0 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} p_{2_x} \\ p_{2_y} \\ p_{2_z} \end{bmatrix} = \begin{bmatrix} \rho\, s_1 s\frac{\theta_2}{2} + a_2 c_1 s_2 - \rho\, s_1 \\ -\rho\, c_1 s\frac{\theta_2}{2} + a_2 s_1 s_2 + \rho\, c_1 \\ \rho\, s\frac{\theta_1}{2} + d_1 - a_2 c_2 \end{bmatrix}. \tag{12}$$

Finally the pose of Σ_3 is

$$^0\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_{3_x} & -c_1 c_{2-3} & s_1 & -c_1 s_{2-3} \\ p_{3_y} & -s_1 c_{2-3} & -c_1 & -s_1 s_{2-3} \\ p_{3_z} & -s_{2-3} & 0 & c_{2-3} \end{bmatrix}, \tag{13}$$

$$\mathbf{p}_3 = \begin{bmatrix} p_{3_x} \\ p_{3_y} \\ p_{3_z} \end{bmatrix} = \begin{bmatrix} \rho\, s_1 s\frac{\theta_2}{2} + a_2 c_1 s_2 - \rho\, s_1 s\frac{\theta_3}{2} \\ -\rho\, c_1 s\frac{\theta_2}{2} + a_2 s_1 s_2 + \rho\, c_1 s\frac{\theta_3}{2} \\ \rho\, s\frac{\theta_1}{2} + d_1 - a_2 c_2 \end{bmatrix}. \tag{14}$$

The position vectors $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ that describe the position of the origin of the corresponding intermediate reference frames are given in Equations (11), (12) and (14), respectively. The joint axes, taken from the respective transformation matrices, are

$$\hat{\mathbf{z}}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \hat{\mathbf{z}}_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}, \hat{\mathbf{z}}_2 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}, \hat{\mathbf{z}}_3 = \begin{bmatrix} -c_1 s_{2-3} \\ -s_1 s_{2-3} \\ c_{2-3} \end{bmatrix}. \tag{15}$$

The vector mapping the rate of actuation of Joint 1 to the linear velocity of the EE due to the rotation of Joint 1 is

$$\begin{aligned} \mathbf{J}_{\mathrm{P}_{1,\mathrm{r}}} &= z_0 \times (\mathbf{p}_e - \mathbf{p}_0) \\ &= \begin{bmatrix} \rho\, c_1 s\frac{\theta_2}{2} - a_2 s_1 s_2 - \rho\, c_1 s\frac{\theta_3}{2} + \rho\, s_1 s_{2-3} s\frac{\theta_4}{2} + d_4 s_1 s_{2-3} \\ \rho\, s_1 s\frac{\theta_2}{2} + a_2 c_1 s_2 - \rho\, s_1 s\frac{\theta_3}{2} - \rho\, c_1 s_{2-3} s\frac{\theta_4}{2} - d_4 c_1 s_{2-3} \\ 0 \end{bmatrix}, \end{aligned} \tag{16}$$

and the vector mapping the rate of actuation of Joint 1 to the linear velocity of the EE due to the translation of Joint 1 is

$$\mathbf{J}_{\mathrm{P}_{1,\mathrm{t}}} = \begin{bmatrix} 0 \\ 0 \\ \frac{\rho}{2}\, c\frac{\theta_1}{2} \end{bmatrix}. \tag{17}$$

The total linear velocity Jacobian component for Joint 1 comes from the summation of Equations (16) and (17) for $i = 1$, giving

$$\mathbf{J}_{\mathrm{P}_1} = \begin{bmatrix} \rho\, c_1 s\frac{\theta_2}{2} - a_2\, s_1 s_2 - \rho\, c_1 s\frac{\theta_3}{2} + \rho\, s_1 s_{2-3} s\frac{\theta_4}{2} + d_4\, s_1 s_{2-3} \\ \rho\, s_1 s\frac{\theta_2}{2} + a_2\, c_1 s_2 - \rho\, s_1 s\frac{\theta_3}{2} - \rho\, c_1 s_{2-3} s\frac{\theta_4}{2} - d_4\, c_1 s_{2-3} \\ \frac{\rho}{2}\, c\frac{\theta_1}{2} \end{bmatrix}, \tag{18}$$

and because only the rotational component of the joint motion impacts the orientation of the EE, the angular velocity component of the Jacobian is

$$\mathbf{J}_{\mathrm{O}_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{19}$$

Similarly for Joints 2, 3, and 4:

$$\mathbf{J}_{\mathrm{P}_2} = \begin{bmatrix} a_2\, s_1 c_2 - \rho\, s_1 c_{2-3} s\frac{\theta_4}{2} - d_4\, s_1 c_{2-3} - \frac{\rho}{2}\, c_1 c\frac{\theta_2}{2} \\ a_2\, s_1 c_2 - \rho\, s_1 c_{2-3} s\frac{\theta_4}{2} - d_4\, s_1 c_{2-3} - \frac{\rho}{2}\, c_1 c\frac{\theta_2}{2} \\ a_2 s_2 - \rho s_{2-3} s\frac{\theta_4}{2} - d_4 s_{2-3} \end{bmatrix}, \text{ and } \mathbf{J}_{\mathrm{O}_2} = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}.$$

$$\mathbf{J}_{\mathrm{P}_3} = \begin{bmatrix} \rho\, c_1 c_{2-3} s\frac{\theta_4}{2} + d_4\, c_1 c_{2-3} - \frac{\rho}{2}\, c\frac{\theta_3}{2} s_1 \\ \rho\, s_1 c_{2-3} s\frac{\theta_4}{2} + d_4\, s_1 c_{2-3} + \frac{\rho}{2}\, c\frac{\theta_3}{2} c_1 \\ \rho s_{2-3} s\frac{\theta_4}{2} + d_4 s_{2-3} \end{bmatrix}, \text{ and } \mathbf{J}_{\mathrm{O}_3} = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}.$$

$$\mathbf{J}_{\mathrm{P}_4} = \begin{bmatrix} -\frac{\rho}{2}\, c\frac{\theta_4}{2} c_1 s_{2-3} \\ -\frac{\rho}{2}\, c\frac{\theta_4}{2} s_1 s_{2-3} \\ \frac{\rho}{2}\, c\frac{\theta_4}{2} c_{2-3} \end{bmatrix}, \text{ and } \mathbf{J}_{\mathrm{O}_4} = \begin{bmatrix} -c_1 s_{2-3} \\ -s_1 s_{2-3} \\ c_{2-3} \end{bmatrix}.$$

The full 6×4 Jacobian is assembled as

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{\mathrm{P}_1} & \mathbf{J}_{\mathrm{P}_2} & \mathbf{J}_{\mathrm{P}_3} & \mathbf{J}_{\mathrm{P}_4} \\ \mathbf{J}_{\mathrm{O}_1} & \mathbf{J}_{\mathrm{O}_2} & \mathbf{J}_{\mathrm{O}_3} & \mathbf{J}_{\mathrm{O}_4} \end{bmatrix}. \tag{20}$$

A full examination of the singularities has yet to be conducted but a simple example of a singular configuration is easily found. With only four joint variables it is no surprise that there will be certain directions in which the EE cannot be moved at a given time, but in certain situations the capabilities are further diminished. When $\theta_{v_1} = \theta_{v_2} = \theta_{v_3} = \theta_{v_4} = 180^\circ$ the Jacobian matrix becomes

$$\mathbf{J} = \begin{bmatrix} 0 & (a_2+\rho+d_4) & -(\rho+d_4) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \quad (21)$$

In this configuration instantaneous linear velocities along the y_0- and z_0-axes and angular velocity about the x_0-axis are not achievable.

4 Conclusions

In this paper the Jacobian for nA-chains in general, and in particular, a novel 4 DOF 4A-chain was derived. These chains are joined by A-pairs, which take advantage of the single DOF self motion of the architecturally singular midline-to-vertex configuration of the Griffis-Duffy platform. The self motion is a sinusoidally coupled rotation and translation. The coupling means that existing techniques for establishing the relationship between the joint rates and the resulting linear and angular velocity of the distal link in the chain have to be adapted. Linear and angular velocity relationships between links were considered distinctly and the results combined to reveal the manipulator Jacobian. With the Jacobian established, the manipulator singular configurations can now be investigated with the starting point based on the method reported in [3].

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