Design Parameter Space of Planar Four-bar Linkages

M. John D, Hayes[1], Mirja Rotzoll[1], and Manfred L. Husty[2]

[1] Carleton University, Department of Mechanical and Aerospace Engineering, Ottawa, ON, Canada
[2] University of Innsbruck, Unit Geometry and CAD, Innsbruck, Austria

Abstract. A new algebraic input-output relation for planar four-bar mechanisms is a quartic curve in the input-output joint angle parameter plane. This equation contains four terms with quadratic coefficients of link lengths which all factor into the product of two linear terms. The structure of these eight linear factors suggests that they are the eight faces of an octahedron in a design parameter space of the link lengths. In this paper we show that the design parameter octahedron space implies a complete classification scheme for all 27 possible planar $4R$ mechanisms, in addition to linkages containing one, or two prismatic joints.

Keywords: Algebraic input-output relation; planar four-bar linkage; design parameter octahedron.

1 Introduction

In the firmament of mechanical design the four-bar linkage burns as its brightest star. This is seen to be true when one considers the tremendous volume of literature investigating analysis and design of four-bar mechanisms, ranging from antiquity to present [1]. In this paper we investigate the geometry of the design parameter space of planar four-bar mechanisms. Since we will be concerned with the input-output (IO) relation, we will use the standard description of a planar $4R$ function generator for reference. Such a function generator correlates driver and follower angles such that the mechanism generates the function $\varphi = f(\psi)$, or vice versa, see Fig. 1.

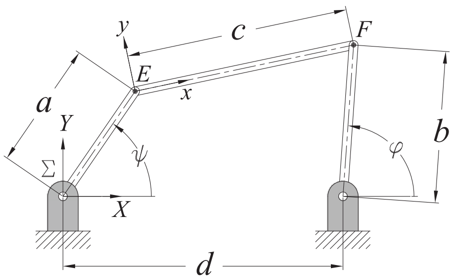

Fig. 1. Planar $4R$ linkage.

Surprisingly, design methods have not focused on algebraic IO equations, rather they generally use the transcendental Freudenstein synthesis equations [3], or variants thereof. The Freudenstein equation relating the input to the

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output angles of a planar $4R$ four-bar mechanism, with link lengths as in Fig. 1, was first put forward in [4]. In the equation the angle ψ is traditionally selected to be the input while φ is the output angle:

$$k_1 + k_2 \cos(\varphi_i) - k_3 \cos(\psi_i) = \cos(\psi_i - \varphi_i). \tag{1}$$

Eq. (1) is linear in the k_i Freudenstein parameters, which are defined in terms of the link length ratios as:

$$k_1 \equiv \frac{(a^2 + b^2 + d^2 - c^2)}{2ab}; \quad k_2 \equiv \frac{d}{a}; \quad k_3 \equiv \frac{d}{b}.$$

In this paper we use instead the algebraic IO relation derived in [7] and the geometric analysis of the quartic algebraic IO curve in [8] to show that it implies a classification scheme for all 27 possible planar $4R$ mechanisms [11]. The classification scheme characterises all Grashof and non-Grashof ranges of motion of the input and output links. Moreover, the structure of the algebraic IO equation suggests a design parameter space [8] that will be examined more fully in this paper.

Study's kinematic mapping image space coordinates and resultants were employed in [7] to derive the IO equation. Then Weierstraß (tangent of the half-angle) substitutions

$$u = \tan\left(\frac{\psi}{2}\right), \ v = \tan\left(\frac{\varphi}{2}\right)$$

were applied to convert the trigonometric equation to an algebraic one, which has the following form:

$$Au^2v^2 + Bu^2 + Cv^2 - 8abuv + D = 0 \tag{2}$$

where;

$$\begin{aligned}
A = (a - b - c + d)(a - b + c + d) &= A_1 A_2;\\
B = (a + b - c + d)(a + b + c + d) &= B_1 B_2;\\
C = (a + b - c - d)(a + b + c - d) &= C_1 C_2;\\
D = (a - b + c - d)(a - b - c - d) &= D_1 D_2.
\end{aligned}$$

Eq. (2) is quartic in the coordinate plane of u and v. Since the distance d between the ground fixed links can be viewed as a scaling factor for function generators, without loss in generality we can normalise a, b, and c by d and consider the design parameter sub-space comprised of three mutually orthogonal bases distances with $d = 1$. Another way of looking at the design parameter sub-space is as the projection of the four dimensional space onto the hyperplane $d = 1$.

Regardless, it is shown in [8] that the quartic curve represented by Eq. (2) has two double points, and therefore possesses genus 1. The double points are the points at infinity of the u and v axes in the u-v plane. Each of these double points can have real or complex tangents depending on the values of the link lengths, which in turn determines the nature of the mobility of the linkage, as well as the number of assembly modes (the maximum is two), and the number of folding assemblies (the maximum is three).

2 Design Parameter Octahedron

In the design parameter space, the eight linear factors in Eq. (2) can be interpreted as the eight faces of a regular octahedron determined by the six vertices $V = (a, b, c)$: $V_1 = (1, 0, 0)$; $V_2 = (-1, 0, 0)$; $V_3 = (0, 1, 0)$; $V_4 = (0, -1, 0)$; $V_5 = (0, 0, 1)$; $V_6 = (0, 0, -1)$, see Fig. 2.

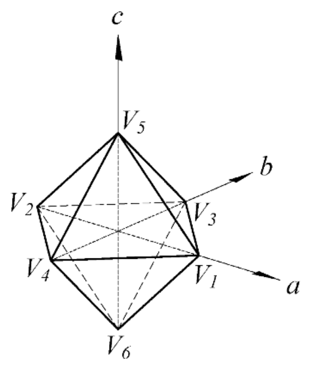

Fig. 2. Design parameter octahedron.

Each face of the octahedron lies entirely in one of the eight quadrants in the parameter space. Given the octahedron, four questions that naturally arise.

1. What do the six vertices imply?
2. What is the significance of points on the octahedron edges?
3. What is the significance of points on the octahedron faces?
4. What is the significance of the location of a general point in the parameter space?

2.1 The Six Octahedron Vertices

With reference to Fig. 2, each of the six octahedron vertices lie at the terminal ends of the design parameter space basis unit vectors, **a**, **b**, and **c**. They comprise the six points $V_{1,2} = (\pm 1, 0, 0)$, $V_{3,4} = (0, \pm 1, 0)$, $V_{5,6} = (0, 0, \pm 1)$. Each vertex is the point common to the planes of four faces and represents a degenerate planar four-bar mechanism with no mobility because it contains two links of zero length and two links of unit length.

2.2 The Twelve Octahedron Edges

Again, referring to Fig. 2, each of the twelve octahedron edges, excluding the vertices, is the line in common with two octahedron faces. Each edge lies entirely in one of eight design parameter sub-space coordinate planes. For example, the edge that lies in the coordinate plane spanned by the positive basis vectors **a** and **b** is the intersection of the face planes defined by the vertices $\{V_1, V_3, V_5\}$ and $\{V_1, V_6, V_3\}$. Each edge represents a degenerate four-bar mechanism with no mobility because it contains one link of zero length.

2.3 Points on the Eight Octahedron Faces

Because of the beautiful structure of the eight linear factors in Eq. (2), it may be shown in a straightforward way that each of the linear factors defines one of eight planes containing one of the octahedron faces. In Euclidean space, E_3, a necessary and sufficient condition that four points, whose homogeneous point coordinates are $(x_0 : x_1 : x_2 : x_3)$, $(y_0 : y_1 : y_2 : y_3)$, $(z_0 : z_1 : z_2 : z_3)$ and $(w_0 : w_1 : w_2 : w_3)$, be coplanar is that [2, 5]

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \\ w_0 & w_1 & w_2 & w_3 \end{vmatrix} = 0. \tag{3}$$

It follows that the plane determined by three distinct points has the equation

$$X_0x_0 + X_1x_1 + X_2x_2 + X_3x_3 = 0, \tag{4}$$

where the *plane coordinates* $[X_0 : X_1 : X_2 : X_3]$ are obtained by Grassmannian expansion [10] of the matrix in Eq. (3), giving

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} x_0 + \begin{vmatrix} y_0 & y_3 & y_2 \\ z_0 & z_3 & z_2 \\ w_0 & w_3 & w_2 \end{vmatrix} x_1 + \begin{vmatrix} y_0 & y_1 & y_3 \\ z_0 & z_1 & z_3 \\ w_0 & w_1 & w_3 \end{vmatrix} x_2 + \begin{vmatrix} y_0 & y_2 & y_1 \\ z_0 & z_2 & z_1 \\ w_0 & w_2 & w_1 \end{vmatrix} x_3 = 0. \tag{5}$$

Employing the Grassmannian expansion we obtain the equation of the plane containing the octahedron face defined by the vertices $\{V_1, V_6, V_3\}$ using their homogeneous coordinates: $V = (1 : a : b : c) \Rightarrow V_1 = (1 : 1 : 0 : 0)$, $V_6 = (1 : 0 : 0 : -1)$, $V_3 = (1 : 0 : 1 : 0)$. Using the determinants in Eq. (5) and the three vertices reveals the corresponding plane coordinates as

$$[X_0 : X_1 : X_2 : X_3] =$$

$$\left[\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} : \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{vmatrix} : \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} \right] = [1 : -1 : -1 : 1]. \tag{6}$$

Hence, the plane equation containing face $\{V_1, V_6, V_3\}$ can be expressed as

$$1 - a - b + c = 0. \tag{7}$$

When the coordinates in Eq. (7) are homogenised, the relation can be expressed as

$$a + b - c - d = 0. \tag{8}$$

Thus, the plane equation determined by the three vertices $\{V_1, V_6, V_3\}$ is precisely the linear factor C_1 in Eq. (2). The remaining seven linear factors in Eq. (2) are, similarly, the plane equations for the seven other octahedron faces. If a point in the design parameter space satisfies Eq. (8), then it lies in the plane of the face spanned by the three vertices $\{V_1, V_6, V_3\}$, and the corresponding mechanism has link lengths constrained by the relation $a + b = c + d$. Depending on the lengths of the individual links satisfying this relation the resulting mechanism can be a double crank, double rocker, or crank rocker, and can have as many as three folding configurations and assembly modes [8, 11].

Similarly, points in the planes of the faces spanned by vertices $\{V_2, V_5, V_3\}$ and by vertices $\{V_1, V_5, V_4\}$ lead to the plane equations

$$1 + a - b - c = 0 \text{ and } 1 - a + b - c = 0,$$

which correspond to the linear factors A_1 and D_1 respectively, when the coordinates are homogenised giving

$$a - b - c + d = 0 \text{ and } a - b + c - d = 0.$$

Points laying in the planes of these two faces correspond to linkages with link lengths constrained by the relations $a + d = b + c$ and $a + c = b + d$. Again, depending on the lengths, the resulting mechanisms can be a double-crank, double-rocker, or crank-rocker, and can have as many as three folding configurations and assembly modes. However, points in the planes spanned by the remaining five faces, corresponding to linear factors A_2, B_1, B_2, C_2, and D_2 represent linkages with zero finite mobility because either the sum of the magnitudes of all the link lengths is identically zero, or one link length is equal to the sum of the lengths of the remaining three links.

2.4 A General Point in the Design Parameter Space

The location of a single point in the design parameter space is a specific planar $4R$ whose link lengths satisfy Eq. (2). The values of the link lengths are directed distances, and hence can be positive or negative. Clearly, if one of the lengths is identically zero, then the resulting $3R$ linkage is a structure. The absolute values of the link lengths identified with Eq. (2) lead to an alternate form of the classification scheme for planar $4R$ linkages first presented in [11] and later refined in [9], and hence to an expression for the Grashof condition. Recall that the Grashof condition states that a planar $4R$ will contain one link that can fully rotate if

$$l + s < p + q, \tag{9}$$

where l and s refer to the lengths of the longest and shortest links, while p and q are the lengths of the two intermediate links.

Input Link, *a*. The limits of angular displacement for the input link, a, if they exist, can be determined using the law of cosines and the two triangles formed by the lengths a and d when the coupler and output link align, giving lengths $c - b$ and $c + b$, respectively, see Fig. 3. In order for $\psi_{\min}$ and $\psi_{\max}$ to exist, then

$$-1 \leq \cos(\psi) \leq 1. \tag{10}$$

It can be shown using the methods in [9, 11] that the conditions leading to

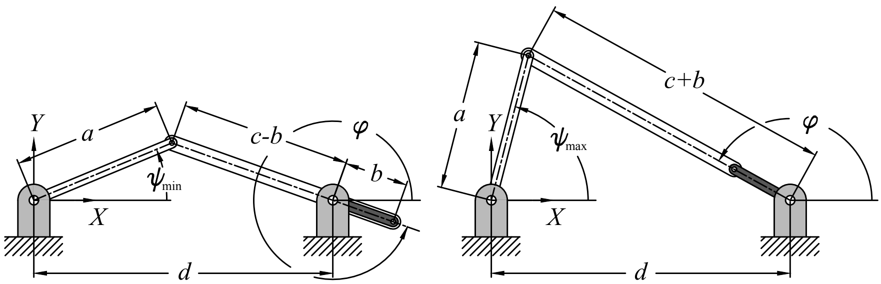

Fig. 3. The angular limits of the output link, if they exist, are $\psi_{\min}$ and $\psi_{\min}$.

$-1 < \cos(\psi) > 1$ can be expressed as the product of two linear factors from Eq. (2). The fist product concerns the existence of $\psi_{\min}$:

$$(a+b-c-d)(a-b+c-d) > 0 \quad \text{(i.e. } C_1 D_1 \ > \ 0). \tag{11}$$

If this condition is satisfied, then both factors must be either positive or negative, and the input link has no $\psi_{\min}$. This implies that the input link can rotate through $\psi = 0$ reaching angles below the line joining the centres of the two ground fixed R-pairs. If this condition is not satisfied then one of either C_1 or D_1 is negative and $\psi_{\min}$ may be computed, using the upper sign $(c-b)^2$, as[1]

$$\psi_{\substack{\min \\ \max}} = \cos^{-1}\left(\frac{a^2+d^2-(c\mp b)^2}{2ad}\right). \tag{12}$$

Referring again to Fig. 3, the second product concerns the existence of $\psi_{\max}$, and can be expressed as:

$$(a-b-c+d)(a+b+c+d) < 0 \quad \text{(i.e. } A_1 B_2 \ < \ 0). \tag{13}$$

If this condition is satisfied then $\psi_{\max}$ does not exist, and the input link can rotate through π. Since B_2 must always be positive, this condition simplifies to

$$a+d < b+c. \tag{14}$$

If the condition in Eq. (13) is not satisfied, then it must be that $a+d \geq b+c$, and $\psi_{\max}$ may be computed using the lower sign $(c+b)^2$ in Eq. (12).

The classification, as in [11], uses the observation that if $C_1D_1 > 0$ and $A_1 < 0$ then neither $\psi_{\min}$ nor $\psi_{\max}$ exist, and the input link is a fully rotatable crank and therefore the link lengths must satisfy the Grashof condition. If $C_1D_1 > 0$ while $A_1 \geq 0$ then $\psi_{\max}$ exists, but not $\psi_{\min}$, and the input link is a 0-rocker because it rocks through 0 between the $\pm\psi_{\max}$ limits. If $C_1D_1 \leq 0$ while $A_1 < 0$

[1] Note that $\cos(\psi)$ returns the same value for $\pm\psi$. Hence, the $\cos^{-1}$ function leads to two limiting values of $\pm\psi_{\min}$ and $\pm\psi_{\max}$, one for each of the *elbow up* and *elbow down* configurations of the linkage.

then $\psi_{\min}$ exists, but not $\psi_{\max}$, and the input link is a π-rocker because it rocks through π between the $\pm\psi_{\min}$ limits. Alternately, if $C_1D_1 \le 0$ and $A_1 \ge 0$ then both $\psi_{\min}$ and $\psi_{\max}$ exist and the input link is a rocker which can pass through neither 0 nor π and rocks in one of two sperate ranges: $\psi_{\min} \le \psi_{\max}$; or $-\psi_{\max} \le -\psi_{\min}$.

Output Link, *b*. The limits of angular displacement for the output link, b, if they exist, can be determined using the law of cosines and the two triangles formed by the lengths b and d when the coupler and input link align, giving lengths $c+a$ and $c-a$, respectively, see Fig. 4. Note that φ in this case is an exterior angle, and the corresponding angle used in the law of cosines is $\pi-\varphi$ necessitating a sign change: $-\cos(\pi-\varphi)=\cos(\varphi)$. In order for $\varphi_{\min}$ and $\varphi_{\max}$ to exist, then

$$-1 \le \cos(\varphi) \le 1. \tag{15}$$

The conditions leading to $-1 > \cos(\varphi) > 1$ can be expressed as the products of

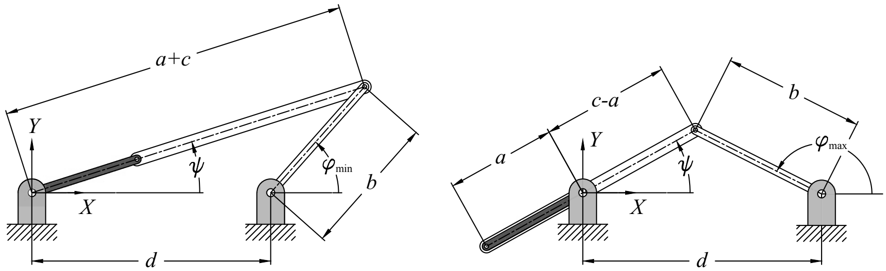

Fig. 4. The angular limits of the output link, if they exist, are $\varphi_{\min}$ and $\varphi_{\min}$.

two linear factors from Eq. (2). If $\varphi_{\min}$ does not exist then a and c can't align and:

$$(a-b+c-d)(a+b+c+d) > 0 \quad \text{(i.e. } D_1B_2 > 0). \tag{16}$$

Since B_2 is always positive, then in order to satisfy Eq. (16) D_1 must also be positive. This leads to the simpler expression for the condition in Eq. (16):

$$a+c > b+d. \tag{17}$$

If this condition is satisfied, then $\varphi_{\min}$ does not exist and the output link can rotate through $\varphi = 0$ reaching angles below the line joining the centres of the two ground fixed R-pairs. When this condition is not satisfied then D_1 is either identically zero or negative meaning that $\varphi_{\min}$ exists and may be computed using the upper sign $(a+c)^2$ in Eq. (18) as

$$\varphi_{\substack{\min\\ \max}} = \cos^{-1}\left(\frac{(a\pm c)^2-(b^2+d^2)}{2bd}\right). \tag{18}$$

Referring again to Fig. 4, the second product concerns the existence of $\varphi_{\max}$, and can be expressed as:

$$(a-b-c+d)(a+b-c-d) < 0 \quad \text{(i.e. } A_1C_1 \; < \; 0). \tag{19}$$

If this condition is satisfied then $\varphi_{\max}$ does not exist, and the output link can rotate through π. Satisfying this condition requires that one factor is positive while the other is negative. If the condition in Eq. (19) is not satisfied, then it must be that A_1 and C_1 are either both positive or negative. In this case $\varphi_{\max}$ may be computed using the lower sign $(a-c)^2$ in Eq. (18).

Again, following [11], the Grashof condition for this case is $A_1C_1 < 0$ and $D_1 > 0$. Using these conditions as indicators, the output link can also be a crank, a 0-rocker, a π-rocker, or a rocker restricted to one of the two separate ranges $\varphi_{\min} \leq \varphi_{\max}$; or $-\varphi_{\max} \leq -\varphi_{\min}$.

Implications of Vanishing Linear Factors. The remaining conditions to consider are if any one, or more, of the three factors are identically zero. Consider the following zeros:

$$\begin{aligned} A_1 = 0 \;&\Rightarrow\; a-b-c+d=0 \;\Rightarrow\; a+d=b+c; \\ C_1 = 0 \;&\Rightarrow\; a+b-c-d=0 \;\Rightarrow\; a+b=c+d; \\ D_1 = 0 \;&\Rightarrow\; a-b+c-d=0 \;\Rightarrow\; a+c=b+d. \end{aligned}$$

If only one of $A_1 = 0$, $C_1 = 0$, or $D_1 = 0$, then the mechanism is a point on one of the planes containing the faces of the octahedron spanned by either vertices $\{V_2, V_5, V_3\}$, $\{V_1, V_6, V_3\}$, or $\{V_4, V_1, V_5\}$, respectively. In each case, the linkage has a single folding configuration. If two of the factors are identically zero, then the mechanism is represented by a point that lies on the line of intersection of the two corresponding faces, which is never an octahedron edge for pairs of these three faces. In this case, the linkage has two folding configurations because of the equality in length of two different sums of pairs of link lengths. Finally, if all three factors are simultaneously identical to zero, the corresponding mechanism is represented by the point common to the planes of all three associated octahedron faces. It is a simple matter to show this leads to a third order equation with only one solution: $a = b = c = d$. In the design parameter space normalised with $d = 1$, this means the point $(1, 1, 1)$, a rhombus linkage possessing three folding configurations.

2.5 Classification

Any planar $4R$ linkage can be classified according to the values of the three linear factors A_1, C_1, and D_1 which can each either be positive, identically zero, or negative. Using the criteria from above the linkage type can be classified according to it's link lengths. All 27 possible mechanisms are listed in Table 1.

#	A_1	C_1	D_1	Input a	Output b	#	A_1	C_1	D_1	Input a	Output b
1	+	+	+	0-rocker	0-rocker	15	0	0	-	crank	π-rocker
2	+	+	0	0-rocker	0-rocker	16	0	-	+	π-rocker	crank
3	+	+	-	rocker	rocker	17	0	-	0	crank	crank
4	+	0	+	0-rocker	crank	18	0	-	-	crank	π-rocker
5	+	0	0	0-rocker	crank	19	-	+	+	crank	crank
6	+	0	-	0-rocker	π-rocker	20	-	+	0	crank	crank
7	+	-	+	rocker	crank	21	-	+	-	π-rocker	π-rocker
8	+	-	0	0-rocker	crank	22	-	0	+	crank	crank
9	+	-	-	0-rocker	π-rocker	23	-	0	0	crank	crank
10	0	+	+	crank	crank	24	-	0	-	crank	π-rocker
11	0	+	0	crank	crank	25	-	-	+	π-rocker	0-rocker
12	0	+	-	π-rocker	π-rocker	26	-	-	0	crank	0-rocker
13	0	0	+	crank	crank	27	-	-	-	crank	rocker
14	0	0	0	crank	crank						

Table 1. Classification of all possible planar $4R$ linkages. Shaded cells satisfy the Grashof condition.

2.6 Continuous Sets of Points in the Design Parameter Space

Planar four-bar linkages however are not exclusively jointed with R-pairs, they often contain P-pairs. However, four-bar mechanisms containing more than two P-pairs cannot move the coupler in general plane motion, rather they can only generate translations and hence are not considered here. A kinematic inversion of an $RRRP$ linkage will possess one variable link length and one variable joint angle, typically called a *slider-crank*. Hence the roles of fixed constant and variable in Eq. 2 can be reassigned to generate a function of the form $b = f(u)$, for example. The important thing to note is that the same IO equation can be used for kinematic synthesis! The resulting mechanism however, will not be represented by a single point in the design parameter space. Rather, it will be represented by a line parallel to the basis vector direction representing the variable link length. The length of the line will be determined by the extremities of the slider translation. This will be interesting to investigate in function generation optimisation problems, but will be left for future work.

The kinematic inversions of the *elliptic-trammel* $PRRP$ linkage are the $RPPR$ and $RRPP$ linkages known as Oldham's coupling and the Scotch yoke, respectively. These linkages possess two variable link lengths. It turns out that Eq. 2 can also be used for function generation synthesis. We believe this to be remarkable! Again, the roles of fixed constant and variable are reassigned. In this case the function generation synthesis problem can be modelled with Eq. 2 to generate functions of the form $b = f(a)$, while the angles represented by u and v are now constants that are identified in the synthesis. In the design parameter space the resulting mechanism will be represented by a curve that is the approximated functional relationship between lengths a and b over the desired maximum input-output range. Again, algorithm development for approximate function generation problems for $PRRP$ type linkages will be left for future work.

3 Conclusions

In this paper we have considered the design and analysis of planar four-bar linkages that can move the coupler in general plane motion in a fundamentally new way. Using the algebraic IO curve from [7, 8] we have shown that the eight linear factors of link lengths can be interpreted as the eight faces of a regular octahedron in the function generator design parameter space of link lengths projected into the hyperplane $d = 1$. We have shown that a point in the design parameter space represents a planar $4R$ linkage, while it's location implies the IO limits of the input and output links yielding the classification from [11]. We believe that this work, together with [8], will lead to a new approach to approximate synthesis optimisation using continuous approximate synthesis as introduced in [6].

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