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A DISCRETE DYNAMIC PROGRAMMING APPROXIMATION TO THE MULTIOBJECTIVE DETERMINISTIC FINITE HORIZON OPTIMAL CONTROL PROBLEM*

A. GUIGUE†, M. AHMADI‡, M. J. D. HAYES‡, AND R. G. LANGLOIS‡

Abstract. This paper addresses the problem of finding an approximation to the minimal element set of the objective space for the class of multiobjective deterministic finite horizon optimal control problems. The objective space is assumed to be partially ordered by a pointed convex cone containing the origin. The approximation procedure consists of a two-step discretization in time and state space. Following the first-order time discretization, the dynamic programming principle is used to find the multiobjective discrete dynamic programming equation equivalent to the resulting discrete multiobjective optimal control problem. The multiobjective discrete dynamic programming equation is finally discretized in the state space. The convergence of the approximation for both discretization steps is discussed.

Key words. multiobjective optimal control, discrete approximation, dynamic programming, partial order generated by a cone, convergence of sequences of sets, topology of families of sets, external stability property

AMS subject classifications. 49M25, 90C29, 49L20, 54C60

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1. Introduction.

Many engineering applications can lead to the optimal control problem formulation [3] with several objectives to be optimized simultaneously. Problems involving multiple objectives present additional difficulties since the *optimal solution* is not as clearly defined as for single objective problems. An example of an application with multiple objectives, which has also motivated this paper, is the generation of optimal joint trajectories for a redundant robotic manipulator operating inside a wind tunnel [13]. Ideally, for this application, the optimal trajectories should minimize both the joint speed and the aerodynamic interference represented by a kinematic measure of the joint configuration. More precisely, the problem presented in [13] is a multiobjective deterministic finite horizon optimal control problem which belongs to the class of problems studied in this paper.

For an optimization problem with a vector-valued objective function, the definition of an optimal solution requires the comparison of any two objective vectors y_1 and y_2 in the *objective space*, which is the set of all possible values that can be taken by the vector-valued objective function. This comparison is provided by a binary relation, generally expressing the preferences of the decision maker. Consider the following example of a simple binary relation: the natural partial order on $\mathbf{R}^p$ when p objective functions are to be minimized. Given two vectors y_1 and y_2 in $\mathbf{R}^p$, y_1 is said to be preferred to y_2 if and only if each component of y_1 is less than or equal to its corresponding component of y_2, or equivalently, if and only if $y_2 \in y_1 + \mathbf{R}^p_+$. For

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†Corresponding author. Mechanical and Aerospace Engineering Department, Carleton University, 1125 Colonel By Drive, Room 3135 MacKenzie Building, Ottawa, ON, K1S 5B6, Canada (aguigue@connect.carleton.ca).

‡Mechanical and Aerospace Engineering Department, Carleton University, 1125 Colonel By Drive, Room 3135 MacKenzie Building, Ottawa, ON, K1S 5B6, Canada (mahmadi@mae.carleton.ca, jhayes@mae.carleton.ca, rlangloi@mae.carleton.ca).

this particular binary relation, an objective vector is defined as optimal (or Pareto optimal in the multiobjective optimization literature [17, 19, 20]) if there is no other objective vector except itself that can be preferred to it. The resolution of an optimization problem with a vector-valued objective function consists of obtaining the set of optimal objective vectors, hereinafter referred to as the minimal element set. In this paper, we will consider the more general binary relation defined in terms of a pointed convex cone $D \subset \mathbf{R}^p$ containing the origin [22].

This paper starts by formulating the multiobjective autonomous deterministic finite horizon optimal control problem in section 2. This formulation does not include any terminal cost; however, our developments can also be applied to nonautonomous systems with terminal cost by including additional assumptions. We propose a two-step numerical approximation procedure for the multiobjective optimal control problem that is applied directly to the original problem rather than to the first-order necessary conditions for optimality [23]. This approximation procedure is built upon the one used for the single objective deterministic discounted infinite horizon optimal control problem as detailed in literature [4, 5, 6, 9]. To the best of our knowledge, this is the first work that provides an approximation to the minimal element set of the multiobjective optimal control problem through discrete dynamic programming. In section 3, we introduce a topology on the family of compact sets of $\mathbf{R}^p$ defined from the Hausdorff distance [15]. With this topology, the minimal element map, which is the map that associates its minimal element set with each compact set, is shown to be continuous. The existence of minimal elements and the external stability property ([17, p. 53], [20, p. 59]) for compact sets are also stated in section 3. The approximation procedure starts with a first-order discretization in time detailed in section 4. This discretization yields a discrete multiobjective optimal control problem, called the *discrete problem*. In section 5, we show that by choosing a particular sequence of time steps and using the external stability property, convergent sequences of minimal elements of the corresponding discrete problems can be constructed. In section 6, using the dynamic programming principle [11, 10], we obtain a discrete multiobjective dynamic programming equation with respect to the ordering cone D. The solution to this equation is shown to be the minimal element set of the discrete problem. The second step of the approximation procedure, presented in section 7, consists of a state-space discretization of the above-mentioned discrete multiobjective dynamic programming equation. Using the continuity of the minimal element map, the solution to the resulting approximate dynamic programming equation is shown to converge towards the minimal element set of the discrete problem in the sense of Hausdorff. This result concludes the presentation of the proposed approximation procedure. The conditions needed for our developments to remain valid for multiobjective nonautonomous problems with terminal cost are discussed in section 8. The approximation procedure proposed in this paper has already been successfully implemented for the optimal joint trajectory generation problem encountered in the robotic application presented in [13].

2. The multiobjective deterministic finite-time horizon optimal control problem. Consider the evolution over a fixed finite-time interval $I = [t_0, t_1]$ $(t_0 < t_1)$ of a dynamical system whose n-dimensional state dynamics are given by a continuous function $\mathbf{f}(\cdot,\cdot,\cdot) : I \times \mathbf{R}^n \times U \to \mathbf{R}^n$, where the control space U is a nonempty compact subset of $\mathbf{R}^m$ [10]. The function $\mathbf{f}(t,\cdot,\mathbf{u})$ is assumed to be Lipschitz:

$$\forall \mathbf{u} \in U,\ \forall t \in I,\ \forall (\mathbf{x},\mathbf{y}) \in \mathbf{R}^{n\times n},\ \|\mathbf{f}(t,\mathbf{x},\mathbf{u}) - \mathbf{f}(t,\mathbf{y},\mathbf{u})\| \le K_{\mathbf{f}}\|\mathbf{x}-\mathbf{y}\|, \tag{2.1}$$

where $\|\cdot\|$ denotes the Euclidian norm. A control $\mathbf{u}(\cdot) : [t, t_1] \subset I \to U$ is a bounded, Lebesgue measurable function. The set of such controls $\mathbf{u}(\cdot)$ is denoted by $\mathcal{U}(t)$, which is nonempty for any t. The Lipschitz condition (2.1) guarantees that, given any control $\mathbf{u}(\cdot)$, the system of differential equations governing the dynamical system

$$\dot{\mathbf{x}}(s) = \mathbf{f}(s, \mathbf{x}(s), \mathbf{u}(s)),\ t \le s \le t_1,$$

with initial conditions

$$\mathbf{x}(t) = \mathbf{x_t},$$

has a unique solution $\tilde{\mathbf{x}}(\cdot) : [t, t_1] \to \mathbf{R}^n$ [21, pp. 467–492], called a trajectory of the dynamical system, $\tilde{\mathbf{x}}(s)$ being the state of the system at time s. The cost of each trajectory $\mathbf{x}(\cdot)$ is evaluated by a p-dimensional vector function $\mathbf{J}(\cdot,\cdot,\cdot) : I \times \mathbf{R}^n \times \mathcal{U}(t) \to \mathbf{R}^p$,

$$\mathbf{J}(t, \mathbf{x_t}, \mathbf{u}(\cdot)) = \int_t^{t_1} \mathbf{L}(s, \mathbf{x}(s), \mathbf{u}(s))ds, \tag{2.2}$$

where the p-dimensional vector function $\mathbf{L}(\cdot,\cdot,\cdot) : I \times \mathbf{R}^n \times U \to \mathbf{R}^p$, usually called the running cost function [10], is assumed to be continuous. The objective space $Y(t, \mathbf{x_t})$ is defined as the set of all possible costs (2.2):

$$Y(t, \mathbf{x_t}) = \{\mathbf{J}(t, \mathbf{x_t}, \mathbf{u}(\cdot)), \mathbf{u}(\cdot) \in \mathcal{U}(t)\}.$$

For simplicity, no terminal cost [10] has been included in (2.2). Moreover, the dynamical system is assumed to be autonomous, $\partial \mathbf{f}/\partial t = 0$, and the running cost function independent of the time, $\partial \mathbf{L}/\partial t = 0$. These simplifications will be discussed later in section 8. Consequently, we shall set $t_0 = 0$, $T = t_1 - t_0$, $I = [0, T]$, $\mathcal{U} = \mathcal{U}(0)$, and $Y(\mathbf{x_0}) = Y(0, \mathbf{x_0})$. Moreover, throughout this paper, we make the following additional assumptions on the functions $\mathbf{f}$ and $\mathbf{L}$.

(i) The function $\mathbf{f}$ is uniformly bounded:

$$\forall \mathbf{x} \in \mathbf{R}^n,\ \forall \mathbf{u} \in U,\ \|\mathbf{f}(\mathbf{x}, \mathbf{u})\| \le M_{\mathbf{f}}. \tag{2.3}$$

(ii) The function $\mathbf{L}(\cdot, \mathbf{u})$ is Lipschitz:

$$\forall \mathbf{u} \in U,\ \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{R}^n \times \mathbf{R}^n,\ \|\mathbf{L}(\mathbf{x}, \mathbf{u}) - \mathbf{L}(\mathbf{y}, \mathbf{u})\| \le K_{\mathbf{L}} \|\mathbf{x} - \mathbf{y}\|. \tag{2.4}$$

(iii) The function $\mathbf{L}$ is uniformly bounded:

$$\forall \mathbf{x} \in \mathbf{R}^n,\ \forall \mathbf{u} \in U,\ \|\mathbf{L}(\mathbf{x}, \mathbf{u})\| \le M_{\mathbf{L}}. \tag{2.5}$$

A set $D \subset \mathbf{R}^p$ is a cone if $\lambda D = D$, for every $\lambda \in \mathbf{R},\ \lambda > 0$. A cone D is pointed if $D \cap -D \subset \{0\}$ [20]. In this paper, $\mathbf{R}^p$ is assumed to be partially ordered by a binary relation defined in terms of a pointed convex cone $D \subset \mathbf{R}^p$ containing the origin [22] (Definition 2.1). The ordering cone D is additionally assumed to be closed.

Definition 2.1. *Let $y_1 \in \mathbf{R}^p$, $y_2 \in \mathbf{R}^p$; y_1 is said to be preferred to y_2, or y_1 is less than y_2 if and only if $y_2 \in y_1 + D$.*

For this particular binary relation, a *minimal element* [7, p. 32] of a given set $Y \subset \mathbf{R}^p$, also called an *efficient solution* in the multiobjective optimization literature ([17, p. 39], [20, p. 33]), is defined.

Definition 2.2. *An element $y_1 \in Y$ is said to be a minimal element if and only if there is no $y_2 \in Y$ $(y_2 \neq y_1)$, such that $y_1 \in y_2 + D$, or equivalently, if and only if there is no y_2 such that $y_1 \in y_2 + D \backslash \{0\}$.*

The set of minimal elements of the set Y with respect to the partial order generated by the cone D is denoted by $\mathcal{E}(Y, D)$. The particular case $D = \mathbf{R}^p_+$ corresponds to the natural partial order on $\mathbf{R}^p$, also called Pareto optimality in the multiobjective optimization literature [17, 19, 20], while a minimal element is called a Pareto optimal solution.

Based on the above, the multiobjective deterministic finite-time horizon optimal control problem denoted by (P) can now be defined.

Problem (P): Determine the minimal element set $V(\mathbf{x_0})$,

$$V(\mathbf{x_0}) = \mathcal{E}(cl(Y(\mathbf{x_0})), D), \tag{2.6}$$

and the corresponding optimal controls $\mathbf{u}^*(\cdot)$ for which these minimal elements are reached.

Considering the closure of the objective space, $cl(Y(\mathbf{x_0}))$, instead of the objective space, $Y(\mathbf{x_0})$, in (2.6) guarantees the existence of minimal elements as shown later in Proposition 3.5. A special case of interest occurs when $p = 1$ and setting $D = \mathbf{R}_+$ in (2.6). The problem (P) then reduces to the single objective deterministic finite-time horizon optimal control problem. The minimal element set $V(\mathbf{x_0})$ becomes a singleton that can be identified with the so-called *value function* [10, p. 9], defined for nonautonomous problems by

$$V(\mathbf{t}, \mathbf{x_t}) = \inf\{J(t, \mathbf{x_t}, \mathbf{u}(\cdot)),\ \mathbf{u}(\cdot) \in \mathcal{U}(t)\}.$$

3. Mathematical preliminaries. Since the minimal elements of the objective space $Y(\mathbf{x_0})$ form a set, the approximation procedure proposed in this paper requires careful attention to the problem of convergence of sequences of sets. In this perspective, we consider the pseudometric space $(\mathcal{M}, \mathcal{H})$ and the metric space $(\mathcal{K}, \mathcal{H})$, where $\mathcal{M} = \{M \subset \mathbf{R}^p,\ M \neq \emptyset,\ M \text{ bounded}\}$, $\mathcal{K} = \{K \subset \mathbf{R}^p,\ K \neq \emptyset,\ K \text{ compact}\}$, and $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff distance. Section 3.1 presents some topological properties of the spaces $(\mathcal{M}, \mathcal{H})$ and $(\mathcal{K}, \mathcal{H})$. We then provide in section 3.2 three important results related to minimal elements. In particular, we show that for compact sets, the existence of minimal elements is guaranteed (Proposition 3.5), and the *external stability* or *domination* property holds ([17, p. 53], [20, p. 59]) (Proposition 3.6). Proposition 3.6, together with the more convenient equivalent definition of the convergence of a sequence of sets in $\mathcal{M}$ in terms of the convergence of sequences of elements of these sets provided by Proposition 3.2, allows us to state the continuity of the *minimal element map* $E(\cdot) : K \in \mathcal{K} \to \mathcal{E}(K, D) \in \mathcal{M}$ (Proposition 3.9). This key result is used in section 7 to prove the convergence of the state-space approximation. In the following, $\|.\|$ denotes the Euclidian norm and B is the unit closed ball in $\mathbf{R}^p$.

3.1. Topological properties of $(\mathcal{M}, \mathcal{H})$ and $(\mathcal{K}, \mathcal{H})$ [15]. Let $(M_1, M_2) \in \mathcal{M} \times \mathcal{M}$; the Hausdorff distance [1, p. 365] $\mathcal{H}(\cdot, \cdot)$ between M_1 and M_2 is

$$\mathcal{H}(M_1, M_2) = \max\left\{\sup_{m_1 \in M_1} d(m_1, M_2),\ \sup_{m_2 \in M_2} d(m_2, M_1)\right\},$$

where, for $M \in \mathcal{M}$,

$$d(x, M) = \inf_{m \in M} \|x - m\|.$$

It is easy to check that the Hausdorff distance $\mathcal{H}(\cdot,\cdot)$ defines a pseudometric on $\mathcal{M}$ (since $\mathcal{H}(M_1, M_2) = 0 \Leftrightarrow cl(M_1) = cl(M_2)$) and a metric on $\mathcal{K}$. We introduce an equivalent definition for the Hausdorff distance $\mathcal{H}(\cdot,\cdot)$ (Proposition 3.1).

PROPOSITION 3.1.

$$\mathcal{H}(M_1, M_2) = \inf_l \{l \geq 0,\ M_1 \subset M_2 + lB \textit{ and } M_2 \subset M_1 + lB\}.$$

Proof. Let $\mathcal{L} = \{l \geq 0,\ M_1 \subset M_2 + lB \text{ and } M_2 \subset M_1 + lB\}$ and $l^* = \inf \mathcal{L}$; we prove below that $\mathcal{H}(M_1, M_2) = l^*$.

First, note that l^* is well defined as M_1 and M_2 belong to $\mathcal{M}$. Let $l \in \mathcal{L}$; then by definition, $M_1 \subset M_2 + lB$. Hence, $\forall m_1 \in M_1,\ \exists m_2 \in M_2,\ \|m_1 - m_2\| \leq l$, which implies $\forall m_1,\ d(m_1, M_2) \leq l$ and $\sup\{d(m_1, M_2),\ m_1 \in M_1\} \leq l$. Similarly, $\sup\{d(m_2, M_1),\ m_2 \in M_2\} \leq l$. Hence, $\mathcal{H}(M_1, M_2) \leq l$. Since the inequality holds for any $l \in \mathcal{L}$, it follows that $\mathcal{H}(M_1, M_2) \leq l^*$.

Conversely, $\mathcal{H}(M_1, M_2) \geq \sup\{d(m_2, M_1),\ m_2 \in M_2\} \geq d(m_2, M_1),\ \forall m_2 \in M_2$. Hence, $\forall \epsilon > 0,\ \forall m_2 \in M_2,\ \exists m_1 \in M_1,\ \mathcal{H}(M_1, M_2) > \|m_1 - m_2\| - \epsilon$, and $M_2 \subset M_1 + (\mathcal{H}(M_1, M_2) + \epsilon)B$. By symmetry, $M_1 \subset M_2 + (\mathcal{H}(M_1, M_2) + \epsilon)B$. Hence, $\forall \epsilon > 0,\ \mathcal{H}(M_1, M_2) + \epsilon \in \mathcal{L}$, which implies $l^* \leq \mathcal{H}(M_1, M_2)$.

Combining the two inequalities yields $\mathcal{H}(M_1, M_2) = l^*$. ∎

Let $(M_n)_{n \in \mathbf{N}}$ be a sequence in $\mathcal{M}$ and $M \in \mathcal{M}$; the sequence (M_n) is said to converge towards M in the sense of Hausdorff if and only if

$$\lim_{n \to \infty} \mathcal{H}(M_n, M) = 0.$$

We introduce a more convenient equivalent definition of the convergence of a sequence of sets in terms of the convergence of *samples* of these sets (Proposition 3.2), where a sample is defined as a sequence (m_n) such that $\forall n \in \mathbf{N},\ m_n \in M_n$.

PROPOSITION 3.2. *The sequence (M_n) converges towards M in the sense of Hausdorff if and only if the two conditions S_1 and S_2 are satisfied:*

(i) *Condition S_1: For all $m \in M$, there exists a sample (m_n) of the sequence (M_n) such that*

$$\lim_{n \to \infty} m_n = m.$$

(ii) *Condition S_2: For any sample (m_n) of the sequence (M_n), there exists a sequence (x_n) in M such that*

$$\lim_{n \to \infty} (m_n - x_n) = 0.$$

Proof. From Proposition 3.1, we have

$$\lim_{n \to \infty} \mathcal{H}(M_n, M) = 0 \Leftrightarrow \forall \epsilon > 0,\ \exists n_0,\ \forall n \geq n_0,\ M \subset M_n + \epsilon B, \text{ and } M_n \subset M + \epsilon B.$$

This equivalence, together with

(i) $\forall n \geq n_0,\ M \subset M_n + \epsilon B \Leftrightarrow \forall m \in M,\ \forall n \geq n_0,\ \exists m_n \in M_n,\ \|m_n - m\| < \epsilon \Leftrightarrow$ condition S_1 holds, and

(ii) $\forall n \geq n_0,\ M_n \subset M + \epsilon B \Leftrightarrow \forall n \geq n_0,\ \forall m_n \in M_n,\ \exists x_n \in M,\ \|m_n - x_n\| < \epsilon \Leftrightarrow$ condition S_2 holds,

yields the result. ∎

COROLLARY 3.3. *If the sequence (M_n) converges towards M, then the following holds:*

(i) $\bigcup M_n$ *is bounded;*

(ii) *Each sample of the sequence* (M_n) *is bounded;*

(iii) *If* $M \in \mathcal{K}$, *any convergent subsequence of a sample of the sequence* (M_n) *has its limit in* M.

Proof. Part (i) follows from the boundedness of the sets M_n and M and the condition S_2 from Proposition 3.2. Part (ii) is a consequence of (i). Part (iii) follows from the closure of the set M and the condition S_2 from Proposition 3.2. □

Proposition 3.4 describes the relation between the set convergence in the sense of Hausdorff and the well-known Kuratowski–Painlevé limit of sets [1, p. 16].

PROPOSITION 3.4. *Let* $(K_n)_{n\in\mathbf{N}}$ *be a sequence of sets in* $\mathcal{K}$ *and* $K \in \mathcal{K}$; *the sequence* (K_n) *converges towards* K *in the sense of Hausdorff if and only if* $\bigcup K_n$ *is bounded and the sequence* (K_n) *converges towards* K *in the sense of Kuratowski–Painlevé.*

3.2. Existence of minimal elements and external stability for compact sets. Three important results related to minimal elements of compact sets are established below.

(i) Proposition 3.5 shows that compactness guarantees the existence of minimal elements. This proposition does not require the assumption that the ordering cone D is closed.

(ii) Proposition 3.6 shows that compactness yields the external stability or domination property. This property states that for each element $k \in K$, there exists a minimal element of K that is preferred to k.

(iii) For a given compact set $K_0 \in \mathcal{K}$, and under the assumption that the minimal elements of the set K_0 with respect to the ordering cone D is equal to the minimal elements of the set K_0 with respect to the ordering cone $int(D)'$, where $int(D)' = int(D) \cup \{0\}$, Proposition 3.9 shows that the minimal element map $E(\cdot) : K \in \mathcal{K} \to \mathcal{E}(K, D) \in \mathcal{M}$ is continuous at K_0. Note that in the definition of the minimal element map, as $\mathcal{E}(K, D)$ is bounded but not necessarily closed, it is only true that $\mathcal{E}(K, D) \in \mathcal{M}$.

PROPOSITION 3.5. *Let* $K \in \mathcal{K}$; *then there exists a minimal element.*

The proof for Proposition 3.5 is omitted here as two different approaches for the proof already exist in the literature. The first consists of an induction argument on p and assumes a weaker property than compactness for K [14]. The second uses Zorn's lemma [8] but requires $cl(D)$ to be pointed, which, for example, is not satisfied by the ordering cone generating the lexicographic order [20, p. 31].

PROPOSITION 3.6. *Let* $K \in \mathcal{K}$; *then for each element* $k \in K$, *there exists a minimal element of* K *that is preferred to* k, *or equivalently,* $\mathcal{E}(K, D) \cap (k - D) \neq \emptyset$.

Proof. The proof is divided into two steps. Let $k \in K$. First, we prove that $\mathcal{E}(K \cap (k - D), D) \neq \emptyset$, and then that $\mathcal{E}(K \cap (k - D), D) \subset \mathcal{E}(K, D)$. These two facts ensure that $\mathcal{E}(K, D) \cap (k - D) \neq \emptyset$. The first part is a consequence of Proposition 3.5 as $K \cap (k - D) \in \mathcal{K}$ from the assumptions on K and D. For the second part, let $k' \in \mathcal{E}(K \cap (k - D), D)$, then $K \cap (k - D) \cap (k' - D) = \{k'\}$. As $k' \in (k - D)$, $(k' - D) \subset (k - D)$. Hence, $K \cap (k' - D) \subset K \cap (k - D) \cap (k' - D) = \{k'\}$, which proves that k' is a minimal element of K. □

LEMMA 3.7. *Let* $Y \subset \mathbf{R}^p$, $Y \neq \emptyset$, *and then* $cl(\mathcal{E}(Y, D)) \subset \mathcal{E}(Y, int(D)')$, *where* $int(D)' = int(D) \cup \{0\}$.

Proof. If $int(D) = \emptyset$, then the result is obvious. Otherwise, let $y \in cl(\mathcal{E}(Y, D))$, and then there exists a sequence (y_n) in $\mathcal{E}(Y, D)$ converging towards y. Assume $y \notin \mathcal{E}(Y, int(D)')$; then there exists $y' \in Y$, $y' \neq y$ such that $y \in y' + int(D)'$. We have

$$\lim_{n\to\infty} y_n - y' = y - y' \in int(D)'.$$

Hence, $y_n - y' \in int(D)' \subset D$ for large enough n, which contradicts the fact that $y_n \in \mathcal{E}(Y, D)$. ∎

COROLLARY 3.8. *Let $K \in \mathcal{K}$, and assume that $\mathcal{E}(K, D) = \mathcal{E}(K, int(D)')$, then $\mathcal{E}(K, D)$ is compact.*

Proof. It is enough to show that $\mathcal{E}(K, D)$ is closed, which is a consequence of Lemma 3.7. Indeed, $cl(\mathcal{E}(K, D)) \subset \mathcal{E}(K, int(D)') \subset \mathcal{E}(K, D)$; hence $cl(\mathcal{E}(K, D)) = \mathcal{E}(K, D)$. ∎

PROPOSITION 3.9. *Let $K \in \mathcal{K}$, and assume that $\mathcal{E}(K, D) = \mathcal{E}(K, int(D)')$; then $E(\cdot)$ is continuous at K.*

Proof. Consider a sequence of sets (K_n) in $\mathcal{K}$ converging towards K in the sense of Hausdorff. Proposition 3.2 is used below to prove that the sequence $(E(K_n)) = (\mathcal{E}(K_n, D))$ converges towards $E(K) = \mathcal{E}(K, D)$ in the sense of Hausdorff. As a result, the proof is divided into two parts.

Part 1 (proof of S_1): Let $k \in \mathcal{E}(K, D)$; we need to find a sample of $(\mathcal{E}(K_n, D))$ that converges towards k. Knowing that $k \in K$ and from S_1, there exists a sample (k'_n) of (K_n) such that the sequence (k'_n) converges towards k. From the external stability property (Proposition 3.6), for all k'_n, there exists $k_n \in \mathcal{E}(K_n, D)$ such that $k'_n \in k_n + D$. The sequence (k_n) can be shown to converge towards k. From Corollary 3.3, the sequence (k_n) is bounded. Therefore, we need only to show that any of its convergent subsequences converges towards k. Let $(k_{\psi(n)})$ be such a convergent subsequence

$$\lim_{n\to\infty} k_{\psi(n)} = a.$$

Since D is closed, $k - a \in D$. By assumption, $k \in \mathcal{E}(K, D)$ and from Corollary 3.3, $a \in K$, which implies $a = k$.

Part 2 (proof of S_2): Let (k_n) be a sample of $(\mathcal{E}(K_n, D))$, hence of (K_n). From S_2, there exists a sequence (x_n) in K such that

$$\lim_{n\to\infty} (k_n - x_n) = 0.$$

From the external stability property (Proposition 3.6), for all x_n, there exists $y_n \in \mathcal{E}(K, D)$ such that $x_n \in y_n + D$. The sequence $(k_n - y_n)$ can be shown to converge towards zero. From the boundedness of the sequence (k_n) (Corollary 3.3) and knowing that the sequence (y_n) is in K, the sequence $(k_n - y_n)$ is bounded. Therefore, we need only to show that any of its convergent subsequences converges towards zero. It is possible to find convergent subsequences $(k_{\psi(n)} - y_{\psi(n)})$, $(k_{\psi(n)})$, and $(y_{\psi(n)})$ such that

$$\lim_{n\to\infty} k_{\psi(n)} - y_{\psi(n)} = a, \ \lim_{n\to\infty} k_{\psi(n)} = k, \text{ and } \lim_{n\to\infty} y_{\psi(n)} = y.$$

Hence, $a = k - y$. It is additionally true that

$$\lim_{n\to\infty} x_{\psi(n)} = k.$$

D being closed, it follows that $a \in D$. Now we claim that $k \in \mathcal{E}(K, D)$. Otherwise, from the assumption of the proposition, $k \notin \mathcal{E}(K, D)$ implies $k \notin \mathcal{E}(K, int(D)')$.

Hence, there exists $v \in K,\ v \neq k$ such that $k \in v + int(D)'$. Applying S_1 to v, there exists a sample (v_n) of (K_n) that converges towards v, and

$$\lim_{n\to\infty} k_{\psi(n)} - v_{\psi(n)} = k - v \in int(D)'.$$

Hence, $k_{\psi(n)} - v_{\psi(n)} \in int(D)' \subset D$ for large enough n, which contradicts the fact that $k_{\psi(n)} \in \mathcal{E}(K_{\psi(n)}, D)$. Finally, from Corollary 3.8, $\mathcal{E}(K, D)$ is compact; hence $y \in \mathcal{E}(K, D)$. To summarize, it has been shown that $a = k - y$ with $a \in D$ and both k and y in $\mathcal{E}(K, D)$, which implies $a = 0$. □

Note that the assumption $\mathcal{E}(K, D) = \mathcal{E}(K, int(D)')$ in Proposition 3.9 is only used in the proof of S_2.

4. A first-order discretization in time. We proceed in this section to a first-order discretization in time with a fixed step h of the problem (P). This discretization yields a discrete multiobjective optimal control problem denoted by (P_h). It is shown, in section 5, how to generate convergent samples of the sequence of sets $(\mathcal{E}(cl(Y_h(\mathbf{x_0})), D))$ as h converges towards zero, where the set $Y_h(\mathbf{x_0})$ is defined as the objective space for the problem (P_h).

Consider a division of I into N intervals of equal length $h = T/N$ and the instants $(t_i)_{i=0\cdots N}$, where $t_i = ih$. We build the discrete multiobjective optimal control problem (P_h) by considering that the controls $\mathbf{u}(\cdot)$, the dynamics $\mathbf{f}(\cdot,\cdot)$, and the running cost $\mathbf{L}(\cdot,\cdot)$ remain constant in any time interval $[t_i, t_{i+1})$. Hence, the discrete control $(\mathbf{u_i})_{i=0\cdots N}$ for the problem (P_h) is defined by

$$\mathbf{u_i} = \mathbf{u}(t_i),\ \mathbf{u}(\cdot) \in \mathcal{U}.$$

The discrete trajectory $(\mathbf{x_i})_{i=1\cdots N}$ is obtained by the recursion

$$\mathbf{x_{i+1}} = \mathbf{x_i} + h\mathbf{f}(\mathbf{x_i}, \mathbf{u_i}) \tag{4.1}$$

with initial conditions $\mathbf{x_0}$. And finally, the discrete cost $\mathbf{J_h}(\mathbf{x_0}, \mathbf{u}(\cdot))$ is given by the series

$$\mathbf{J_h}(\mathbf{x_0}, \mathbf{u}(\cdot)) = h \sum_{i=0}^{N-1} \mathbf{L}(\mathbf{x_i}, \mathbf{u_i}). \tag{4.2}$$

Therefore, the *discrete objective space* $Y_h(\mathbf{x_0})$ is defined by

$$Y_h(\mathbf{x_0}) = \{\mathbf{J_h}(\mathbf{x_0}, \mathbf{u}(\cdot)), \mathbf{u}(\cdot) \in \mathcal{U}\},$$

and the set of minimal elements of the discrete objective space, $V_h(\mathbf{x_0})$, hereinafter referred to as the *discrete minimal element set*, is

$$V_h(\mathbf{x_0}) = \mathcal{E}(cl(Y_h(\mathbf{x_0})), D).$$

Note that the final value of the trajectory $\mathbf{x_N}$ and the final control $\mathbf{u_N}$ do not play any role in the proposed discretization as they do not appear in (4.2).

For the error estimates that will follow in section 5.1, it is convenient to consider the piecewise constant extension $\mathbf{u_h}(\cdot)$ to I of the discrete control:

$$\forall t \in I,\ \mathbf{u_h}(t) = \mathbf{u_i},\ i = \left[\frac{t}{h}\right], \tag{4.3}$$

and similarly, the piecewise constant extension $\mathbf{x_h}(\cdot)$ to I of the discrete trajectory:

$$\forall t \in I,\ \mathbf{x_h}(t) = \mathbf{x_i},\ i = \left[\frac{t}{h}\right].$$

The piecewise constant extensions $\mathbf{u_h}(\cdot)$ and $\mathbf{x_h}(\cdot)$ are also referred to as discrete control and discrete trajectory. If $\mathcal{U}_h \subset \mathcal{U}$ denotes the set of discrete controls (4.3), the discrete objective space $Y_h(\mathbf{x_0})$ is equivalently defined by

$$Y_h(\mathbf{x_0}) = \{\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot)), \mathbf{u_h}(\cdot) \in \mathcal{U}_h\}.$$

Evidently, the definition of the discrete minimal element set $V_h(\mathbf{x_0})$ remains the same.

The existence of minimal elements (Proposition 3.5) and the external stability property (Proposition 3.6) for the discrete objective space $Y_h(\mathbf{x_0})$ are needed in section 5 to build convergent samples of the sequence of discrete minimal element sets $V_h(\mathbf{x_0})$ as the time step h converges towards zero. For this purpose, we state in Proposition 4.1 the compactness of the discrete objective space $Y_h(\mathbf{x_0})$, from which follows that $V_h(\mathbf{x_0}) = \mathcal{E}(Y_h(\mathbf{x_0}), D)$.

PROPOSITION 4.1. *The discrete objective space $Y_h(\mathbf{x_0})$ is a compact set.*

Proof. Let $\mathbf{g}(\cdot) : \mathbf{R}^N \to \mathbf{R}^N$ be the function that associates a discrete control to the discrete trajectory and $\mathbf{h}(\cdot,\cdot) : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^p$ be the function that associates a discrete control and the discrete trajectory to the cost. Both these functions are continuous as $\mathbf{f}(\cdot,\cdot)$ and $\mathbf{L}(\cdot,\cdot)$ are continuous by assumption. The discrete objective space $Y_h(\mathbf{x_0})$ can be viewed as the image of the compact set U^N by the continuous function $\mathbf{h}(\mathbf{g}(\cdot),\cdot) : \mathbf{R}^N \to \mathbf{R}^p$. Hence, it is itself compact. ◻

5. A direct convergence proof. We propose in this section a recursive procedure which generates convergent samples of the sequence of discrete minimal element sets $V_h(\mathbf{x_0})$ as the time step h converges towards zero. It is worth mentioning that under certain assumptions, the discrete objective space $Y_h(\mathbf{x_0})$ can be shown to converge towards the objective space $Y(\mathbf{x_0})$ in the sense of Hausdorff. It then follows from the continuity of the minimal element map (Proposition 3.9) that the discrete minimal element set $V_h(\mathbf{x_0})$ converges towards the minimal element set $V(\mathbf{x_0})$. This guarantees that the samples generated by the recursive procedure converge in the minimal element set $V(\mathbf{x_0})$. The key idea behind this procedure is to use the sequence $(h_r),\ h_r = T/2^r,\ r \in \mathbf{N}$ for the time step [2]. Using this sequence, it is possible to obtain an error estimate between the minimal elements of the discrete objective space $Y_{2h}(\mathbf{x_0})$ and elements of the discrete objective space $Y_h(\mathbf{x_0})$ as any discrete control $\mathbf{u_{2h}}(\cdot)$ in $\mathcal{U}_{2h}$ can always be viewed as a discrete control $\mathbf{u_h}(\cdot) \in \mathcal{U}_h$ satisfying $\mathbf{u_h}(t_{2i}) = \mathbf{u_h}(t_{2i+1}),\ i = 0 \cdots N-1$. A minimal element of the discrete objective space $Y_h(\mathbf{x_0})$ can finally be obtained using the external stability property (Proposition 3.6). The error estimate between the elements of the discrete objective space $Y_{2h}(\mathbf{x_0})$ and elements of the discrete objective space $Y_h(\mathbf{x_0})$ is derived in section 5.1, while section 5.2 contains the proposed procedure and the proof of convergence for the samples generated by the procedure.

5.1. Error estimation. Let $\mathbf{u_{2h}}(\cdot)$ be a discrete control in $\mathcal{U}_{2h}$ and choose the discrete control $\mathbf{u_h}(\cdot)$ in $\mathcal{U}_h$ such that $\mathbf{u_{2h}}(t) = \mathbf{u_h}(t),\ \forall t \in I$. An intermediate step in the derivation of an error estimate between the discrete costs $\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot))$ and $\mathbf{J_{2h}}(\mathbf{x_0}, \mathbf{u_{2h}}(\cdot))$ is the derivation of an error estimate between the discrete trajectories $\mathbf{x_h}(\cdot)$ and $\mathbf{x_{2h}}(\cdot)$. This step will be achieved in Proposition 5.1 using the Gronwall–Bellman inequality. The derivation of the error estimate between the discrete costs

$\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot))$ and $\mathbf{J_{2h}}(\mathbf{x_0}, \mathbf{u_{2h}}(\cdot))$ will follow in Proposition 5.2. Note that both the error estimates obtained in Proposition 5.1 and in Proposition 5.2 are of order h and uniform in $\mathbf{x_0}$.

PROPOSITION 5.1 (error estimate for the discrete trajectories). *Under the assumption* (2.3) *that the function* $\mathbf{f}$ *is uniformly bounded, and for two discrete controls* $\mathbf{u_h}(\cdot) \in \mathcal{U}_h$ *and* $\mathbf{u_{2h}}(\cdot) \in \mathcal{U}_{2h}$ *satisfying* $\mathbf{u_h}(t) = \mathbf{u_{2h}}(t)$, $\forall t \in I$, *the following error estimate between the discrete trajectories* $\mathbf{x_h}(\cdot)$ *and* $\mathbf{x_{2h}}(\cdot)$ *at time* $t \in I$ *holds:*

$$\|\mathbf{x}_h(t) - \mathbf{x}_{2h}(t)\| \leq C_1 h \exp^{K_\mathbf{f} t},$$

where C_1 *is a constant.*

Proof. Equation (4.1) can be rewritten as

$$\mathbf{x_h}(t) = \int_0^{\left[\frac{t}{h}\right]h} \mathbf{f}(\mathbf{x_h}(s), \mathbf{u_h}(s))ds + \mathbf{x}_0.$$

Similarly,

$$\mathbf{x_{2h}}(t) = \int_0^{\left[\frac{t}{2h}\right]2h} \mathbf{f}(\mathbf{x_{2h}}(s), \mathbf{u_{2h}}(s))ds + \mathbf{x}_0.$$

Now, we have

$$\|\mathbf{x}_h(t) - \mathbf{x}_{2h}(t)\| \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_0^t \|\mathbf{f}(\mathbf{x_h}(s), \mathbf{u_h}(s)) - \mathbf{f}(\mathbf{x_{2h}}(s), \mathbf{u_{2h}}(s))\| ds, \\ I_2 &= \int_t^{\left[\frac{t}{h}\right]h} \|\mathbf{f}(\mathbf{x_h}(s), \mathbf{u_h}(s))\| ds, \\ I_3 &= \int_t^{\left[\frac{t}{2h}\right]2h} \|\mathbf{f}(\mathbf{x_{2h}}(s), \mathbf{u_{2h}}(s))\| ds. \end{aligned}$$

The uniform boundedness assumption on $\mathbf{f}$ leads directly to

$$\begin{aligned} I_2 &\leq hM_\mathbf{f}, \\ I_3 &\leq 2hM_\mathbf{f}. \end{aligned}$$

Knowing that $\mathbf{u_{2h}}(s) = \mathbf{u_h}(s)$, $\forall s \in I$, and using the Lipschitz condition on $\mathbf{f}$, we obtain

$$I_1 \leq K_\mathbf{f} \int_0^t \|\mathbf{x_h}(s) - \mathbf{x_{2h}}(s)\| ds.$$

Finally,

$$\|\mathbf{x}_h(t) - \mathbf{x}_{2h}(t)\| \leq 3hM_\mathbf{f} + K_\mathbf{f} \int_0^t \|\mathbf{x}_h(s) - \mathbf{x}_{2h}(s)\| ds.$$

Applying the Gronwall–Bellman inequality [16] yields

$$\|\mathbf{x}_h(t) - \mathbf{x}_{2h}(t)\| \leq 3M_\mathbf{f} h \exp^{K_\mathbf{f} t} = C_1 h \exp^{K_\mathbf{f} t},$$

where $C_1 = 3M_\mathbf{f}$. ∎

PROPOSITION 5.2 (error estimate for the discrete costs). *Under the same assumptions as in Proposition* 5.1 *and the assumption* (2.4) *that the function* $\mathbf{L}(\cdot, \mathbf{u})$ *is Lipschitz, the following error estimate between the discrete costs* $\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot))$ *and* $\mathbf{J_{2h}}(\mathbf{x_0}, \mathbf{u_{2h}}(\cdot))$ *holds:*

$$\|\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot)) - \mathbf{J_{2h}}(\mathbf{x_0}, \mathbf{u_{2h}}(\cdot))\| \leq C_2 h,$$

where C_2 *is a constant.*

Proof. Equation (4.2) can be rewritten as

$$\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot)) = \int_0^T \mathbf{L}(\mathbf{x_h}(s), \mathbf{u_h}(s)) ds.$$

Knowing that $\mathbf{u_{2h}}(s) = \mathbf{u_h}(s),\ \forall s \in I$, and using the Lipschitz condition on $\mathbf{L}$, we obtain

$$\|\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot)) - \mathbf{J_{2h}}(\mathbf{x_0}, \mathbf{u_{2h}}(\cdot))\| \leq K_\mathbf{L} \int_0^T \|\mathbf{x_h}(s) - \mathbf{x_{2h}}(s)\|.$$

Substituting the error estimate for the discrete trajectories obtained in Proposition 5.1 and integrating yields

$$\|\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot)) - \mathbf{J_{2h}}(\mathbf{x_0}, \mathbf{u_{2h}}(\cdot))\| \leq \frac{3M_\mathbf{f} K_\mathbf{L}}{K_\mathbf{f}}(\exp(K_\mathbf{f} T) - 1)h = C_2 h,$$

where $C_2 = \frac{3M_\mathbf{f} K_\mathbf{L}}{K_\mathbf{f}}(\exp(K_\mathbf{f} T) - 1)$. ◻

5.2. Generating convergent samples of the sequence $(V_h(\mathbf{x_0}))$. The proposed procedure to generate samples of the sequence $V_h(\mathbf{x_0})$, detailed below as Procedure 1, is a recursive procedure consisting of two main steps. Starting from an element $\mathbf{J_{2h}}(\mathbf{x_0}, \mathbf{u_h}(\cdot))$ of the minimal element set $V_{2h}(\mathbf{x_0})$, the discrete cost $\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot))$ corresponding to the discrete control $\mathbf{u_h}(\cdot)$ satisfying $\mathbf{u_h}(t) = \mathbf{u_{2h}}(t), \forall t \in I$, is calculated. Then, the application of the external stability property (Proposition 3.6) to the discrete objective set $Y_h(\mathbf{x_0})$ yields an element in the minimal element set $V_h(\mathbf{x_0})$ preferred to the discrete cost $\mathbf{J_h}(\mathbf{x_0}, \mathbf{u_h}(\cdot))$.

Procedure 1: let $h \to 0$ through the sequence (h_r) with $h_r = T/2^r,\ r \in N$. Let the set Z_r be defined by $Z_r = V_{h_r}(\mathbf{x_0})$ ($\mathbf{x_0}$ is dropped for brevity), and let $(\mathbf{z_r})$ be any sample of the sequence (Z_r) built recursively as follows.

Step 1 Let $r = 0$. From Propositions 3.5 and 4.1, the set Z_0 is nonempty; hence let $\mathbf{z_0} \in Z_0$. Note that it is possible to initialize Procedure 1 at any $r = r_0 > 0$.

Step 2 Proposition 5.2 yields an element $\mathbf{y_{r+1}}$ in the discrete objective space $Y_{h_{r+1}}(\mathbf{x_0})$ such that

$$\|\mathbf{z_r} - \mathbf{y_{r+1}}\| \leq C_2 h_{r+1}.$$

Step 3 From the external stability property (Proposition 3.6) and Proposition 4.1, there exists $\mathbf{z_{r+1}} \in Z_{r+1}$ such that $\mathbf{y_{r+1}} \in \mathbf{z_{r+1}} + D$.

Step 4 Repeat Step 2.

The convergence of the sequence $(\mathbf{z_r})$ built from Procedure 1 will be proved in Proposition 5.7. The main idea behind this proof is to show that every element of the sequence $(\mathbf{z_r})$ is in an appropriate neighborhood of elements of any converging subsequence, from which the convergence of the whole sequence can be concluded. This proof requires the following preliminaries.

LEMMA 5.3 (Lemma 3.2.3, [20, p. 52]). *Let $K \in \mathcal{K}$, and $Y \subset \mathbf{R}^p$ be a closed set; then the set $K+Y$ is closed.*

Define $\widehat{D} = \{d \in D,\ \|d\| = 1\}$ and $\widetilde{D}(d, D) = D \cap (d - D),\ \forall d \in D$. We also introduce the scalar $\alpha(D) = \inf\{\|x-y\|,\ x \in \widehat{D},\ y \in -D\}$.

LEMMA 5.4. $\alpha(D) > 0$.

Proof. Assume $\alpha(D) = 0$; then there exists a sequence (x_n) in $\widehat{D}$ and a sequence (y_n) in $-D$ such that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

$\widehat{D}$ being compact, there exists a convergent subsequence $(x_{\psi(n)})$ such that

$$\lim_{n\to\infty} x_{\psi(n)} = x \in \widehat{D}.$$

It follows that the sequence $(y_{\psi(n)})$ is bounded. Hence, there exists a convergent subsequence $(y_{\phi\circ\psi(n)})$ such that

$$\lim_{n\to\infty} y_{\phi\circ\psi(n)} = y \in -D.$$

Finally, $x + (-y) = 0$ with both x and $-y$ in D. Knowing that D is pointed implies that $x = y = 0$, which is a contradiction. □

LEMMA 5.5. $\forall y \in \widetilde{D}(d, D),\ \|y\| \le \|d\|/\alpha(D)$.

Proof. If $y = 0$, then the result is obvious. Let $y \in \widetilde{D}(d, D),\ y \neq 0$, and $y \in d - D$; hence there exists $x \in D$ such that $y + x = d$. Divide this last expression by $\|y\|$, and rewrite it as

$$\frac{y}{\|y\|} - \frac{-x}{\|y\|} = \frac{d}{\|y\|},$$

$y/\|y\| \in \widehat{D}$, and $-x/\|y\| \in -D$. The proof is completed by taking the norm on both sides and using the definition of $\alpha(D)$. □

PROPOSITION 5.6. *Consider $(K_1, K_2) \in \mathcal{K} \times \mathcal{K}$ such that $(K_1 + D) \cap K_2 \neq \emptyset$ and define $K = (K_1 + D) \cap (K_2 - D)$. Then, $K \in \mathcal{K}$ and $\forall (k, \widetilde{k}) \in K \times K$,*

$$\|k - \widetilde{k}\| \le \frac{2}{\alpha(D)} \sup\{\|k_1 - k_2\|,\ (k_1, k_2) \in K_1 \times K_2\} + \mathit{diam}(K_1),$$

where $\mathit{diam}(K_1)$ denotes the diameter of the set K_1.

Proof. Let $k \in K$; then there exists $k_1 \in K_1$, $k_2 \in K_2$, $(d_1, d_2) \in D \times D$ such that $k = k_1 + d_1$ and $k = k_2 - d_2$. Hence, $k_2 - k_1 = d_1 + d_2$, and therefore $k_2 - k_1 \in D$. Now, we can write $k = k_1 + (k_2 - k_1) - d_2$. Hence, $k - k_1 \in (k_2 - k_1) - D$. Knowing that $k - k_1 \in D$ yields $k - k_1 \in \widetilde{D}(k_2 - k_1, D)$. Similarly, let $\tilde{k} \in K$; then there exists $\widetilde{k_1} \in K_1$ and $\widetilde{k_2} \in K_2$ such that $\widetilde{k} - \widetilde{k_1} \in \widetilde{D}(\widetilde{k_2} - \widetilde{k_1}, D)$. We have

$$\|k - \widetilde{k}\| \le \|k - k_1\| + \|k_1 - \widetilde{k_1}\| + \|\widetilde{k_1} - \widetilde{k}\|.$$

From Lemma 5.5, $\|k - k_1\| \le \|k_2 - k_1\|/\alpha(D)$ and $\|\widetilde{k_1} - \widetilde{k}\| \le \|\widetilde{k_2} - \widetilde{k_1}\|/\alpha(D)$. Hence,

$$\|k - k_1\| + \|k_1 - \widetilde{k_1}\| + \|\widetilde{k_1} - \widetilde{k}\| \le \frac{1}{\alpha(D)}\|k_2 - k_1\| + \|k_1 - \widetilde{k_1}\| + \frac{1}{\alpha(D)}\|\widetilde{k_2} - \widetilde{k_1}\|.$$

Finally,

$$\|k - \widetilde{k}\| \leq \frac{2}{\alpha(D)} \sup\{\|k_1 - k_2\|,\ (k_1, k_2) \in K_1 \times K_2\} + \mathrm{diam}(K_1),$$

which shows that K is bounded. The closure is a consequence of Lemma 5.3. Hence, K is compact. ▯

PROPOSITION 5.7. *Under the assumption* (2.5) *that the function* $\mathbf{L}$ *is uniformly bounded, the sequence* $(\mathbf{z_r})$ *converges.*

Proof. The uniform boundedness assumption on $\mathbf{L}$ implies that

$$\forall \mathbf{x_0} \in \mathbf{R}^n,\ \forall \mathbf{u}(\cdot) \in \mathcal{U}_h,\ \|\mathbf{J_h}(\mathbf{x_0}, \mathbf{u}(\cdot))\| \leq T M_{\mathbf{L}},$$

which shows that the sets $Y_h(\mathbf{x_0})$, and consequently Z_r, are uniformly bounded. The sequence $(\mathbf{z_r})$ being bounded has at least one accumulation point $\mathbf{z}$. From the definition of $\mathbf{z}$,

$$\forall \epsilon > 0,\ \forall r_0,\ \exists r > r_0,\ \|\mathbf{z_r} - \mathbf{z}\| < \epsilon.$$

Repeatedly applying this definition with $\epsilon_p = 1/p,\ p = 1 \cdots \infty$ yields an increasing sequence (r_p) such that $\|\mathbf{z_{r_p}} - \mathbf{z}\| < 1/p$. Setting $\psi(p) = r_p$, a subsequence $(\mathbf{z}_{\psi(\mathbf{p})})$ which converges towards $\mathbf{z}$ and satisfies $\|\mathbf{z}_{\psi(\mathbf{p})} - \mathbf{z}\| < 1/p,\ \forall p \geq 1$ is obtained. The key point is to observe that necessarily, from the construction of the sequence $(\mathbf{z_r})$,

$$\mathbf{z_r} \in K,\ \forall r \in [\psi(p), \psi(p+1)],$$

where $B(\mathbf{x}, r)$ denotes the closed ball centered at $\mathbf{x}$ with radius r,

$$K = (B(\mathbf{z}_{\psi(\mathbf{p+1})}, l) + D) \bigcap (B(\mathbf{z}_{\psi(\mathbf{p})}, l) - D),$$

and

$$l = C_2 \sum_{r=\psi(p)+1}^{\psi(p+1)} h_r \leq C_2 \frac{T}{2^{\psi(p)}}.$$

Applying Proposition 5.6 with $K_1 = B(\mathbf{z}_{\psi(\mathbf{p+1})}, l)$ and $K_2 = B(\mathbf{z}_{\psi(\mathbf{p})}, l)$ leads to

$$\|\mathbf{z_r} - \mathbf{z}_{\psi(\mathbf{p})}\| \leq \frac{2}{\alpha(D)} \left(2l + \|\mathbf{z}_{\psi(\mathbf{p+1})} - \mathbf{z}_{\psi(\mathbf{p})}\| \right) + l.$$

By applying the triangular inequality,

$$\|\mathbf{z}_{\psi(\mathbf{p})} - \mathbf{z}_{\psi(\mathbf{p+1})}\| < \|\mathbf{z}_{\psi(\mathbf{p})} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{z}_{\psi(\mathbf{p+1})}\| < \frac{1}{p} + \frac{1}{p+1},$$

which implies that $\|\mathbf{z_r} - \mathbf{z}_{\psi(\mathbf{p})}\| < \beta_p$, where the sequence (β_p) converges towards zero.

Let now $r \in \mathbf{N}$; then there exists a unique p such that $r \in [\psi(p), \psi(p+1))$. By applying the triangular inequality,

$$\|\mathbf{z_r} - \mathbf{z}\| \leq \|\mathbf{z_r} - \mathbf{z}_{\psi(\mathbf{p})}\| + \|\mathbf{z}_{\psi(\mathbf{p})} - \mathbf{z}\| \leq \beta_p + \frac{1}{p},$$

which shows that the sequence $(\mathbf{z_r})$ converges towards $\mathbf{z}$. ▯

6. A discrete dynamic programming formulation. After having performed the time discretization, the problem is to determine the discrete minimal element set $V_h(\mathbf{x_0})$. This is realized in two steps. First, we prove in Corollary 6.6, using the dynamic programming principle [10], that the discrete minimal element set $V_h(\mathbf{x_0})$ is also the solution to a discrete multiobjective dynamic programming equation with respect to the ordering cone D denoted by (HJ_h). Second, in section 7, we proceed to a discretization in the state space of the discrete multiobjective dynamic programming equation (HJ_h) yielding an approximate multiobjective dynamic programming equation denoted by (HJ_h^d). The solution to this approximate equation is shown in Corollary 7.2 to converge towards the discrete minimal element set $V_h(\mathbf{x_0})$ in the sense of Hausdorff.

The statement of the discrete multiobjective dynamic programming equation (HJ_h) in Proposition 6.1 requires the introduction of a few additional notations, which are introduced below. For $0 \le k \le m \le N-1$, the discrete objective space $Y_h^{k,m}(\mathbf{x_k})$ is defined by

$$Y_h^{k,m}(\mathbf{x_k}) = \left\{ \mathbf{J}_h^{k,m}(\mathbf{x_k}, \mathbf{u}(\cdot)) = h \sum_{i=k}^{m} \mathbf{L}(\mathbf{x_i}, \mathbf{u_i}), \mathbf{u}(\cdot) \in \mathcal{U}_h \right\},$$

and the corresponding discrete minimal element set by

$$V_h^{k,m}(\mathbf{x_k}) = \mathcal{E}\left(cl\left(Y_h^{k,m}(\mathbf{x_k}) \right), D \right).$$

With this notation, it can be seen that the discrete objective space $Y_h(\mathbf{x_0})$ is the same as the set $Y_h^{0,N-1}(\mathbf{x_0})$ and the discrete minimal element set $V_h(\mathbf{x_0})$ is the same as the set $V_h^{0,N-1}(\mathbf{x_0})$. The set $\widetilde{Y}_h^{k,m}(\mathbf{x_k})$ that will also be called discrete objective space is defined by

$$\widetilde{Y}_h^{k,m}(\mathbf{x_k}) = \left\{ \mathbf{J}_h^{k,m}(\mathbf{x_k}, \mathbf{u}(\cdot)) + V_h^{m+1,N-1}(\mathbf{x_{m+1}}), \mathbf{u}(\cdot) \in \mathcal{U}_h \right\}, \tag{6.1}$$

and the corresponding discrete minimal element set by

$$\widetilde{V}_h^{k,m}(\mathbf{x_k}) = \mathcal{E}\left(cl\left(\widetilde{Y}_h^{k,m}(\mathbf{x_k}) \right), D \right).$$

In the definition of the discrete objective space $\widetilde{Y}_h^{k,m}(\mathbf{x_k})$, the particular case $m = N-1$ yields the discrete minimal element set $V_h^{N,N-1}(\mathbf{x_N})$, which is usually referred to as the terminal data condition [10], and is set to

$$V_h^{N,N-1}(\mathbf{x_N}) = \{0\}. \tag{6.2}$$

The definition of the discrete objective space $\widetilde{Y}_h^{k,m}(\mathbf{x_k})$ can be justified by noting that the discrete objective space $Y_h^{m+1,N-1}(\mathbf{x_{m+1}})$ is bounded from the uniform boundedness of the running cost function $\mathbf{L}(\cdot,\cdot)$, which guarantees the existence of minimal elements from Proposition 3.5. The existence of minimal elements for the discrete objective space $\widetilde{Y}_h^{k,m}(\mathbf{x_k})$ can be justified using a similar argument.

PROPOSITION 6.1. *For $0 \le k \le m \le N-1$, the discrete multiobjective dynamic programming equation denoted by* (HJ_h) *is*

$$V_h^{k,N-1}(\mathbf{x_k}) = \widetilde{V}_h^{k,m}(\mathbf{x_k}),$$

which yields, together with (6.1),

$$\widetilde{V}_h^{k,m}(\mathbf{x_k}) = \mathcal{E}\left(cl\left(\left\{\mathbf{J}_h^{k,m}(\mathbf{x_k},\mathbf{u}(\cdot)) + \widetilde{V}_h^{m+1,m'}(\mathbf{x_{m+1}}), \mathbf{u}(\cdot) \in \mathcal{U}_h\right\}\right), D\right),$$

with $m+1 \leq m' \leq N-1$*. From* (6.2)*, the terminal data condition is*

$$\widetilde{V}_h^{N,N}(\mathbf{x_N}) = \{0\}.$$

Proof. For clarity, we prove the following three lemmas to be used in the proof of Proposition 6.1 and postpone this proof to the end of this section. ∎

LEMMA 6.2. $\widetilde{Y}_h^{k,m}(\mathbf{x_k}) \subset cl(Y_h^{k,N-1}(\mathbf{x_k}))$.

Proof. Let $\widetilde{\mathbf{y}} \in \widetilde{Y}_h^{k,m}(\mathbf{x_k})$ and $\epsilon > 0$; then there exists $\widetilde{\mathbf{u}}(\cdot) \in \mathcal{U}_h$ and $\mathbf{y_{m+1}} \in Y_h^{m+1,N-1}(\mathbf{x_{m+1}})$ such that

$$\left\|\widetilde{\mathbf{y}} - \mathbf{J}_h^{k,m}(\mathbf{x_k}, \widetilde{\mathbf{u}}(\cdot)) - \mathbf{y_{m+1}}\right\| \leq \epsilon,$$

with $\mathbf{y_{m+1}} = \mathbf{J}_h^{m+1,N-1}(\mathbf{x_{m+1}}, \widehat{\mathbf{u}}(\cdot))$ for some $\widehat{\mathbf{u}}(\cdot) \in \mathcal{U}_h$. Define the new control $\mathbf{u}(\cdot) \in \mathcal{U}_h$ as

$$\mathbf{u} = \begin{cases} \widetilde{\mathbf{u}}, & j = 1 \cdots m, \\ \widehat{\mathbf{u}}, & j = m+1 \cdots N-1. \end{cases}$$

Then, $\mathbf{J}_h^{k,m}(\mathbf{x_k}, \widetilde{\mathbf{u}}(\cdot)) + \mathbf{y_{m+1}} = \mathbf{J}_h^{k,N-1}(\mathbf{x_k}, \mathbf{u}(\cdot)) = \mathbf{y} \in Y_h^{k,N-1}(\mathbf{x_k})$, and $\mathbf{y}$ verifies

$$\|\widetilde{\mathbf{y}} - \mathbf{y}\| \leq \epsilon.$$

Hence, $\widetilde{\mathbf{y}} \in cl(Y_h^{k,N-1}(\mathbf{x_k}))$. ∎

LEMMA 6.3. $cl(Y_h^{k,N-1}(\mathbf{x_k})) \subset cl(\widetilde{Y}_h^{k,m}(\mathbf{x_k}) + D)$.

Proof. Let $\overline{\mathbf{y}} \in cl(Y_h^{k,N-1}(\mathbf{x_k}))$ and $\epsilon > 0$; then there exists $\mathbf{y} \in Y_h^{k,N-1}(\mathbf{x_k})$ such that $\|\overline{\mathbf{y}} - \mathbf{y}\| \leq \epsilon$. Writing $\mathbf{y} = \mathbf{J}_h^{k,m}(\mathbf{x_k}, \mathbf{u}(\cdot)) + \mathbf{y_{m+1}}$ with $\mathbf{y_{m+1}} \in Y_h^{m+1,N-1}(\mathbf{x_{m+1}})$ yields

$$\left\|\overline{\mathbf{y}} - \mathbf{J}_h^{k,m}(\mathbf{x}_k, \mathbf{u}(\cdot)) - \mathbf{y_{m+1}}\right\| \leq \epsilon.$$

We also have $\mathbf{y_{m+1}} \in cl(Y_h^{m+1,N-1}(\mathbf{x_{m+1}}))$. From the external stability property, there exists $\mathbf{y_{m+1}}^* \in V_h^{m+1,N-1}(\mathbf{x_{m+1}})$ such that $\mathbf{y_{m+1}} \in \mathbf{y_{m+1}}^* + D$. Therefore,

$$\left\|\overline{\mathbf{y}} - \mathbf{J}_h^{k,m}(\mathbf{x_k}, \mathbf{u}(\cdot)) - \mathbf{y_{m+1}}^* - d\right\| \leq \epsilon,$$

which leads to $\mathbf{J}_h^{k,m}(\mathbf{x_k}, \mathbf{u}(\cdot)) + \mathbf{y_{m+1}}^* \in \widetilde{Y}_h^{k,m}(\mathbf{x_k})$. Hence, $\overline{\mathbf{y}} \in cl(\widetilde{Y}_h^{k,m}(\mathbf{x_k}) + D)$. ∎

LEMMA 6.4. $cl(\widetilde{Y}_h^{k,m}(\mathbf{x_k}) + D) \subset cl(\widetilde{Y}_h^{k,m}(\mathbf{x_k})) + D$.

Proof. This is a consequence of the facts that the discrete objective space $\widetilde{Y}_h^{k,m}(\mathbf{x_k})$ is bounded and D is closed. ∎

LEMMA 6.5. *Let* $K_1 \subset \mathcal{K}$*,* $K_2 \subset \mathcal{K}$ *satisfying* $K_1 \subset K_2$ *and* $K_2 \subset K_1 + D$*; then* $\mathcal{E}(K_1, D) = \mathcal{E}(K_2, D)$.

Proof. From Proposition 3.5, $\mathcal{E}(K_1, D) \neq \emptyset$. Let $k_1 \in \mathcal{E}(K_1, D)$, and hence $k_1 \in K_1 \subset K_2$. Assume that $k_1 \notin \mathcal{E}(K_2, D)$, and hence there exists $k_2 \in K_2$,

$d \in D,\ d \neq 0$ such that $k_1 = k_2 + d$. By assumption, $k_2 \in K_1 + D$, and hence there exists $k_1' \in K_1$, $d' \in D$ such that $k_2 = k_1' + d'$, which yields $k_1 = k_1' + d + d'$. D being convex, $d + d' \in D$ with $d + d' \neq 0$, which contradicts the fact that $k_1 \in \mathcal{E}(K_1, D)$. Therefore, $\mathcal{E}(K_1, D) \subset \mathcal{E}(K_2, D)$.

From Proposition 3.5, $\mathcal{E}(K_2, D) \neq \emptyset$. Let $k_2 \in \mathcal{E}(K_2, D)$; then $k_2 \in K_2$. By assumption, $k_2 \in K_1 + D$; hence there exists $k_1 \in K_1$, $d \in D$ such that $k_2 = k_1 + d$. By assumption, $k_1 \in K_2$. Hence, necessarily, as $k_2 \in \mathcal{E}(K_2, D)$, we have $d = 0$, $k_2 = k_1$, and $k_2 \in K_1$. From the assumption $K_1 \subset K_2$ follows that $k_2 \in \mathcal{E}(K_1, D)$. Therefore, $\mathcal{E}(K_2, D) \subset \mathcal{E}(K_1, D)$.

Combining the two inclusions yields $\mathcal{E}(K_1, D) = \mathcal{E}(K_2, D)$. ∎

We can now proceed with the proof of Proposition 6.1.

Proof. [Proposition 6.1] Apply Lemma 6.5 with $K_1 = cl(\widetilde{Y}_h^{k,m}(\mathbf{x_k}))$ and $K_2 = cl(Y_h^{k,N-1}(\mathbf{x_k}))$. The sets $cl(\widetilde{Y}_h^{k,m}(\mathbf{x_k}))$ and $cl(Y_h^{k,N-1}(\mathbf{x_k}))$ are compact. The inclusion $K_1 \subset K_2$ comes from Lemma 6.2, while the inclusion $K_2 \subset K_1 + D$ comes from Lemmas 6.3 and 6.4. ∎

COROLLARY 6.6. *The discrete minimal element set $V_h(\mathbf{x_0})$ is equal to the discrete minimal element set $\widetilde{V}_h^{0,0}(\mathbf{x_0})$, which can be obtained from the discrete multiobjective dynamic programming equation* (HJ_h)

$$\widetilde{V}_h^{k,k}(\mathbf{x_k}) = \mathcal{E}\left(cl\left(\left\{h\mathbf{L}(\mathbf{x_k}, \mathbf{u}) + \widetilde{V}_h^{k+1,k+1}(\mathbf{x_k} + hf(\mathbf{x_k}, \mathbf{u})), \mathbf{u} \in U\right\}\right), D\right), \tag{6.3}$$

with terminal data condition

$$\widetilde{V}_h^{N,N}(\mathbf{x}_N) = \{0\}. \tag{6.4}$$

Proof. Recalling that the discrete element set $V_h(\mathbf{x_0})$ is the same as the set $V_h^{0,N-1}(\mathbf{x_0})$, this result follows directly from Proposition 6.1. ∎

7. A discretization in the state space. This section describes the last step of the approximation procedure proposed in this paper. It consists of a discretization in the state space of the discrete multiobjective dynamic programming equation (6.3) of Corollary 6.6, which yields an approximate multiobjective dynamic programming equation (HJ_h^d). Using the continuity of the minimal element map (Proposition 3.9), the solution to the approximate multiobjective dynamic programming equation (HJ_h^d) is shown in Corollary 7.2 to converge towards the discrete minimal element set $V_h(\mathbf{x_0})$ in the sense of of Hausdorff as the state-space discretization converges towards zero.

We first proceed with the discretization of the state space. If we denote $X_k,\ k = 0 \cdots N$ the set of possible values for the state at the instant t_k and assume that the set X_0 is compact, then it can be observed, using the continuity of the function $\mathbf{f}(\cdot,\cdot)$, that each set $X_k,\ k = 1 \cdots N$ is compact. By compactness, each set $X_k,\ k = 0 \cdots N$ can be covered by a finite number M_k of closed balls $B(\mathbf{x_{k,j}}, \epsilon_k),\ j = 1 \cdots M_k$. Define $d = \max\{\epsilon_k,\ k = 0 \cdots N\}$ as the size of the state-space discretization.

Let $k = 0 \cdots N-1$ and $\mathbf{x_k} \in X_k$; then there always exists j such that $\mathbf{x_k} \in B(\mathbf{x_{k,j}}, \epsilon_k)$. The set $\widehat{Y}_h^{k,k}(\mathbf{x_k})$, representing the state-space approximation to the discrete objective space $\widetilde{Y}_h^{k,k}(\mathbf{x_k})$, is defined by

$$\widehat{Y}_h^{k,k}(\mathbf{x_k}) = \left\{h\mathbf{L}(\mathbf{x_{k,j}}, \mathbf{u}) + \mathcal{E}\left(cl\left(\widehat{Y}_h^{k+1,k+1}(\mathbf{x_k} + hf(\mathbf{x_k}, \mathbf{u}))\right), D\right), \mathbf{u} \in U\right\}, \tag{7.1}$$

while the set $\widehat{Y}_h^{N,N}(\mathbf{x_N})$ is defined by

$$\widehat{Y}_h^{N,N}(\mathbf{x_N}) = \{0\}. \tag{7.2}$$

Proposition 3.5, together with the boundedness assumption on the running cost function $\mathbf{L}(\cdot,\cdot)$, justify the definition of the set $\widehat{Y}_h^{k,k}(\mathbf{x_k})$. The minimal element set $\widehat{V}_h^{k,k}(\mathbf{x_k})$, representing the state-space approximation to the minimal element set $\widetilde{V}_h^{k,k}(\mathbf{x_k})$, is defined by

$$\widehat{V}_h^{k,k}(\mathbf{x_k}) = \mathcal{E}\left(cl\left(\widehat{Y}_h^{k,k}(\mathbf{x_k})\right), D\right).$$

From (7.1)–(7.2), we can write

$$\widehat{V}_h^{k,k}(\mathbf{x_k}) = \mathcal{E}\left(cl\left(\left\{h\mathbf{L}(\mathbf{x_{k,j}},\mathbf{u}) + \widehat{V}_h^{k+1,k+1}(\mathbf{x_k} + hf(\mathbf{x_k},\mathbf{u})), \mathbf{u} \in U\right\}\right), D\right), \tag{7.3}$$

and

$$\widehat{V}_h^{N,N}(\mathbf{x_N}) = \{0\}. \tag{7.4}$$

The multiobjective dynamic programming equation (7.3) denoted by (HJ_h^d), together with the terminal data condition (7.4), is an approximation to the multiobjective dynamic programming equation (6.3), together with the terminal data condition (6.4). Using the continuity of the minimal element map, it is shown in Proposition 7.1 that the minimal element set $\widehat{V}_h^{k,k}(\mathbf{x_k})$ converges towards the discrete minimal element set $\widetilde{V}_h^{k,k}(\mathbf{x_k})$ in the sense of Hausdorff as the state-space discretization d converges towards zero.

PROPOSITION 7.1. *With the assumption (see Proposition* 3.9*)*

$$\widetilde{V}_h^{k,k}(\mathbf{x_k}) = \mathcal{E}\left(cl\left(\widetilde{Y}_h^{k,k}(\mathbf{x_k})\right), int(D)'\right),\ \forall k = 0\cdots N-1,\ \forall \mathbf{x_k} \in X_k,$$

we have

$$\lim_{d\to 0} \mathcal{H}\left(\widehat{V}_h^{k,k}(\mathbf{x_k}), \widetilde{V}_h^{k,k}(\mathbf{x_k})\right) = 0,\ \forall \mathbf{x_k} \in X_k.$$

Proof. The proposed proof is a proof by induction.

Step 1: Proof for the case $k = N-1$. Let $\mathbf{x_{N-1}} \in X_{N-1}$. We have

$$\widetilde{Y}_h^{N-1,N-1}(\mathbf{x_{N-1}}) = \{h\mathbf{L}(\mathbf{x_{N-1}},\mathbf{u}), \mathbf{u} \in U\},$$

and

$$\widehat{Y}_h^{N-1,N-1}(\mathbf{x_{N-1}}) = \{h\mathbf{L}(\mathbf{x_{N-1,j}},\mathbf{u}), \mathbf{u} \in U\}.$$

To be able to apply Proposition 3.9, we must verify that the sets $cl(\widetilde{Y}_h^{N-1,N-1}(\mathbf{x_{N-1}}))$ and $cl(\widehat{Y}_h^{N-1,N-1}(\mathbf{x_{N-1}}))$ are compact, and they are as both these sets are bounded. It is also required that

$$\lim_{\epsilon_{N-1}\to 0} \mathcal{H}\left(cl\left(\widehat{Y}_h^{N-1,N-1}(\mathbf{x_{N-1}})\right), cl\left(\widetilde{Y}_h^{N-1,N-1}(\mathbf{x_{N-1}})\right)\right) = 0.$$

This follows from the Lipschitz assumption on the running cost $\mathbf{L}(\cdot,\cdot)$.

Step 2: Proof for the case $k-1$ assuming that the result is true for $k > 1$, i.e.,

$$\lim_{(\epsilon_k,\ldots,\epsilon_{N-1})\to 0} \mathcal{H}\left(\widehat{V}_h^{k,k}(\mathbf{x_k}), \widetilde{V}_h^{k,k}(\mathbf{x_k})\right) = 0, \forall \mathbf{x_k} \in X_k. \tag{7.5}$$

Let $\mathbf{x_{k-1}} \in X_{k-1}$. To be able to apply Proposition 3.9, we must verify that the sets $cl(\widetilde{Y}_h^{k-1,k-1}(\mathbf{x_{k-1}}))$ and $cl(\widehat{Y}_h^{k-1,k-1}(\mathbf{x_{k-1}}))$ are compact, and they are as both these sets are bounded. It is also required that

$$\lim_{(\epsilon_{k-1},\ldots,\epsilon_{N-1})\to 0} \mathcal{H}\left(cl\left(\widehat{Y}_h^{k-1,k-1}(\mathbf{x_{k-1}})\right), cl\left(\widetilde{Y}_h^{k-1,k-1}(\mathbf{x_{k-1}})\right)\right) = 0.$$

This follows from comparing (6.1), which can be rewritten, using Proposition 6.1,

$$\widetilde{Y}_h^{k-1,k-1}(\mathbf{x_{k-1}}) = \left\{h\mathbf{L}(\mathbf{x_{k-1}},\mathbf{u}) + \widetilde{V}_h^{k,k}(\mathbf{x_{k-1}} + hf(\mathbf{x_{k-1}},\mathbf{u})), \mathbf{u} \in U\right\},$$

with (7.1), which can be rewritten,

$$\widehat{Y}_h^{k-1,k-1}(\mathbf{x_{k-1}}) = \left\{h\mathbf{L}(\mathbf{x_{k-1,j}},\mathbf{u}) + \widehat{V}_h^{k,k}(\mathbf{x_{k-1}} + hf(\mathbf{x_{k-1}},\mathbf{u})), \mathbf{u} \in U\right\},$$

and using both the induction assumption (7.5) and the Lipschitz assumption on the running cost $\mathbf{L}(\cdot,\cdot)$. ◻

COROLLARY 7.2. *The minimal element set* $\widehat{V}_h^{0,0}(\mathbf{x_0})$*, solution to the multiobjective dynamic programming equation approximation* (7.3) *with terminal condition* (7.4)*, converges towards the discrete minimal element set* $V_h(\mathbf{x_0})$ *in the sense of Hausdorff as the state-space discretization* d *converges towards zero.*

Proof. From Proposition 7.1, we know that the minimal element set $\widehat{V}_h^{0,0}(\mathbf{x_0})$ converges towards the discrete minimal element set $\widetilde{V}_h^{0,0}(\mathbf{x_0})$. From Corollary 6.6, the discrete minimal element set $\widetilde{V}_h^{0,0}(\mathbf{x_0})$ has been shown to be equal to the discrete minimal element set $V_h(\mathbf{x_0})$, which completes the proof. ◻

8. Extensions. The results obtained in this paper remain valid, with minimal changes, for nonautonomous problems, including terminal cost. At the expense of adding constraints on the terminal state, a terminal cost can be reformulated as an integral cost [11, pp. 25–26]. However, such constraints are not considered in this paper. Another alternative is to directly include the terminal cost in the formulation of the multiobjective problem. In this case, the terminal cost function is required to be Lipschitz for the error estimate (Proposition 5.2), and uniformly bounded for the proof of convergence (Proposition 5.7). Another important modification concerning the terminal data condition (6.4) and (7.4) is to include the terminal cost evaluated at the terminal state $\mathbf{x_N}$. For nonautonomous problems, the time variable appears explicitly at each step of the approximation procedure. In this case, the Lipschitz assumption in time is also required [12, 18].

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