Multi-modal Continuous Approximate Algebraic Input-output Synthesis of Planar Four-bar Function Generators

Zachary A. Copeland and M. John D. Hayes

Carleton University, 1125 Colonel By Drive, Ottawa K1S 5B6, Canada
zackcopeland@cunet.carleton.ca, johnhayes@cunet.carleton.ca

Abstract. This paper introduces a novel multi-modal function generator synthesis algorithm for planar four-bar mechanisms. Multi-modal in this sense means concurrently optimising multiple functions approximately generated between different joint parameters in a four-bar linkage over the desired continuous input-output ranges. Every planar four-bar function generator explicitly generates six functions uniquely determined by the relationships among the constant link parameters. We will instead investigate the simultaneous synthesis of two distinct functions between different joint variables. An example is described where two different functions in a planar RRRP linkage are generated over continuous ranges between the ground-fixed R-pair and the R-pair connected to the P-pair, as well as between the same ground-fixed R-pair and the P-pair. The multi-modal synthesis equation is the sum of the squared input-output equations integrated over the desired ranges. We compare the generated continuous approximate synthesis algorithm results and the multi-modal continuous synthesis results by comparing the areas between the synthesised planar algebraic curves in the parameter planes of the input and output joint variables to those of the desired input-output functions over their continuous ranges, thereby evaluating the structural error.

Keywords: Planar four-bar mechanisms · multi-modal function generation · algebraic input-output (IO) equations · continuous approximate synthesis.

1 Introduction

The study of planar four-bar linkages involves a large variety of problems: these range from guiding a point along a specific curve or path, known as coupler curve or path generation; guiding a rigid body through a sequence of positions and orientations, known as the Burmester problem [7]; guiding a rigid body along a time-dependent sequence of positions and orientations, usually called trajectory generation [1]; problems concerning the transmission of forces and torques [4]; or designing an optimally balanced linkage [5]. An additional important subset of this gamut is the function generation problem [3]. It consists of identifying a mechanism which is able to generate approximately, in some sense, a mathematical function between an input and output (IO) pair of joint variables for a given planar linkage kinematic architecture comprising RR-, RP-, PR-, or PP-dyads[1].

[1]R and P indicate revolute and prismatic joints connecting a pair of rigid links, also known as R- and P-pairs.

The IO function generation problem is the focus of this paper. All movable mechanical systems generate functions between the joint variables. Figure 1 illustrates one such planar four-bar 4R linkage. If links a_1 and a_3 are the input and output links, respectively, the IO function is specified as $\theta_4 = f(\theta_1)$. Once the four a_i link lengths are identified, the corresponding mechanism generates five additional functions, one for each of the six distinct angle pairings $\theta_j = f(\theta_i)$. All six of these functions are determined by the identified values for the link lengths that approximately generate the lone desired function.

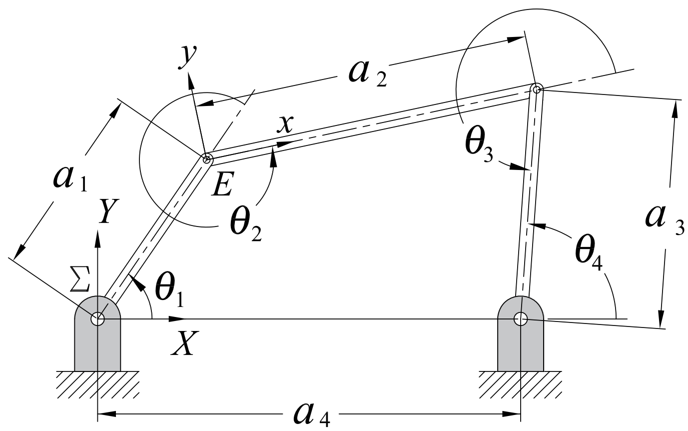


Fig. 1: A general planar 4R function generator.

Optimal function generation synthesis problems typically consist of minimising one of the *design* or *structural* errors [7]. The design error is the residual error of the identified linkage in satisfying the Freudenstein Equation [3], and can be solved in an exact sense using $n = 3$ IO pairs, or by using an overconstrained linear system of equations representing the finite number of $n > 3$ precision points, or poses, that the mechanism is to approximately generate. Exact synthesis results in a linkage that precisely generates the desired function, but only for the three precision IO pairs. Approximate synthesis leads to a linkage that approximates the desired function, in general, over the desired range so that the norm of the design error is minimised in some least-squares sense. The structural error is defined as the difference between the desired output angle, and the output angle that is generated by the linkage at each precision point. This problem is typically solved by minimising the norm of the array of the structural error evaluated at each precision point using some form of Gauss-Newton non-linear minimisation, usually requiring an iterative solution procedure that terminates when a desired minimum norm threshold is obtained.

It was observed in [9] that as the cardinality of the data set used to compute the design error minimising linkage becomes large, on the order of $n \geq 40$, the design error minimising linkage converges to the structural error minimising linkage as n increases further. Hence, one could avoid the need for the non-linear structural error computation provided a sufficient number of precision points was specified. The natural question is then "how large must n be?" The obvious response is to extend the cardinality of the data set used to compute the design error minimising linkage to infinity by way of integration. Unfortunately, while it was demonstrated in [6] that this extension is possible through the integration of the trigonometric Freudenstein equation, the generalisation of the process is computationally prohibitive and any advantage obtained through the elimination of the need for an explicit solution to the non-linear structural error problem is lost to the numerical complexity of the integration. A less cumbersome continuous approximation method was desired, and was realised with the algebraic IO equations [8, 11, 12]. These IO equations have been used for function generator synthesis problems in [11], and subsequently used to extend the observations made in [9] to create the *continuous approximate algebraic input-output synthesis* technique described in [2], which will be relied upon and further expanded in this paper.

2 Multi-Modal Continuous Approximate Synthesis

The concept of continuous approximate function generation synthesis from [2] will be extended in order to enable the simultaneous approximate generation of functions between multiple different pairs of IO parameters, which we call *multi-modal continuous approximate function generator synthesis*. The standard function generation problem concerns the θ_1-θ_4 IO equation; however, considering Figure 1, it is a simple extension to consider the θ_1-θ_3 pair of angles, or any other pair. The motivation for this extension results from subsequent analyses of the kinetic and dynamic properties of a planar four-bar linkage designed to generate a specific function after the kinematic synthesis has occurred. Most notably, this involves the determination of the transmission angle extreme values of the four-bar linkage, which is used as a metric to separate linkages which have practical use from those which do not.

2.1 Planar 4R Multi-modal Function Generation

While an example of the 4R linkage will not be considered, see [11, 12] for detailed examples, the established algebraic IO equations are presented here for completeness. First, the IO equation obtained by the tangent half-angle representation of corresponding joint angles, the v_1-v_4 IO equation is [12],

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{1}$$

where,

$$A = A_1A_2 = (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4), \tag{2}$$

$$B = B_1B_2 = (a_1 - a_2 + a_3 + a_4)(a_1 + a_2 + a_3 + a_4), \tag{3}$$

$$C = C_1C_2 = (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4), \tag{4}$$

$$D = D_1D_2 = (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4), \tag{5}$$

$$v_1 = \tan\frac{\theta_1}{2}, \tag{6}$$

$$v_4 = \tan\frac{\theta_4}{2}. \tag{7}$$

The coefficients A, B, C, and D are products of bilinear factors of the a_i directed link lengths representing the constants to be identified in the synthesis. Following modified derivation steps listed in [12], the remaining five v_i-v_j IO equations are, respectively [10],

$$A_1B_2v_1^2v_2^2 + A_2B_1v_1^2 + C_1D_2v_2^2 + 8a_2a_4v_1v_2 + C_2D_1 = 0, \tag{8}$$
$$A_1B_1v_1^2v_3^2 + A_2B_2v_1^2 + C_2D_2v_3^2 + C_1D_1 = 0, \tag{9}$$
$$A_1D_2v_2^2v_3^2 + B_2C_1v_2^2 + B_1C_2v_3^2 - 8a_1a_3v_2v_3 + A_2D_1 = 0, \tag{10}$$
$$A_1C_1v_2^2v_4^2 + B_2D_2v_2^2 + A_2C_2v_4^2 + B_1D_1 = 0, \tag{11}$$
$$A_1C_2v_3^2v_4^2 + B_1D_2v_3^2 + A_2C_1v_4^2 + 8a_2a_4v_3v_4 + B_2D_2 = 0. \tag{12}$$

2.2 Multi-Modal RRRP Function Generator Synthesis

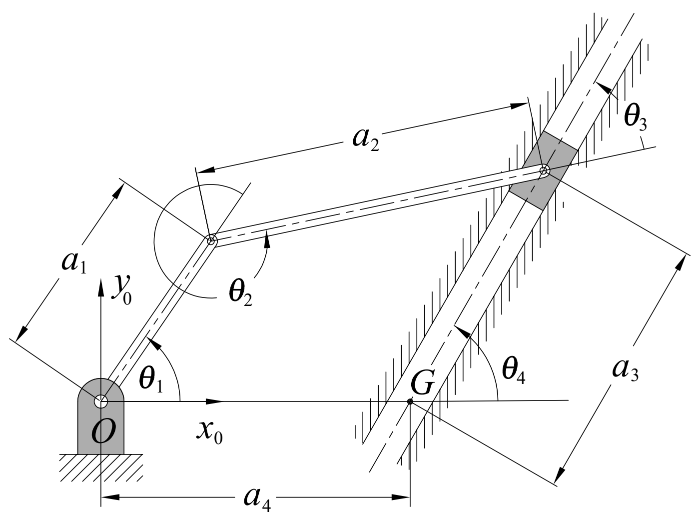


Fig. 2: The RRRP function generator architecture.

Here we shall consider a multi-modal RRRP function generation problem. Such a linkage is illustrated in Figure 2. The RRRP v_1-a_3 IO equation is [11],

$$Aa_3{}^2v_1{}^2 + Ca_3v_1{}^2 - 8a_1a_3v_1v_4 + Ba_3{}^2 + Ev_1{}^2 + Da_3 + F = 0, \tag{13}$$

where $v_1 = \tan(\theta_1/2)$ and, expressed as an array, the coefficients are

$$\begin{bmatrix} A \\ B \\ C \\ -8a_1v_4 \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} v_4^2+1 \\ v_4^2+1 \\ -2(v_4-1)(v_4+1)(a_1+a_4) \\ -8a_1v_4 \\ 2(v_4-1)(v_4+1)(a_1-a_4) \\ (v_4^2+1)(a_1+a_2+a_4)(a_1-a_2+a_4) \\ (v_4^2+1)(a_1+a_2-a_4)(a_1-a_2-a_4) \end{bmatrix}. \tag{14}$$

The constant angle of inclination parameter of the slider, $v_4 = \tan(\theta_4/2)$, and variable link length a_3, have changed roles as constant and variable here compared to the planar 4R. The desired v_1-a_3 IO function is,

$$a_3 = f(v_1) = \frac{1 - {v_1}^2}{{v_1}^2 + 1}. \tag{15}$$

The error minimising linkage parameters, identified in detail in [2], after the application of the continuous approximate IO synthesis algorithm over the range $-3 \le v_1 \le 3$ are

$$\begin{bmatrix} a_1 \\ a_2 \\ a_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0.9554 \\ 1.1894 \\ 1 \\ 1.17 \times 10^{-10} \end{bmatrix}, \tag{16}$$

where the a_i are in generic units of length and v_4 is the dimensionless tangent half-angle parameter.

While six different options for the IO relationship exist, it is not uncommon for a designer to be concerned about the force transmission from input to the output link. Therefore, to demonstrate the multi-modal algorithm the v_1-v_3 pair will also be selected. With reference to Figure 2, the input angle parameter v_1 generates a function between itself and the output tangent half-angle parameter, $v_3 = \tan(\theta_3/2)$: the angle the coupler makes with respect to the output slider. The corresponding IO equation is [11],

$$A_2\, {v_1}^2 {v_3}^2 + A_1\, {v_1}^2 - B_1\, {v_3}^2 - B_2 = 0, \tag{17}$$

where,

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_4 \\ a_1 - a_2 + a_4 \\ a_1 + a_2 - a_4 \\ a_1 - a_2 - a_4 \end{bmatrix}. \tag{18}$$

After squaring this equation according to [2], the corresponding array of joint angle parameters is defined as the array of variables that are scaled by the link length coefficients, giving

$$\mathbf{s}_{v_1 v_3} = \begin{bmatrix} {v_1}^4 {v_3}^4 \\ {v_1}^4 {v_3}^2 \\ {v_1}^4 \\ {v_1}^2 {v_3}^4 \\ {v_1}^2 {v_3}^2 \\ {v_1}^2 \\ {v_3}^4 \\ {v_3}^2 \\ 1 \end{bmatrix}, \tag{19}$$

while the corresponding array of link length coefficients in the same equation is,

$$\mathbf{p}_{v_1 v_3} = \begin{bmatrix} (a_1 - a_2 + a_4)^2 \\ 2\ (a_1 + a_2 + a_4)\,(a_1 - a_2 + a_4) \\ (a_1 + a_2 + a_4)^2 \\ -2\ (a_1 - a_2 + a_4)\,(a_1 + a_2 - a_4) \\ -2(a_1 + a_2 + a_4)(a_1 + a_2 - a_4) - 2(a_1 - a_2 + a_4)(a_1 - a_2 - a_4) \\ -2\ (a_1 + a_2 + a_4)\,(a_1 - a_2 - a_4) \\ (a_1 + a_2 - a_4)^2 \\ 2\ (a_1 + a_2 - a_4)\,(a_1 - a_2 - a_4) \\ (a_1 - a_2 - a_4)^2 \end{bmatrix}. \tag{20}$$

While it would be a simple task to proceed with the concurrent multi-modal continuous approximate synthesis problem from here using a general v_1-v_3 IO function, it remains to be seen what functional IO relationship is actually generated as a necessary result of the previously identified error minimising linkage parameters from the $a_3 = f(v_1)$ function used in this example. It is a simple matter to solve for the algebraic IO relationship which is generated by these linkage parameters, which will be done first to guide the remainder of this process. Once the linkage parameters from Equation (16) have been substituted into Equation (17), it is solved for v_3, yielding:

$$v_3 = \pm \frac{\sqrt{-\left(2251198413\, {v_1}^2 - 3364669033\right)\left(9242383263\, {v_1}^2 + 3626515817\right)}}{2251198413\, {v_1}^2 - 3364669033}. \tag{21}$$

Equation (21) is the $v_3 = g(v_1)$ function generated by this RRRP linkage with link constants listed in Equation (16). Deviation from this function will cause error in both the desired $a_3 = f(v_1)$ and $v_3 = g(v_1)$ functions, however, for a proof of concept we will choose some function which approximates Equation (21). A set of n points will be taken in order to develop a polynomial interpolant of the function. Such methods have been well known for several centuries, see [13] for example. We have chosen $n = 4$ points to be used to generate a different, though related function. The resulting Lagrange polynomial is,

$$v_3 = g(v_1) = -0.223383\, {v_1}^3 - 1.64840\, {v_1}^2 + 0.0300868\, v_1 - 1.03736 = 0. \tag{22}$$

Both polynomials are plotted in Figure 3, the Lagrange interpolant in Equation (22), and the function represented by Equation (21).

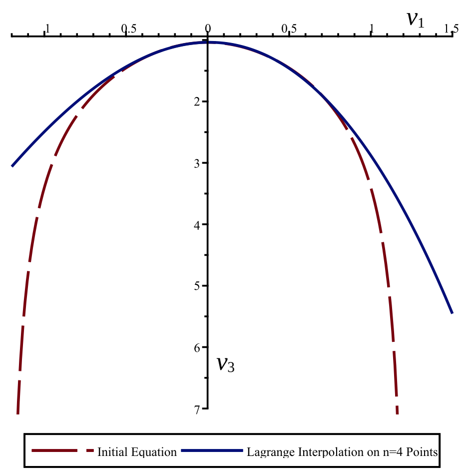


Fig. 3: Lagrange interpolant with $n = 4$ and the function represented by Equation (21).

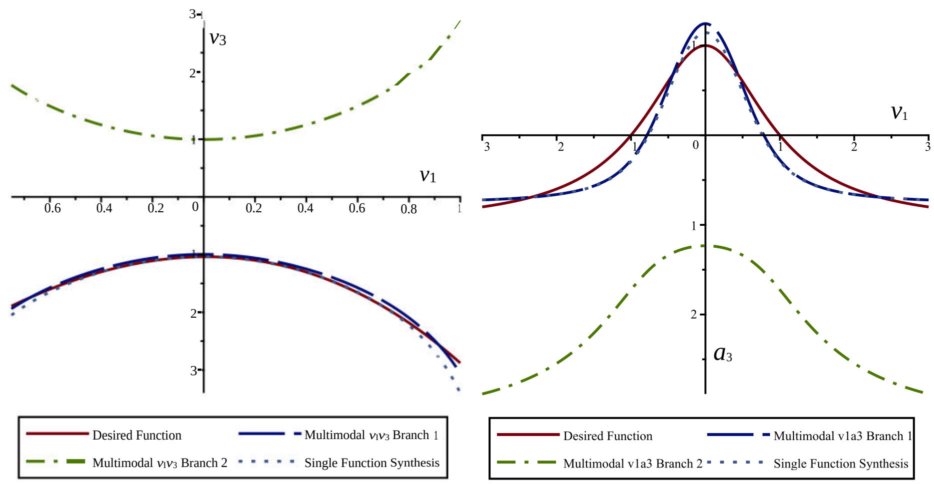
For the $v_3 = g(v_1)$ function the input range is $-0.75 \le v_1 \le 1.1$, and for the $a_3 = f(v_1)$ function the input range is $-3 \le v_1 \le 3$. Using these design requirements, the resulting multi-modal continuous synthesis equation is the sum of the definite integrals

$$\min_{(a_1,a_2,a_4,v_4)\in\mathbb{R}} \left(\mathbf{p}_{v_1 a_3} \int_{-3}^{3} \mathbf{s}_{v_1 a_3}(v_1, f(v_1)) + \mathbf{p}_{v_1 v_3} \int_{-0.75}^{1.1} \mathbf{s}_{v_1 v_3}(v_1, g(v_1)) \right). \tag{23}$$

Equation (23) converges to the following four linkage parameters, which concurrently minimise the residuals of both functions and, presumably, the structural error:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1.0057 \\ 1.2391 \\ 1 \\ 2.1604 \times 10^{-12} \end{bmatrix}. \tag{24}$$

Figure 4 compares the desired and multi-modal generated synthesis IO curves, corresponding to Equation (23), but also plots one mechanism assembly mode of the single function generation synthesis IO curves in each of Figures 4a and 4b. As expected, the multi-modal generated IO curves have different structural errors compared to the linkages synthesised to generate a single function. Moreover, the non-linear structural



(a) Synthesis results for the $v_3 = g(v_1)$ function with and without multi-modal synthesis.

(b) Synthesis results for the $a_3 = f(v_1)$ function with and without multi-modal synthesis.

Fig. 4: Planar RRRP desired and generated multi- and single-modal IO curves.

error evaluation has been simplified to the difference between the areas under the desired and generated IO curves in the relevant parameter planes.

While the multi-modal continuous approximate algebraic input-output synthesis procedure increased the error present for the $a_3 = f(v_1)$ function generation by approximately 31%, it also decreased the error associated with the $v_3 = g(v_1)$ function generator by approximately 81% comparing the areas between the associated curve intervals, thereby reducing the structural error. Furthermore, despite the increase in error for the v_1-a_3 function generator, it is clear from Figure 4b that the only characteristic of the generated function that changed appreciably is the overshoot of the desired maximum value corresponding to the peak.

3 Conclusion

The main goal of this paper was to describe a novel four-bar planar mechanism algorithm that implicitly makes the cardinality of the IO data set used to generate the over constrained set of synthesis equations infinite, and use it to identify link parameters to simultaneously satisfy two desired functions between different IO pairs. This is accomplished by integrating the square of the algebraic IO equation for the desired kinematic architecture over the specified range of input parameter, v_i or a_i, where the output parameter, v_j or a_j, depending on the kinematic architecture, is expressed in terms of the desired function, $v_j = f(v_i)$, et c., for each desired function. The synthesis equation, which we have termed multi-modal, is the sum of the squared IO equations integrated over the desired ranges. Because of this, we denote the entire procedure as multi-modal continuous approximate algebraic function generator synthesis. The algorithm was demonstrated with a synthesis example, in a proof-of-concept fashion, to simultaneously generate two functions in an RRRP planar linkage. Comparing the desired and generated continuous functions over their specified ranges we have observed that the structural error is simply the difference in the areas under the desired and generated IO curves in the parameter planes.

Certainly, any planar four-bar mechanism generates an output parameter that is a distinct function of the input, the actuated variable joint parameter. The linkage that generates this distinct function also determines five additional functions between the remaining pairs of variable joint parameters. The example in this paper has demonstrated that it is possible to approximately generate two distinct IO functions between different variable joint pairs that have not been already determined by the linkage geometry. This simple result illustrates the tremendous value represented by the algebraised IO equations as a design tool. Indeed, the algebraic IO equations described herein, together with the multi-modal continuous approximate synthesis algorithm, stands to enable designers of industrial automated production and assembly systems to approach optimisation in a new way: different linkages in the mechanical system capable of generating different desired functions so that each link in the chain can simultaneously perform different tasks. While the practicality of this is, of course, conjecture, it does suggest the continued generalisation and development of multi-modal continuous approximate synthesis is justified and worth the investigative effort.

Bibliography

[1] Angeles, J., Alivizatoss, A., Akhras, R.: An Unconstrained Nonlinear Least-squares Method of Optimization of RRRR Planar Path Generators. Mechanism and Machine Theory **23**(5), 343–353 (1988)

[2] Copeland, Z., Rotzoll, M., Hayes, M.J.D.: Concurrent Type and Dimensional Continuous Approximate Function Generator Synthesis for All Planar Four-bar Mechanisms. In: 11th CCToMM Symposium on Mechanisms, Machines, and Mechatronics, eds. S. Nokleby, and P. Cardou, Ontario Tech University, Oshawa, ON, Canada (2021)

[3] Freudenstein, F.: Design of Four-link Mechanisms. Ph.D. thesis, Columbia University, New York, N.Y., USA (1954)

[4] Gosselin, C.M., Angeles, J.: Optimization of Planar and Spherical Function Generators as Minimum-defect Linkages. Mechanism and Machine Theory **24**(4), 293–307 (1989)

[5] Gosselin, C.M., Moore, B., Schicho, J.: Dynamic Balancing of Planar Mechanisms Using Toric Geometry. Journal of Symbolic Computation **44**(9), 1346–1358 (2009)

[6] Guigue, A., Hayes, M.J.D.: Continuous Approximate Synthesis of Planar Function-generators Minimising the Design Error. Mechanism and Machine Theory **101**, 158–167 (2016)

[7] Hartenberg, R., Denavit, J.: Kinematic Synthesis of Linkages. McGraw-Hill, Book Co., New York, N.Y., U.S.A. (1964)

[8] Hayes, M.J.D., Husty, M.L., Pfurner, M.: Input-output Equation for Planar Four-bar Linkages. In: Advances in Robot Kinematics, 2018. pp. 12–19. Springer (2018)

[9] Hayes, M., Parsa, K., Angeles, J.: The Effect of Data-set Cardinality on the Design and Structural Errors of Four-bar Function-generators. In: Proceedings of the Tenth World Congress on the Theory of Machines and Mechanisms, Oulu, Finland. pp. 437–442 (1999)

[10] Rotzoll, M., Bucciol, Q., Hayes, M.J.D.: Algebraic Input-output Angle Equation Derivation Algorithm for the Six Distinct Angle Pairings in Arbitrary Planar 4R Linkages. In: 20th International Conference on Advanced Robotics, ICAR 2021, Ljubljana, Slovenia (2021)

[11] Rotzoll, M., Hayes, M.J.D., Husty, M.L.: An Algebraic Input-output Equation for Planar RRRP and PRRP linkages. Transactions of the Canadian Society for Mechanical Engineering **44**(4), 520–529 (2019)

[12] Rotzoll, M., Hayes, M.J.D., Husty, M.L., Pfurner, M.: A General Method for Determining Algebraic Input-output Equations for Planar and Spherical 4R inkages. In: International Symposium on Advances in Robot Kinematics. pp. 90–97 (2020)

[13] Waring, E.: Problems Concerning Interpolations. Philosophical Transactions of the Royal Society **69**, 59–67 (1779)