A General Method for Determining Algebraic Input-output Equations for Planar and Spherical 4R Linkages

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Abstract A new and completely general method for determining the algebraic input-output (*IO*) equations for planar and spherical 4R linkages is presented in this paper. First, the forward kinematics transformation matrix of an arbitrary planar or spherical open 4R kinematic chain is computed in terms of its Denavit-Hartenberg parameters, where the link twist and joint angles are converted to their tangent half-angle parameters. This transformation matrix is mapped to its corresponding eight Study coordinates. The serial kinematic chain is conceptually closed by equating the forward kinematics transformation to the identity matrix. Equating the two corresponding Study arrays yields four equations in terms of the four revolute joint angle parameters. Gröbner bases are then used to eliminate the two intermediate joint angle parameters leaving an algebraic polynomial in terms of the input and output joint angle parameters and the four twist angle or link length parameters. In the limit, as the sphere radius becomes infinite and the link twist angle parameters are expressed as ratios of arc length and sphere radius in the general spherical algebraic *IO* equation, the only terms that remain are those in the planar 4R *IO* equation.

Key words: Algebraic input-output equation, planar and spherical four-bar linkage, Study coordinates, kinematic mapping.

1 Introduction

Four-bar linkages, consisting of four rigid bodies connected by revolute (R) joints have fascinated mathematicians and engineers for centuries. One of the greatest successes was the establishment of an input-output (*IO*) equation by F. Freudenstein, which correlates the driver input angle ψ to the follower output angle φ according

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to a function $\varphi = f(\psi)$ [5]. The *IO* equation developed in [5] is trigonometric, whereas in [6] an algebraic version is derived by mapping the constraint equations of the driver and follower into Study's kinematic image space [1, 11]. Let a, b, c, d be the link lengths of the four-bar mechanism, and ψ and φ the respective input and output angles, then the movement of the mechanism is governed by the following *IO* equation

$$Au^2v^2 + Bu^2 + Cv^2 - 8abuv + D = 0 \quad (1)$$

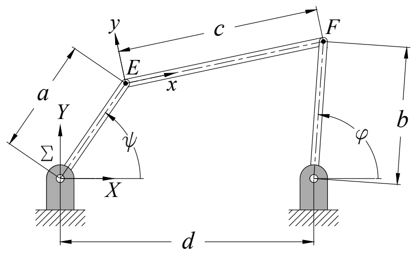

Fig. 1 Planar 4R function generator.

where

$$\begin{aligned} A &= (a-b-c+d)(a-b+c+d), \\ B &= (a+b-c+d)(a+b+c+d), \\ C &= (a+b-c-d)(a+b+c-d), \\ D &= (a-b+c-d)(a-b-c-d), \\ u &= \tan\frac{\psi}{2}, \\ v &= \tan\frac{\varphi}{2}. \end{aligned}$$

In addition, it was shown in [10] that Equation (1) is not only valid for planar four-bar linkages containing revolute joints, but also for planar four-bar linkages containing up to two prismatic joints. In this paper we will describe a method that can be applied to both planar and spherical four-bar linkages to derive the respective *IO* equations. Moreover, we will show that planar linkages can be interpreted as special cases of spherical linkages, thus, expanding the generality of the *IO* equation obtained in [6].

2 Study's kinematic mapping

Consider a coordinate system Σ_2 that moves with a rigid body relative to a stationary reference frame Σ_1. The Euclidean displacement group $\mathscr{D} \in SO(3)$ can be represented by

$$\mathbf{p}' = \mathbf{A}\mathbf{p} + \mathbf{t} \quad (2)$$

where $\mathbf{p}$ is a 3×1 position vector in Σ_2, $\mathbf{A}$ is a proper orthogonal 3×3 rotation matrix, $\mathbf{t}$ is a 3×1 position vector of the origin of Σ_2 expressed in Σ_1, and $\mathbf{p}'$ is the 3×1 position vector of $\mathbf{p}$ expressed in Σ_1 [7, 9].

Displacements of kinematic chains are often parametrised using the Denavit-Hartenberg (DH) convention [3]. The four associated DH parameters are the link lengths a_i, link twist angles τ_i, joint angles θ_i, and link offsets d_i. According to this convention the coordinate transformation from the coordinate system for joint i relative to the coordinate system of the previous joint $i-1$ is given by

$$ {}^{i-1}_{i}\mathbf{T} = \left[\begin{array}{ccc|c} \cos\theta_i & -\sin\theta_i\cos\tau_i & \sin\theta_i\sin\tau_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\tau_i & -\cos\theta_i\sin\tau_i & a_i\sin\theta_i \\ 0 & \sin\tau_i & \cos\tau_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{t} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]. \tag{3} $$

With Study's kinematic mapping distinct Euclidean displacements can be represented as distinct points $x = [x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3]^{\mathsf{T}} \in P^7$ where the first four entries are obtained using the matrix elements a_{ij} of $\mathbf{A}$

$$ x_0 : x_1 : x_2 : x_3 = \begin{cases} 1 + a_{11} + a_{22} + a_{33} : a_{32} - a_{23} : a_{13} - a_{31} : a_{21} - a_{12}, \\ a_{32} - a_{23} : 1 + a_{11} - a_{22} - a_{33} : a_{12} + a_{21} : a_{31} + a_{13}, \\ a_{13} - a_{31} : a_{12} + a_{21} : 1 - a_{11} + a_{22} - a_{33} : a_{23} + a_{32}, \\ a_{21} - a_{12} : a_{31} + a_{13} : a_{23} + a_{32} : 1 - a_{11} - a_{22} + a_{33}. \end{cases} \tag{4} $$

Four different combinations of the rotation matrix elements are needed since certain displacements make one or more of the relations lead to $(0:0:0:0)$, the exceptional generator in P^7. Once the x_i have been determined, the remaining four entries are computed as linear combinations of the vector elements of the translation $\mathbf{t}$ and the x_i determined above, giving

$$ \begin{aligned} y_0 &= \tfrac{1}{2}(t_3x_3 + t_2x_2 + t_1x_1), & y_1 &= \tfrac{1}{2}(t_3x_2 - t_2x_3 - t_1x_0), \\ y_2 &= \tfrac{1}{2}(-t_3x_1 + t_1x_3 - t_2x_0), & y_3 &= \tfrac{1}{2}(-t_3x_0 + t_2x_1 - t_1x_2). \end{aligned} \tag{5} $$

These eight Study parameters must fulfill the Study condition in order to represent a Euclidean displacement, meaning the eight ratios represent a point on Study's seven dimensional quadric $\mathscr{S}$

$$ x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0 \tag{6} $$

excluding the exceptional generator $\mathscr{E}$

$$ (x_0 : x_1 : x_2 : x_3) = (0:0:0:0). \tag{7} $$

3 Planar four-bar linkage

To derive the algebraic *IO* equation for planar four-bar mechanisms using the DH convention [3] and Study's kinematic mapping [11], we first consider the four-bar mechanism to be an open kinematic chain connected by four rotational joints as shown in Fig. 2. The respective DH parameters are listed in Table 1. Note that for planar mechanisms all link twists and all link offsets are identically zero. This simplifies the overall transformation matrix ${}^0_4\mathbf{T}$, which maps the coordinates of points described in the end-link coordinate frame to those of the base frame:

$$ {}^0_4\mathbf{T} = {}^0_1\mathbf{T}\,{}^1_2\mathbf{T}\,{}^2_3\mathbf{T}\,{}^3_4\mathbf{T}, \tag{8} $$

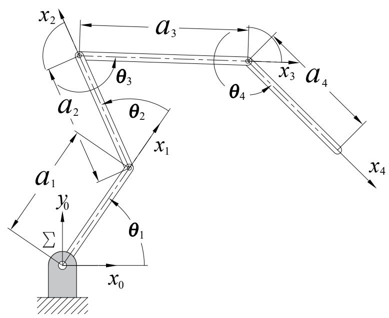

Fig. 2 Open 4R chain.

Table 1 DH parameters for open 4R chain.

joint axis i	link length a_i	link angle θ_i	link offset d_i	link twist τ_i
1	a_1	θ_1	0	0
2	a_2	θ_2	0	0
3	a_3	θ_3	0	0
4	a_4	θ_4	0	0

where the transformation matrices ${}^0_1\mathbf{T}$, ${}^1_2\mathbf{T}$, ${}^2_3\mathbf{T}$ and ${}^3_4\mathbf{T}$ are evaluated according to Equation (3). The computed transformation matrix can be mapped onto Study's quadric using Equations (4, 5) resulting in a Study array with zero entries for x_1, x_2, y_0 and y_3. After normalizing, the remaining four Study parameters become

$$
\begin{aligned}
x_0 &= (2v_2v_3v_4 - 2v_2 - 2v_3 - 2v_4)v_1 + (-2v_3 - 2v_4)v_2 - 2v_3v_4 + 2, && (9)\\
x_3 &= ((-2v_3 - 2v_4)v_2 - 2v_3v_4 + 2)v_1 - 2v_2v_3v_4 + 2v_2 + 2v_3 + 2v_4, && (10)\\
y_1 &= ((v_4(a_1 - a_2 + a_3 - a_4)v_3 - a_1 + a_2 + a_3 + a_4)v_2 + (-a_1 - a_2 + a_3 + a_4)v_3 \\
&\quad - v_4(a_1 + a_2 + a_3 - a_4))v_1 + ((a_1 - a_2 + a_3 + a_4)v_3 + v_4(a_1 - a_2 - a_3 + a_4))v_2 \\
&\quad + v_4(a_1 + a_2 - a_3 + a_4)v_3 - a_1 - a_2 - a_3 - a_4, && (11)\\
y_2 &= (((a_1 - a_2 + a_3 + a_4)v_3 + v_4(a_1 - a_2 - a_3 + a_4))v_2 + v_4(a_1 + a_2 - a_3 + a_4)v_3 \\
&\quad - a_1 - a_2 - a_3 - a_4)v_1 + (-v_4(a_1 - a_2 + a_3 - a_4)v_3 + a_1 - a_2 - a_3 - a_4)v_2 \\
&\quad + (a_1 + a_2 - a_3 - a_4)v_3 + v_4(a_1 + a_2 + a_3 - a_4), && (12)
\end{aligned}
$$

where $v_i = \tan(\theta_i/2)$.

To close the four-bar mechanism, ${}^0_4\mathbf{T}$ is equated to the identity matrix which we also map using Equations (4, 5) onto Study's quadric, resulting in the following Study parameters after normalising:

$$I \mapsto [1:0:0:0:0:0:0:0]^{\mathsf{T}} \in P^7 \qquad (13)$$

Equating the Study array of the overall transformation ${}^0_4\mathbf{T}$ to the Study array of the identity matrix, i.e. setting Equations (10-12) equal to zero, forces the coordinate frame of the end-effector to align with that of the base; but to satisfy the DH

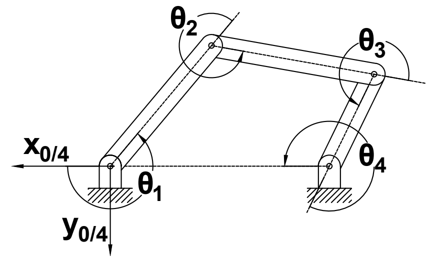

Fig. 3 Closed 4R kinematic chain.

convention they are both rotated by π. Thus, the joint angles, θ_i, are measured as illustrated in Fig. 3. We select the three Equations that are equal to zero, (10-12), and manipulate them with Gröbner bases to eliminate the intermediate joint angle parameters, v_2 and v_3. After collecting the input and output angle parameters v_1 and v_4, the following algebraic *IO* equation emerges

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{14}$$

where

$$\begin{aligned}
A &= (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4) &= A_1A_2,\\
B &= (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4) &= B_1B_2,\\
C &= (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4) &= C_1C_2,\\
D &= (a_1 + a_2 + a_3 + a_4)(a_1 - a_2 + a_3 + a_4) &= D_1D_2.
\end{aligned}$$

It can be shown that Equation (14) is identical to Equation (1) if the phase shift of the input and output angle as well as the different notation are considered.

4 Spherical four-bar linkage

It will now be demonstrated that the same procedure can be applied to determine the *IO* equation for spherical linkages. The DH parameters for a spherical open 4R kinematic chain are listed in Table 2. Note that in the spherical case all link lengths, a_i, and offsets, d_i, are zero with strict adherence to the DH conventions for assigning parameters [3]. After evaluating the overall transformation matrix in terms of DH parameters by applying Equation (3), the result can be mapped with Equations (4, 5) onto Study's quadric. Then setting $v_i = \tan(\theta_i/2)$ and $\alpha_i = \tan(\tau_i/2)$ into the result

Table 2 Open spherical 4R kinematic chain DH parameters.

joint axis i	link length a_i	link angle θ_i	link offset d_i	link twist τ_i
1	0	θ_1	0	τ_1
2	0	θ_2	0	τ_2
3	0	θ_3	0	τ_3
4	0	θ_4	0	τ_4

gives a Study array with non-zero entries for x_0, x_1, x_2 and x_3, while the y_i are all identically zero, as expected:

$$\begin{aligned}
x_0 = {} & ((2\alpha_4((v_2v_3v_4+v_2-v_3+v_4)v_1+(v_3-v_4)v_2+v_3v_4+1)\alpha_3 \\
& +(2v_2v_3v_4-2v_2+2v_3+2v_4)v_1+(-2v_3-2v_4)v_2+2v_3v_4-2)\alpha_2 \\
& +((-2v_2v_3v_4-2v_2-2v_3+2v_4)v_1+(2v_3-2v_4)v_2-2v_3v_4-2)\alpha_3 \\
& +(2((v_2v_3v_4-v_2-v_3-v_4)v_1+(v_3+v_4)v_2+v_3v_4-1))\alpha_4)\alpha_1 \\
& +(((2v_2v_3v_4+2v_2-2v_3+2v_4)v_1+(-2v_3+2v_4)v_2-2v_3v_4-2)\alpha_3 \\
& -(2((v_2v_3v_4-v_2+v_3+v_4)v_1+(v_3+v_4)v_2-v_3v_4+1))\alpha_4)\alpha_2 \\
& +(2((v_2v_3v_4+v_2+v_3-v_4)v_1+(v_3-v_4)v_2-v_3v_4-1))\alpha_4\alpha_3 \\
& +(2v_2v_3v_4-2v_2-2v_3-2v_4)v_1+(-2v_3-2v_4)v_2-2v_3v_4+2; \qquad (15)
\end{aligned}$$

$$\begin{aligned}
x_1 = {} & ((((-2v_2v_3v_4-2v_2+2v_3-2v_4)v_1+(-2v_3+2v_4)v_2-2v_3v_4-2)\alpha_3 \\
& +(2((v_2v_3v_4-v_2+v_3+v_4)v_1+(-v_3-v_4)v_2+v_3v_4-1))\alpha_4)\alpha_2 \\
& -2\alpha_4((v_2v_3v_4+v_2+v_3-v_4)v_1+(-v_3+v_4)v_2+v_3v_4+1)\alpha_3 \\
& +(-2v_2v_3v_4+2v_2+2v_3+2v_4)v_1+(-2v_3-2v_4)v_2-2v_3v_4+2)\alpha_1 \\
& +((2((v_2v_3v_4+v_2-v_3+v_4)v_1+(-v_3+v_4)v_2-v_3v_4-1))\alpha_4\alpha_3 \\
& +(2v_2v_3v_4-2v_2+2v_3+2v_4)v_1+(2v_3+2v_4)v_2-2v_3v_4+2)\alpha_2 \\
& +((-2v_2v_3v_4-2v_2-2v_3+2v_4)v_1+(-2v_3+2v_4)v_2+2v_3v_4+2)\alpha_3 \\
& +(2((v_2v_3v_4-v_2-v_3-v_4)v_1+(-v_3-v_4)v_2-v_3v_4+1))\alpha_4; \qquad (16)
\end{aligned}$$

$$\begin{aligned}
x_2 = {} & (((((-2v_3+2v_4)v_2-2v_3v_4-2)v_1+2v_2v_3v_4+2v_2-2v_3+2v_4)\alpha_3 \\
& -(2(((v_3+v_4)v_2-v_3v_4+1)v_1+v_2v_3v_4-v_2+v_3+v_4))\alpha_4)\alpha_2 \\
& +(2(((v_3-v_4)v_2-v_3v_4-1)v_1+v_2v_3v_4+v_2+v_3-v_4))\alpha_4\alpha_3 \\
& +((-2v_3-2v_4)v_2-2v_3v_4+2)v_1+2v_2v_3v_4-2v_2-2v_3-2v_4)\alpha_1 \\
& +(-(2(((v_3-v_4)v_2+v_3v_4+1)v_1+v_2v_3v_4+v_2-v_3+v_4))\alpha_4\alpha_3 \\
& +((2v_3+2v_4)v_2-2v_3v_4+2)v_1-2v_2v_3v_4+2v_2-2v_3-2v_4)\alpha_2 \\
& +(((-2v_3+2v_4)v_2+2v_3v_4+2)v_1+2v_2v_3v_4+2v_2+2v_3-2v_4)\alpha_3 \\
& -(2(((v_3+v_4)v_2+v_3v_4-1)v_1+v_2v_3v_4-v_2-v_3-v_4))\alpha_4; \qquad (17)
\end{aligned}$$

$$\begin{aligned}
x_3 = {} & (((2(((v_3-v_4)v_2+v_3v_4+1)v_1-v_2v_3v_4-v_2+v_3-v_4))\alpha_4\alpha_3 \\
& +((-2v_3-2v_4)v_2+2v_3v_4-2)v_1-2v_2v_3v_4+2v_2-2v_3-2v_4)\alpha_2 \\
& +(((2v_3-2v_4)v_2-2v_3v_4-2)v_1+2v_2v_3v_4+2v_2+2v_3-2v_4)\alpha_3 \\
& +2\alpha_4(((v_3+v_4)v_2+v_3v_4-1)v_1-v_2v_3v_4+v_2+v_3+v_4))\alpha_1 \\
& +((((-2v_3+2v_4)v_2-2v_3v_4-2)v_1-2v_2v_3v_4-2v_2+2v_3-2v_4)\alpha_3 \\
& -(2(((v_3+v_4)v_2-v_3v_4+1)v_1-v_2v_3v_4+v_2-v_3-v_4))\alpha_4)\alpha_2 \\
& +2\alpha_4(((v_3-v_4)v_2-v_3v_4-1)v_1-v_2v_3v_4-v_2-v_3+v_4)\alpha_3 \\
& +((-2v_3-2v_4)v_2-2v_3v_4+2)v_1-2v_2v_3v_4+2v_2+2v_3+2v_4. \qquad (18)
\end{aligned}$$

Again, the open kinematic chain is closed by equating the Study array to the corresponding identity array in Study coordinates, i.e. setting Equations (16-18) equal to zero. Subsequently, we use Gröbner bases to eliminate the intermediate angle parameters v_2 and v_3 from Equations (16-18), and obtain the desired *IO* equation

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 + 8\alpha_1\alpha_3\left(\alpha_4^2+1\right)\left(\alpha_2^2+1\right)v_1v_4 + D = 0, \tag{19}$$

where

$$\begin{aligned}
A = {} & (\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)\\
& (\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 - \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4),\\
B = {} & (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)\\
& (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 - \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),\\
C = {} & (\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 - \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4)\\
& (\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4),\\
D = {} & (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)\\
& (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1 - \alpha_2 + \alpha_3 + \alpha_4).
\end{aligned}$$

It can be shown that Equation (19) is identical to the corresponding trigonometric *IO* equation for spherical four-bar linkages found in [9].

5 Planar 4R linkages as a special case of the spherical 4R linkage

The two algebraic *IO* equations for planar and spherical 4R linkages already suggest some similarities. As demonstrated in [8], the motion of the planar 4R linkage represents a special case of the spherical 4R linkage. To show that the same relationship is true for the *IO* equations, we consider the directions of the joint axes. While the joint axes of the spherical 4R linkage intersect in the centre of the sphere, the joint axes of the planar 4R linkage are all parallel. In Euclidean space E^3 parallel lines never intersect, however, they do meet in a point at infinity in any projective extension of E^3 [2, 4]. This suggests that if the radius of a spherical linkage approaches infinity, the linkage becomes a planar mechanism in the limit [8]. As the link twist parameters α_i of the spherical *IO* equation are proportional to the ratios of the arc lengths to the sphere radius [12], we can make the following substitution in Equation (19)

$$\alpha_i \propto \frac{a_i}{r}. \tag{20}$$

In the resulting equation the first two cubic factors simplify to

$$\begin{aligned}
\lim_{r\to\infty} -\frac{1}{r}&\left(\frac{a_1a_2a_3}{r^2} - \frac{a_1a_2a_4}{r^2} + \frac{a_1a_3a_4}{r^2} - \frac{a_2a_3a_4}{r^2} + a_1 - a_2 + a_3 - a_4\right)\\
&\left(-\frac{a_1a_2a_3}{r^2} + \frac{a_1a_2a_4}{r^2} + \frac{a_1a_3a_4}{r^2} + \frac{a_2a_3a_4}{r^2} + a_1 + a_2 + a_3 - a_4\right).
\end{aligned} \tag{21}$$

In the limit the only terms remaining inside the parentheses in Equation (21) are

$$(a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4) = A_1A_2. \tag{22}$$

Proceeding with the other cubic factors in the same manner the algebraic *IO* equation for a spherical 4R mechanism leads directly to that of a planar 4R, Equation (14), in the limit. As mentioned, this aligns with the results from [8], and further confirms the validity of the derived *IO* equations as well as the observation in [9] that there exists a connection between the planar and the spherical 4R *IO* equations via the RSSR linkage.

6 Conclusions

We have successfully demonstrated a general method to derive the algebraic *IO* equations for spherical and planar 4R linkages. It requires defining the DH parameters for an open 4R kinematic chain, mapping its coordinate transformation matrix onto Study's quadric, conceptually closing the 4R chain by equating the corresponding Study coordinates to the identity array and eliminating the intermediate joint angles using Gröbner bases. Moreover, we have shown that the planar 4R *IO* equation represents a special case of the spherical 4R by evaluating the limit at infinity of the equation.

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