Angular Velocity and Acceleration Extrema: Implications for Force Analysis in Planar 4R Mechanisms

Matthew John D. Hayes, Twe Tar O. Hninn, and Rishad A. Irani[1]

Abstract The algebraic input-output (IO) equation expresses one joint angle tangent half-angle parameter in terms of another as an implicit function scaled by the link lengths. There are six distinct IO equations for any given four-bar linkage. Using one of them, we propose algorithms leading to the mechanism configurations corresponding to extreme angular velocities and accelerations which, in turn, directly implies the values for the magnitudes of the extreme shaking forces and moments in the non-moving ground-fixed link. A detailed example is presented where the mechanism configurations for extreme shaking force and moment magnitudes are compared to the configurations corresponding to extreme values of angular velocity and acceleration.

Key words: Planar 4R mechanism, extreme angular velocity and acceleration, extreme shaking forces and moments.

1 Introduction

An algorithm that generalises the procedure for determining the algebraic IO equations for any four-bar linkage kinematic architecture, planar, spherical, and spatial, using only algebraic means can be found in the Ph.D. thesis of Mirja Rotzoll [7, 10]. The linkage is initially considered as an open kinematic chain, and Fig. 1a describes its kinematic geometry using the original Denavit-Hartenberg convention [4, 5]. The resulting forward kinematics transformation matrix for the open chain is then mapped to Study's kinematic mapping image space as eight soma coordinates as defined in [8]. This mapping allows one to characterise the displacement from the base to the end-effector frame as a set of linearly independent algebraic varieties in the

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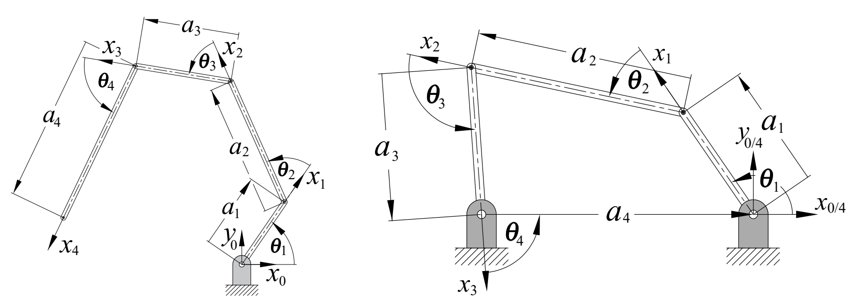
seven-dimensional projective kinematic mapping image space. To obtain a simple closed kinematic chain that represents the mechanism, the open chain is conceptually closed by equating the obtained algebraic parametrisation to its identity, leading to a set of equations that completely describe the linkage kinematics. We have chosen to close the kinematic chain such that there is a counter-clockwise circulation of joints, see Fig. 1b for example. With the help of elimination theory, the given system of polynomial equations is solved such that an equation is obtained that depends on only two joint variables. These equations represent the desired algebraic IO equations for the respective linkage. In total, there are six algebraic IO equations for each four-bar linkage kinematic architecture.



(a) Generic open 4R kinematic chain. (b) Generic closed 4R kinematic chain.

Fig. 1: Serial and parallel planar 4R linkages.

In this paper, attention is focused on planar 4R mechanisms, where 4R indicates four sequential revolute joints connecting four rigid links, a_1, a_2, and a_3 that are two-force members, and the non-moving rigidly fixed base link a_4. Let the input angle parameter be v_1 and the output angle parameter be v_4, where $v_i = \tan\theta_i/2$. In [10] two elimination steps were applied to the Gröbner bases of the ideal generated by the Study soma coordinates to eliminate the angle parameters v_2 and v_3 from the equations yielding the algebraic IO equation relating the v_1 and v_4 angle parameters, which we call the v_1-v_4 IO equation:

$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{1}$$

where

$$\begin{aligned}
A = A_1A_2 &= (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4),\\
B = B_1B_2 &= (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4),\\
C = C_1C_2 &= (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4),\\
D = D_1D_2 &= (a_1 + a_2 + a_3 + a_4)(a_1 - a_2 + a_3 + a_4),\\
v_1 = \tan\frac{\theta_1}{2},&\ v_4 = \tan\frac{\theta_4}{2}.
\end{aligned}$$

The coefficients A, B, C, and D all factor into a pair of bi-linear factors each.

One of the advantages of this IO equation formulation is that the five remaining algebraic IO equations relating the five remaining distinct angle parameter pairings contain coefficients comprised of pairs of the same bi-linear factors, but in different distinct permutations:

$$A_1B_2v_1^2v_2^2+A_2B_1v_1^2+C_1D_2v_2^2-8a_2a_4v_1v_2+C_2D_1=0; \tag{2}$$

$$A_1B_1v_1^2v_3^2+A_2B_2v_1^2+C_2D_2v_3^2+C_1D_1=0; \tag{3}$$

$$A_1D_2v_2^2v_3^2+B_2C_1v_2^2+B_1C_2v_3^2-8a_1a_3v_2v_3+A_2D_1=0; \tag{4}$$

$$A_1C_1v_2^2v_4^2+B_2D_2v_2^2+A_2C_2v_4^2+B_1D_1=0; \tag{5}$$

$$A_1C_2v_3^2v_4^2+B_1D_2v_3^2+A_2C_1v_4^2+8a_2a_4v_3v_4+B_2D_1=0. \tag{6}$$

It is a straightforward exercise to differentiate the six algebraic IO equations revealing the angular velocity and acceleration level kinematics. Again, because these are algebraic polynomials, it is equally straight forward to determine the critical input angles leading to the mechanism configurations where extreme output angular velocities and accelerations occur given a constant input angular velocity. We propose that the extreme magnitudes of shaking force and moment in a planar 4R mechanism occur in configurations of extreme output angular velocity and acceleration for a given constant input angular velocity, where the input is θ_1 and the output is θ_4, see Fig. 1b. Papers examining extreme shaking forces and moments and papers examining extreme angular velocities and accelerations are readily available in the literature, see [1, 3, 6] for example. But work examining the configurations for extreme output angular velocity and acceleration, and the implications for shaking forces and moments appear to be absent from the archival literature; hence, the present work.

2 Differential Kinematics

The algebraic angular velocity parameter IO equations are obtained by differentiating Equations (1-6) with respect to time. Since we only require the time derivatives of the v_1-v_4 IO equation for the analysis in this paper we will not discuss the remaining five, still the interested reader can find all six angular velocity and acceleration IO equations in [6]. However, the details of the first time derivative need a few words of discussion. Because the angle parameter is $v=\tan(\theta/2)$, the time derivative is configuration dependent in the following way

$$\dot{v} \;=\; \frac{d}{dt}\tan(\theta/2) \;=\; \frac{\dot{\theta}}{2}\sec^2(\theta/2) \;=\; \frac{\dot{\theta}}{2}\left(\frac{\cos^2(\theta/2)+\sin^2(\theta/2)}{\cos^2(\theta/2)}\right) \;=\; \frac{\dot{\theta}}{2}\left(1+v^2\right).$$

Therefore, the $\dot{v}_1$-$\dot{v}_4$ angular velocity parameter IO equation is

$$\left(\left(Av_4^2+B\right)v_1-4a_1a_3v_4\right)\dot{v}_1+\left(\left(Av_1^2+C\right)v_4-4a_1a_3v_1\right)\dot{v}_4=0, \quad (7)$$

where the coefficients A, B, and C are defined in Equation (1).

The six angular acceleration parameter IO equations are obtained as the time derivatives of the angular velocity parameter IO equations. The time derivative of $\dot{v}$ is a somewhat more complicated compound function requiring a combination of the chain and power rules from elementary differential calculus [2] to determine

$$\ddot{v}=\frac{1}{2}\left(\ddot{\theta}+\dot{\theta}^2\right)\left(1+v^2\right),$$

which reveals that the angular acceleration parameter $\ddot{v}$ depends not only on angular acceleration, but on angular velocity and instantaneous position of the linkage as well. The six angular acceleration parameter equations are to be found in [6], but we shall only list the $\ddot{v}_1$-$\ddot{v}_4$ angular acceleration parameter IO equation:

$$\begin{aligned}\left((Av_4^2+B)v_1-4a_1a_3v_4\right)\ddot{v}_1+\left((Av_1^2+C)v_4-4a_1a_3v_1\right)\ddot{v}_4+\\(Av_4^2+B)\dot{v}_1^2+(Av_1^2+C)\dot{v}_4^2+4(Av_1v_4-2a_1a_3)\dot{v}_1\dot{v}_4=0.\end{aligned} \quad (8)$$

3 Extreme Angular Velocity and Accelerations

Using Equations (7) and (8) it is possible to determine the mechanism configurations where the extreme output angular velocities and accelerations occur [6]. The extreme angular velocities, along with the configurations in which they occur in both assembly modes, can be obtained computationally with the following algorithm, where θ_i and θ_j are the input and output, respectively.

Extreme planar 4R angular velocity algorithm.

If values for a_1, a_2, a_3, and a_4 are given and the input angular velocity is a constant specified value, we wish to determine the critical values $\theta_{i_{\text{crit}}}$ that result in $\dot{\theta}_{j_{\text{min/max}}}$, so θ_j must be eliminated from both the position and angular velocity IO equations.

1. Convert v_i and v_j in the IO equation to angles as $v=\tan(\theta/2)$ and solve for θ_j. There will be two solutions, one for each assembly mode.
2. Substitute the expression for θ_j from Step 1 into the $\dot{\theta}_i$-$\dot{\theta}_j$ equation and solve for $\dot{\theta}_j$, which gives $\dot{\theta}_j=f(\theta_i)$ since $\dot{\theta}_i$ is a specified constant.
3. Solve $\dfrac{d\dot{\theta}_j}{d\theta_i}=0$ for $\theta_{i_{\text{crit}}}$ and determine the values of $\dot{\theta}_{j_{\text{min/max}}}$ corresponding to each distinct value of $\theta_{i_{\text{crit}}}$.

Similarly, the extreme angular accelerations, along with the configurations in which they occur in both assembly modes, can be obtained computationally with the following algorithm.

Extreme planar 4R angular acceleration algorithm.

If values for a_1, a_2, a_3, and a_4 are given and the input angular velocity is a constant specified value, we wish to determine the critical values $\theta_{i_{\text{crit}}}$ that result in $\ddot{\theta}_{j_{\text{min/max}}}$, so both θ_j and $\dot{\theta}_j$ must be eliminated from the position, angular velocity, and acceleration IO equations.

1. Convert v_i and v_j in the IO equation to angles as $v = \tan(\theta/2)$ and solve for θ_j.
2. Substitute the expression for θ_j from Step 1 into the $\dot{\theta}_i$-$\dot{\theta}_j$ equation and solve for $\dot{\theta}_j$, which gives $\dot{\theta}_j = f(\theta_i)$, since $\dot{\theta}_i$ is a specified constant.
3. Substitute the expressions for θ_j and $\dot{\theta}_j$ into the $\ddot{\theta}_i$-$\ddot{\theta}_j$ equation.
4. Solve the resulting equation for $\ddot{\theta}_j$, which gives $\ddot{\theta}_j = f(\theta_i)$, since $\ddot{\theta}_i = 0$.
5. Solve $\dfrac{d\ddot{\theta}_j}{d\theta_i} = 0$ for $\theta_{i_{\text{crit}}}$ and determine the values of $\ddot{\theta}_{j_{\text{min/max}}}$ corresponding to each distinct value of $\theta_{i_{\text{crit}}}$.

4 Implications for Extreme Shaking Forces and Moments

To demonstrate that the configurations where the extreme angular velocity and acceleration outputs, $\dot{\theta}_{4_{\text{max}}}$ and $\ddot{\theta}_{4_{\text{max}}}$, given a constant angular velocity input, $\dot{\theta}_1$, imply the configurations for the extreme shaking forces and moments, we will consider the following example of a planar 4R mechanism. The relevant mechanism properties are listed in Tab. 1 where the density and radius of each link is 7750 kg/m^3 and 0.01 m respectively.

Table 1: Mechanism properties.

Link	Length	Volume	Mass
a_1	0.4 m	$V_1 = \pi/25000$ m^3	$m_1 = 31\pi/100$ kg
a_2	1.2 m	$V_2 = 3\pi/25000$ m^3	$m_2 = 93\pi/100$ kg
a_3	0.8 m	$V_3 = \pi/12500$ m^3	$m_3 = 31\pi/50$ kg
a_4	1.0 m	Not relevant	Not relevant

To test our proposal we need the extreme angular velocity and acceleration, $\dot{\theta}_{4_{\text{max}}}$ and $\ddot{\theta}_{4_{\text{max}}}$, as functions of the input angle θ_1. Using Equations (7) and (8) and the algorithms for determining the configuration dependent extreme values for $\dot{\theta}_{4_{\text{max}}}$ and $\ddot{\theta}_{4_{\text{max}}}$, the resulting IO functions for assembly mode 2 are plotted in Fig. 3 and the extreme values for both assembly modes are listed in Tab. 2.

The force analysis follows the matrix method for identifying joint reaction forces, as well as shaking forces and moments acting on the ground-fixed, non-moving frame [9], link a_4. For our example, given in Fig. 2, the centres of gravity (CG) are located at the geometric centres of each link for simplicity, but they may be located

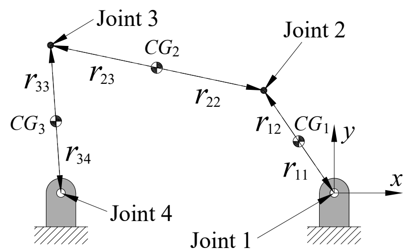


Fig. 2: Locating the centres of gravity (CG) for the force analysis.

Table 2: Mechanism angular velocity and acceleration extrema for $\dot{\theta}_1 = 25$ rad/s.

Assembly mode	$\theta_{1_{\text{crit}}}$ rad (deg)	$\dot{\theta}_{4_{\text{min/max}}}$ rad/s	$\theta_{1_{\text{crit}}}$ rad (deg)	$\ddot{\theta}_{4_{\text{min/max}}}$ rad/s^2
1	0.36933536508 (21.16135764°)	-23.95938176	0.8382412814 (48.02768764°)	462.6986574
	-1.667165012 (-95.52151894°)	13.51343678	-0.1482557001 (-8.494425904°)	-1062.775829
2	1.667165012 (95.52151894°)	13.51343678	0.1482557001 (8.494425904°)	1062.775829
	5.913849943 (338.8386424°)	-23.95938176	5.444944024 (311.9723124°)	-462.6986574

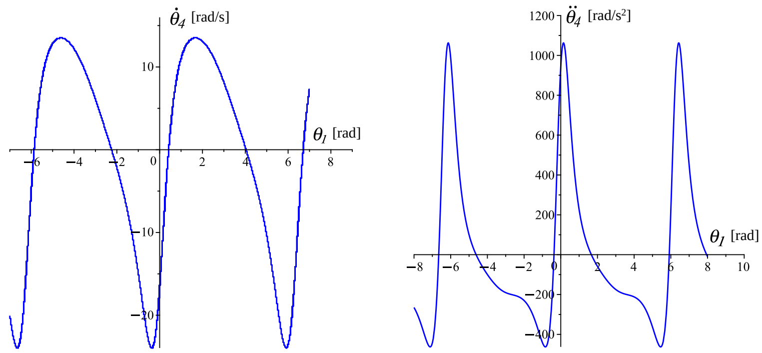
in any location relative to the two joints of each link. The vectors $\mathbf{r}_{ij}$ specify the direction cosines of Joint j relative to CG$_i$, expressed in the base-fixed, non-moving x-y coordinate system, see Fig. 2.

The reaction forces in the joints and the shaking force and moment transferred to the base by the motion of the mechanism are determined using the system of linear equations collected in the linear algebraic equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a sparse 9×9 matrix of constant coefficients scaling the vector of unknown joint reaction forces and moment, $\mathbf{x}$, while the vector $\mathbf{b}$ contains the inertial forces and moments. The matrix and vector elements are determined from free body diagrams of links a_1, a_2, a_3, and a_4. The matrix equation is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -r_{11_y} & r_{11_x} & -r_{12_y} & r_{12_x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & r_{22_y} & -r_{22_x} & -r_{23_y} & r_{23_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & r_{33_y} & -r_{33_x} & -r_{34_y} & r_{34_x} & 0 \end{bmatrix} \begin{bmatrix} F_{41_x} \\ F_{41_y} \\ F_{21_x} \\ F_{21_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ T_{41} \end{bmatrix} = \begin{bmatrix} m_1 a_{CG_{1_x}} \\ m_1 a_{CG_{1_y}} \\ I_{CG_1} \alpha_1 \\ m_2 a_{CG_{2_x}} \\ m_2 a_{CG_{2_y}} \\ I_{CG_2} \alpha_2 \\ m_3 a_{CG_{3_x}} \\ m_3 a_{CG_{3_y}} \\ I_{CG_3} \alpha_3 \end{bmatrix}. \quad (9)$$

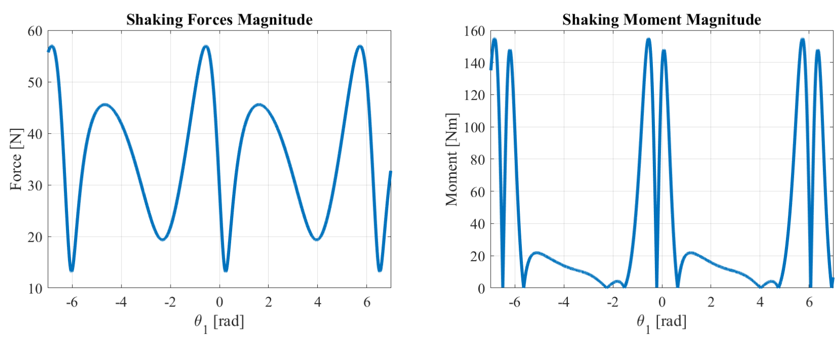
and the unknown reaction forces for a given input angle are computed as

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{10}$$



(a) Angular velocity output $\dot{\theta}_4 = f(\theta_1)$. (b) Angular acceleration output $\ddot{\theta}_4 = f(\theta_1)$.

Fig. 3: Angular velocity and acceleration output as a function of input angle.



(a) Shaking force magnitude. (b) Shaking moment magnitude.

Fig. 4: Shaking force and moment magnitudes as functions of input angle.

5 Discussion and Conclusions

The shaking force is computed as the vector sum of $\mathbf{F}_{14} + \mathbf{F}_{34}$, the forces transferred to the base link a_4 through the input and output links, a_1 and a_3, respectively, noting that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. The shaking moment is computed as $\mathbf{T}_{14} + \mathbf{r}_{34} \times \mathbf{F}_{34}$, where $\mathbf{T}_{14}$ is the reaction torque that a_1 exerts on a_4 through Joint 1. The maximum shaking force and moment magnitudes as well as corresponding input angle θ_1 given the mechanism data listed in Tab. 2 for a constant input angular velocity of $\dot{\theta}_1 = 25$ rad/s are listed in Tab. 3. Comparing the peaks in angular velocity and acceleration output for assembly mode 2, Fig. 3, with those of the peaks in shaking force and moment magnitude for assembly mode 2, Fig. 4, it is clear that the extreme values of shaking force and moment occur at input joint angles very close to those where the extreme values of angular velocity and acceleration occur. For example, the maximum shaking force of 56.9764 N occurs when $\theta_1 = 5.7299$ rad, while the maximum shaking moment 154.9566 Nm occurs when $\theta_1 = 5.7352$ rad. The maximum value of $\dot{\theta}_4 = -23.9594$ rad/s occurs when $\theta_1 = 5.9138$ rad, which is 3.2% different from $\theta_1 = 5.7299$ when the maximum shaking force occurs. Similar agreement can be observed between the angles θ_1 where the extreme shaking moments and output angular accelerations, $\ddot{\theta}_4$, occur. These results suggest that the output angular velocity and acceleration are the dominant contributors to shaking forces and moments, and that this tendency should be further investigated. If this is so, then for many applications, if the mechanism instantaneous configuration for $\dot{\theta}_{4_{\max}}$ is known, the linear accelerations can be computed and the ground forces estimated with reasonable accuracy.

Table 3: Shaking force and moment extrema for assembly mode 2.

Max. Shaking Force [N]	θ_1 [rad]	Max. Shaking Moment [Nm]	θ_1 [rad]
56.9764	$5.7299 \pm n2\pi$	154.9566	$5.7352 \pm n2\pi$

References

1. Arda, M.: Dynamic analysis of a four-bar linkage mechanism. International Scientific Journal “Machines. Technologies. Materials.” **5**, 186–190 (2020)
2. Bradley, G.L., Smith, K.J.: Calculus. Prentice Hall, Englewood Cliffs, NJ, U.S.A. (1995)
3. Briot, S., Arakelian, V.: Complete shaking force and shaking moment balancing of in-line four-bar linkages by adding a class-two RRR or RRP Assur group. Mechanism and Machine Theory **57**, 13–26 (2012)
4. Denavit, J., Hartenberg, R.S.: A Kinematic Notation for Lower-pair Mechanisms Based on Matrices. Trans ASME J. Appl. Mech. **23**, 215–221 (1955)
5. Hartenberg, R.S., Denavit, J.: Kinematic Synthesis of Linkages. McGraw-Hill, Book Co., New York, N.Y., U.S.A. (1964)
6. Hayes, M.J.D., Rotzoll, M.: Dimensional Kinematic Synthesis of Angular Velocity Function Generators. 12^{th} CCToMM Symposium on Mechanisms, Machines, and Mechatronics, Université Laval, Québec, QC, Canada (2023)
7. Hayes, M.J.D., Rotzoll, M., Bucciol, Q., Copeland, Z.A.: Planar and Spherical Four-bar Linkage v_i–v_j Algebraic Input-output Equations. Mechanism and Machine Theory **182** (2023)
8. Husty, M.L., Walter, D.R.: Mechanism constraints and singularities — the algebraic formulation. pp. 101–180. Singular Configurations of Mechanisms and Manipulators, eds. A. Müller D. Zlatanov, CISM International Centre for Mechanical Sciences (2019)
9. Norton, R.L.: Design of Machinery, *5^{th} edition*. McGraw-Hill, Book Co., New York, N.Y., U.S.A. (2012)
10. Rotzoll, M.: Algebraic Input-output Equations of Four-bar Kinematic Chains: Planar; Spherical; Spatial. Ph.D. thesis, Carleton University, Ottawa, ON, Canada (2023)