KINEMATIC MAPPING OF 3-LEGGED PLANAR PLATFORMS WITH HOLONOMIC HIGHER PAIRS

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Abstract.
A planar parallel manipulator with holonomic higher pairs is introduced. The end effector is a circular disk which rolls with out slip along the straight lines of the non–grounded rigid links of each of three 2R legs. The contact points between the disk and legs are holonomic higher pairs. The forward kinematic problem of this manipulator is unlike that of planar Stewart–Gough type platforms because the initial assembly configuration must be included in the analysis. A procedure using kinematic mapping to solve the forward kinematics is discussed and a numerical example is given.

1. Introduction

Planar kinematic mappings map the set of all planar displacements onto the points of a three dimensional projective space with Cartesian homogeneous coordinates X_i $(i = 1, 2, 3, 4)$. It has recently been shown that this mapping has important applications in robotics, specifically, in the solution of the forward kinematics (FK) problem of planar and spatial Stewart–Gough (SG) type platforms [4].

In this paper, a kinematic mapping of planar displacements will be discussed. Its application will be demonstrated by an example wherein the FK problem of a planar parallel manipulator with holonomic higher pairs are solved. An algebraic approach was successfully used in [8] to obtain the FK solutions of the general planar three legged platform, but is computationally incompatible for manipulators with higher pairs of the type introduced in this paper.

2. A Kinematic Mapping of Planar Displacements

A general displacement in the plane requires three independent coordinates to fully characterise it. It is convenient to think of the relative planar motion between two rigid bodies as the motion of a Cartesian reference coordinate system, E attached to one of the bodies, with respect to the Cartesian coordinate system, Σ attached to the other, [1]. Without loss of generality, Σ may be considered as fixed while E is free to move. Then the position of a point in E relative to Σ can be given by

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}, \tag{1}$$

where

i. (x', y') are the Cartesian coordinates of a point in E.
ii. (X', Y') are the Cartesian coordinates of the same point in Σ.
iii. (a, b) are the Cartesian coordinates of the origin of E measured in Σ, ie, the components of the position vector of the origin of E in Σ.
iv. ϕ is the rotation angle measured from the X'–axis to the x'–axis, the positive sense being counter–clockwise.

Equation (1) does not represent a linear transformation. This fact is computationally inconvenient, and can be remedied by the use of Cartesian homogeneous coordinates [7]

$$x' = \frac{x}{z} \quad , \quad y' = \frac{y}{z}$$
$$X' = \frac{X}{Z} \quad , \quad Y' = \frac{Y}{Z}.$$

Substituting these homogeneous coordinates in equation (1) and setting the homogenising coordinates to be equal, i.e. set $Z = z$, then multiplying through by z yields the following linear transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{2}$$

which represents a displacement of E with respect to Σ.

All general planar displacements may be represented as a single rotation through a finite angle about a fixed axis normal to the plane. The coordinates of the piercing point of this axis is the *pole* of the displacement.

If E and Σ are initially coincident then after the displacement the pole has the same coordinates in both E and Σ. The location of the pole of a displacement along with the rotation angle convey sufficient information to characterise the displacement.

The value of the homogenising coordinate is arbitrary. If it is set equal to $\sin\phi/2$ it can be shown that the homogeneous coordinates of the pole, which are identical in each of the two coordinate systems Σ and E, in terms of the three displacement parameters a, b and ϕ are determined by the set of equations (3) [1, 6]

$$\begin{aligned} X_p = \quad x_p = \quad & \frac{1}{2}a\sin(\phi/2) - \frac{1}{2}b\cos(\phi/2) \\ Y_p = \quad y_p = \quad & \frac{1}{2}a\cos(\phi/2) + \frac{1}{2}b\sin(\phi/2) \\ Z_p = \quad z_p = \quad & \sin\phi/2. \end{aligned} \tag{3}$$

Many mappings can be defined that map a position (a, b, ϕ) of the moving coordinate system E with respect to the fixed system Σ in the plane to a point described by the homogeneous coordinates $(X_1 : X_2 : X_3 : X_4)$ of a three dimensional projective *image space*, Σ'. The mapping used here is as follows [1, 4, 6]:

$$(X_1 : X_2 : X_3 : X_4) \quad = \quad (X_p : Y_p : Z_p : \tau Z_p), \tag{4}$$

where

$$\begin{aligned} (X_1 : X_2 : X_3 : X_4) \quad &\neq \quad (0:0:0:0) \\ \tau \quad &= \quad \cot(\phi/2) \\ 0 \leq \quad \phi \quad &< 2\pi. \end{aligned}$$

$(X_p : Y_p : Z_p)$ depend on (a, b, ϕ) as given by the set of equations (3). The image of the pole coordinates under the kinematic mapping is called the *image point* of the displacement $D(a, b, \phi)$. The image point is given by

$$\begin{aligned} (X_1 : X_2 : X_3 : X_4) \quad = \quad & [(a\sin(\phi/2) - b\cos(\phi/2) : \\ & (a\cos(\phi/2) + b\sin(\phi/2) : \\ & 2\sin(\phi/2) : 2\cos(\phi/2)]. \end{aligned} \tag{5}$$

By virtue of the relationships expressed in (5), the linear transformation operator, the matrix from equation (2), may be expressed in terms of the homogeneous coordinates of the image space, Σ'. This means that we now have a linear transformation to express a displacement of E with respect to Σ in terms of the image point as given by (5):

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (X_4^2 - X_3^2) & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & (X_4^2 - X_3^2) & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & (X_4^2 + X_3^2) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (6)$$

Since equation (6) is a linear transformation, for each unique displacement described by (a, b, ϕ) there is a corresponding point in the image space. From equation (5), the inverse mapping is obtained. That is, for a given point of the image space, the displacement parameters are obtained from

$$\begin{aligned} \tan(\phi/2) &= X_3/X_4 \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2) \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2). \end{aligned} \quad (7)$$

Any image point with $X_3 = X_4 = 0$ can not be mapped to a displacement of the plane, and must be disregarded. It can be seen from equation (5) that this condition requires $\phi = 0°$ and $\phi = 180°$ simultaneously.

3. An Application for Planar Parallel Manipulators

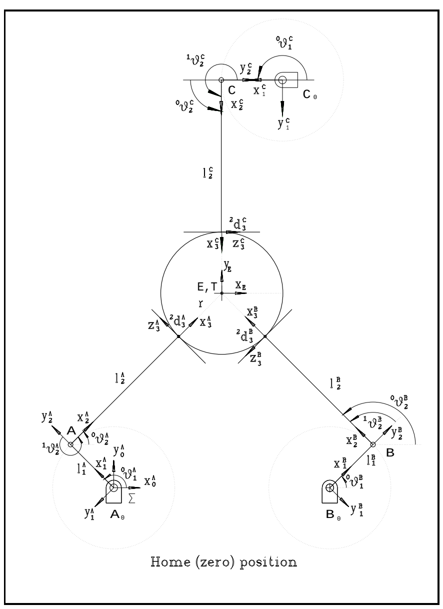

Figure 1. A planar manipulator with three DOF.

Consider the planar manipulator shown in Fig. 1. It consists of three closed kinematic chains. The disk, which is modelled as a pinion gear, rolls without slip on each of the three racks tangent to it. The rolling constraints are holonomic due to the pure rolling constraint and because the motion is planar, hence the constraint equations can be expressed in terms of displacement, i.e. in *integral form*. Each of the three legs connect a rack to a base point via two revolute (R) pairs. The leg links are rigid and the rack is welded at a right angle to the second leg link. The R–pairs connecting two links in a leg shall be referred to as *knee joints* A, B and C. The three base points A_0, B_0, C_0, are rigidly fixed.

Joint and fixed link parameters along with link reference frames are identified by left and right sub and superscripts. The generic parameter

$$ {}^{k}\Psi_{i}^{j} $$

is identified as follows:

1a. For a joint variable the right sub–script i, $i \in \{1, 2, 3\}$ identifies the joint number. For each manipulator leg, the joint number at the connection between the first link and the base is 1. Between the first and second links is 2. The higher pair between link 2 and the disk is 3.
1b. For a coordinate axis, the right sub–script i, $i \in \{0, 1, 2, 3\}$ represents the link to which the coordinate system is attached. 0 is for the base, 1 is for the first link, etc..
2. The right super–script, j, $j \in \{A, B, C\}$ denotes a particular manipulator leg.
3. The left super–script, k, $k \in \{\Sigma, 0, 1, 2, E, T\}$ refers to the reference frame in which the variable is represented.
4. l_i^j is the length of link i in leg j.
5. r is the radius of the disk.

The link reference frames, save for E and T, were assigned using the Denavit and Hartenberg procedure [2], modified to accommodate the higher pairs. For reasons discussed below, it is convenient to have these two reference frames with origins incident on the disk centre. Both E and T translate with the disk, but only E rotates with it.

3.1. THE FORWARD KINEMATICS PROBLEM

The FK problem is conventionally expressed as a transformation of the position and orientation of the end effector from a joint space representation to a Cartesian space representation. That is, given a set of n joint variables, one for each n degrees of freedom, determine the position and orientation of the end effector with respect to a non–moving reference coordinate system. The pure rolling nature of the higher pairs make this manipulator markedly different from planar SG type platforms because the pure rolling condition renders FK solutions completely dependent on the initial assembly configuration (IAC). The FK analysis can not be reduced to the planar SG case because no equivalent mechanism exists which can exactly reproduce a rack–and–pinion motion, p.106 [3], hence the method in [8] can not be used. Furthermore, some displacements require a combination of two distinct types of rolling with respect to the fixed reference frame: 1) The disk rolls on a rack, or 2), a rack rolls on the disk. Each type of rolling produces a change in the location of the contact point but yields an entirely different displacement. As a result, conventional joint variable inputs can not be used.

We propose to modify the problem by using instead a set of *pseudo inputs* from which the position and orientation of the disk in the non–moving reference frame can be determined. The pseudo inputs are the positions of the knee joints in the rotating disk frame, E. These positions are

$$\begin{bmatrix} {}^{E}A \\ {}^{E}B \\ {}^{E}C \end{bmatrix}. \tag{8}$$

Each position is specified by a 2×1 array of Cartesian coordinates, having the form

$$\begin{bmatrix} {}^{E}x_2^j \\ {}^{E}y_2^j \end{bmatrix}.$$

Hence, six pseudo input variables are required. Because the knee joints are constrained to move on circles, the position and orientation of the disk in the non-moving frame Σ can be determined with the kinematic mapping discussed earlier and the procedure introduced by Husty [4].

The actual joint inputs are the change in variable joint lengths $\Delta^2 d_3^j = {}^2d_3^j - {}^2d_{30}^j$, $j \in \{A, B, C\}$, where the subscript '30' denotes the initial condition. These lengths are the change in distance of the contact point measured along the y_2^j coordinate axis, which is always parallel to the rack. This is why the solution is coupled with the ICA. They are related to the pseudo inputs in the following way:

$$\begin{bmatrix} {}^{E}x_2^j \\ {}^{E}y_2^j \end{bmatrix} = \begin{bmatrix} c^{\Sigma}\phi & s^{\Sigma}\phi \\ -s^{\Sigma}\phi & c^{\Sigma}\phi \end{bmatrix} \begin{bmatrix} {}^{T}x_2^j \\ {}^{T}y_2^j \end{bmatrix} = \begin{bmatrix} (l_2^j + r)c^{\Sigma}\vartheta_2^j - \Delta^2 d_3^j s^{\Sigma}\vartheta_2^j \\ (l_2^j + r)s^{\Sigma}\vartheta_2^j + \Delta^2 d_3^j c^{\Sigma}\vartheta_2^j \end{bmatrix} \tag{9}$$

where $c \equiv cos$, $s \equiv sin$, and ${}^{\Sigma}\phi$ is the orientation angle of the disk with respect to Σ. Since the reference frame T translates with the disk, ${}^{T}\phi = {}^{\Sigma}\phi$ and, of course, ${}^{T}\vartheta = {}^{\Sigma}\vartheta$. So, the pseudo inputs are theoretically *valid* as input parameters, except that the actual inputs can not be specified until the disk orientation is known. The higher pair variables along with the ICA must be specified or the disk orientation can not be determined. A cart–before–horse scenario, to be sure. While this approach to the FK problem is not necessarily practical, it is a start. To the best of our knowledge the FK of such a planar parallel platform with higher pairs have never been addressed.

Using the pseudo inputs, the FK problem of the manipulator shown in Fig. 1 can be stated in the following way: Given the coordinates of the three base points A_0, B_0, C_0 in an arbitrary fixed coordinate system, Σ, the coordinates of the knee joints ${}^EA, {}^EB, {}^EC$ expressed in an arbitrary coordinate system, E, which moves with the disk, the fixed lengths of each link, l_i^j, $i \in \{1,2\}$, $j \in \{A,B,C\}$, and the radius of the disk, find the positions and orientations of the disk such that the knee joints ${}^EA, {}^EB, {}^EC$ can be joined to the base points A_0, B_0, C_0 with legs of the given lengths.

To obtain the solutions for a given set of inputs and ICA, begin by removing the disk connections with legs B and C. Observe that the higher pairs are locked by virtue of the specified input parameter. The knee joint ${}^EA$ is constrained to move on a circle with A_0 as its centre and radius l_1^A. Furthermore, the rigid body comprised of link l_2^A and the disk can rotate about ${}^EA$. This two parameter motion corresponds to a two parameter set of points in the image space called a *constraint surface*, H. The equation of H is found using equation (6) and the fact that the moving point ${}^EA$ is bound to a circle, which gives

$$\begin{aligned} H: \quad 0 = {} & z^2(X_1^2 + X_2^2) + (1/4)[(x^2+y^2) - 2C_1xz - 2C_2yz + C_3z^2]X_3^2 + \\ & (1/4)[(x^2+y^2) + 2C_1xz + 2C_2yz + C_3z^2]X_4^2 + (C_1z - x)zX_1X_3 + \\ & (C_2z - y)zX_2X_3 - (y + C_2z)zX_1X_4 + (C_1z + x)zX_2X_4 + \\ & (C_2x - C_1y)zX_3X_4. \end{aligned} \tag{10}$$

It is shown in [1] that this constraint surface is a hyperboloid which contains the isotropic points $J_1(1 : i : 0 : 0)$ and $J_2(1 : -i : 0 : 0)$. Recall that any point with $X_3 = X_4 = 0$ can not be mapped to a displacement of the plane. When the other two points B and C are examined in turn, three hyperboloidal surfaces are generated, H_A, H_B, and H_C, which correspond to the complete range of possible displacements around the points still connected. The points of intersection of H_A, H_B and H_C represent the positions of the end–effector where its three knee joints are on their respective circles. Therefore, these points of intersection constitute the solutions to the FK problem. It is to be noted that three hyperboloids can intersect in at most eight points. However, all hyperboloidal constraint surfaces corresponding to planar displacements with one point that moves on a circle contain the points J_1 and J_2 and hence, these two points are always in the solution set, and must be disregarded. Therefore, there is a maximum of six real solutions to the FK problem for manipulators of this type.

3.2. EXAMPLE

Table 1 gives the coordinates of the base points A_0, B_0, C_0 in the fixed frame Σ with origin at A_0, the input variable coordinates of the knee joints ${}^EA, {}^EB, {}^EC$ in the moving frame E, with origin at centre of the disk, along with the ICA. The ICA parameters are distinguished by adjoining a '0' to the right subscript. The fixed link lengths, in generic units are $r = 4$, $l_1^j = 4$, $l_2^j = 10$, $j \in \{A, B, C\}$.

TABLE 1. Input parameters and ICA.

j_0	${}^\Sigma x$	${}^\Sigma y$	${}^E j$	${}^E x$	${}^E y$	j	${}^2 d_{30}^j$	${}^0\vartheta_{10}^j$	${}^1\vartheta_{20}^j$
A_0	0	0	${}^EA$	-9	-11	A	0	135°	270°
B_0	13	0	${}^EB$	9	-11	B	0	45°	90°
C_0	10	26	${}^EC$	9.5	10.5	C	0	180°	270°

Substituting the input data from Table 1 into equation (10) gives the following three constraint surfaces in the image space:

$$
\begin{aligned}
H_A: \quad & X_1^2 + X_2^2 + 46.5X_3^2 + 46.5X_4^2 + 9X_1X_3 + 11X_2X_3 + \\
& 11X_1X_4 - 9X_2X_4 = 0 && (11)\\
H_B: \quad & X_1^2 + X_2^2 + 147.25X_3^2 + 30.25X_4^2 - 22X_1X_3 - 4X_2X_4 + \\
& 11X_2X_3 + 11X_1X_4 = 0 && (12)\\
H_C: \quad & X_1^2 + X_2^2 + 424.125X_3^2 + 56.125X_4^2 - 19.5X_1X_3 - \\
& 36.5X_2X_3 + 15.5X_1X_4 - .5X_2X_4 - 142X_3X_4 = 0. && (13)
\end{aligned}
$$

Since X_4 is the homogenising coordinate, its value is arbitrary, hence it is set $X_4 = 1$. Fig. 2 is a view of the resulting constraint hyperboloids where one of the intersections is visible. The set of three equations $H_A = 0, H_B = 0, H_C = 0$ can now be solved for the variables X_1, X_2, X_3. The following solutions are obtained:

$$
\begin{aligned}
S_1: \quad & X_1 = -5.35817508, X_2 = 1.69375244, X_3 = 0.18597447\\
S_2: \quad & X_1 = -4.23444169, X_2 = 3.13635972, X_3 = 0.15325037\\
S_3: \quad & X_1 = -4.71288212, X_2 = 2.25800666, X_3 = 0.20703047\\
S_4: \quad & X_1 = -6.90743973, X_2 = 2.76957064, X_3 = 0.03229377\\
S_5: \quad & X_1 = -4.306063 + 2.801994i, X_2 = 0.652824 + 0.102659i,\\
& X_3 = -0.043999 + 0.180029i\\
S_6: \quad & X_1 = -4.306063 - 2.801994i, X_2 = 0.652824 - 0.102659i,\\
& X_3 = -0.043999 - 0.180029i
\end{aligned}
$$

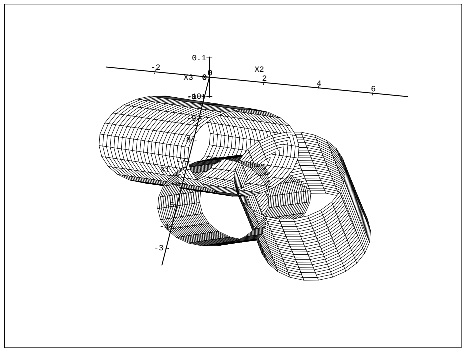

Figure 2. The constraint hyperboloids in the image space.

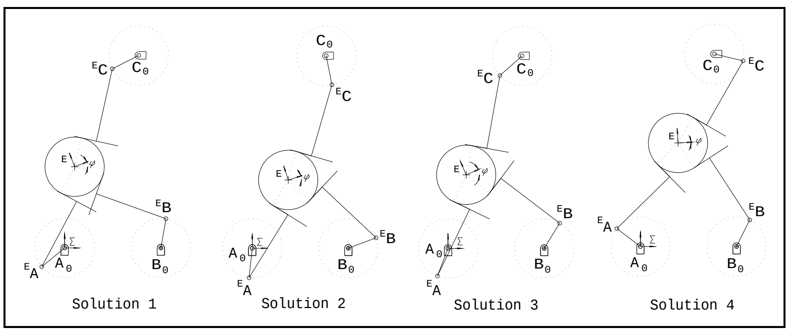

Figure 3. The four real solutions.

There are four real and one set of complex conjugate solutions for a total of six solutions, as expected. The position and orientation of the end–effector corresponding to each of these solutions in terms of the displacement parameters a, b, and ϕ can be found by substituting the solutions for X_1, X_2, X_3, along with $X_4 = 1$ into equations (7). The subsequent four sets of displacement parameters are given in Table 2. It is a simple matter of planar trigonometry to determine the the link parameters $\Delta^2 d_3^j$ and ${}^T\vartheta_2^j$, $j \in \{A, B, C\}$, also printed in Table 2, given the locations of the knee joints in E along with the fixed link lengths, disk radius and ICA. Recall that because the reference frame T translates with the disk but does not rotate, ${}^T\vartheta_2^j = {}^\Sigma\vartheta_2^j$. The four real solutions are illustrated in Figure 3.

As anticipated, the actual inputs, $\Delta^2 d_3^j$ are consistent in magnitude.

TABLE 2. Four real solutions.

	Sol'n 1	Sol'n 2	Sol'n 3	Sol'n 4
a	1.347918	4.860703	2.459188	5.087701
b	10.967028	9.213788	9.934891	13.979180
ϕ (deg.)	21.070388	17.425626	23.393454	3.699307
$\Delta^2 d_3^A$	2.449489	2.449489	2.449489	2.449489
$\Delta^2 d_3^B$	-2.449489	2.449489	2.449489	2.449489
$\Delta^2 d_3^C$	-2.121320	-2.121320	-2.121320	-2.121320
${}^T\vartheta_2^A$ (deg.)	241.856762	238.211999	244.179828	224.485681
${}^T\vartheta_2^B$ (deg.)	-19.715986	-43.209187	-37.241359	-56.935505
${}^T\vartheta_2^C$ (deg.)	77.548874	73.904112	79.871940	60.177794

There is one anomaly in the single negative $\Delta^2 d_3^B$ in the first solution. This anomaly appears to expose a flaw in the algorithm, but we are provided with three solutions where the magnitudes and sense of the actual inputs agree.

4. Conclusions

A kinematic mapping for planar displacements has been presented. An important application of this mapping is the solution of the forward kinematics problem of a planar parallel manipulator with holonomic higher pairs.

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